Dr Oliver Mathematics
Further Mathematics
Kinematics of a Particle moving in a Straight Line or a Plane
Past Examination Questions

This booklet consists of 67 questions across a variety of examination topics.
The total number of marks available is 693.

1. A particle P moves on the x-axis. The acceleration of P at time t seconds is $(4t-8)$ ms^{-2}, measured in the direction of x increasing. The velocity of P at time t seconds is v ms^{-1}. Given that $v = 6$ when $t = 0$, find

   (a) v in terms of t, (4)

   Solution

   $$a = 4t - 8 \Rightarrow v = 2t^2 - 8t + c.$$

   Now,

   $$v = 6, t = 0 : 6 = 0 - 0 + c \Rightarrow c = 6$$

   and

   $$\underline{\underline{v = 2t^2 - 8t + 6}}.$$

   (b) the distance between the two points where P is instantaneously at rest. (7)

   Solution

   $$\begin{aligned} v = 0 &\Rightarrow 2t^2 - 8t + 6 = 0 \\ &\Rightarrow 2(t^2 - 4t + 3) = 0 \\ &\Rightarrow 2(t-1)(t-3) = 0 \\ &\Rightarrow t = 1 \text{ or } t = 3. \end{aligned}$$

   Now,

   $$\begin{aligned} s &= \int_1^3 (2t^2 - 8t + 6)\,\mathrm{d}t \\ &= \left[\tfrac{2}{3}t^3 - 4t^2 + 6t\right]_{t=1}^3 \\ &= (18 - 36 + 18) - (\tfrac{2}{3} - 4 + 6) \\ &= -2\tfrac{2}{3}; \end{aligned}$$

   hence, they are $\underline{\underline{2\tfrac{2}{3}\text{ m}}}$ apart.

2. A ball B of mass 0.4 kg is struck by a bat at a point O which is 1.2 m above horizontal ground. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are respectively horizontal and vertical. Immediately before being struck, B has velocity $(-20\mathbf{i} + 4\mathbf{j})$ ms^{-1}. Immediately after being struck it has velocity $(15\mathbf{i} + 16\mathbf{j})$ ms^{-1}. After B has been struck, it moves freely under gravity and strikes the ground at the point A, as shown in Figure 1.

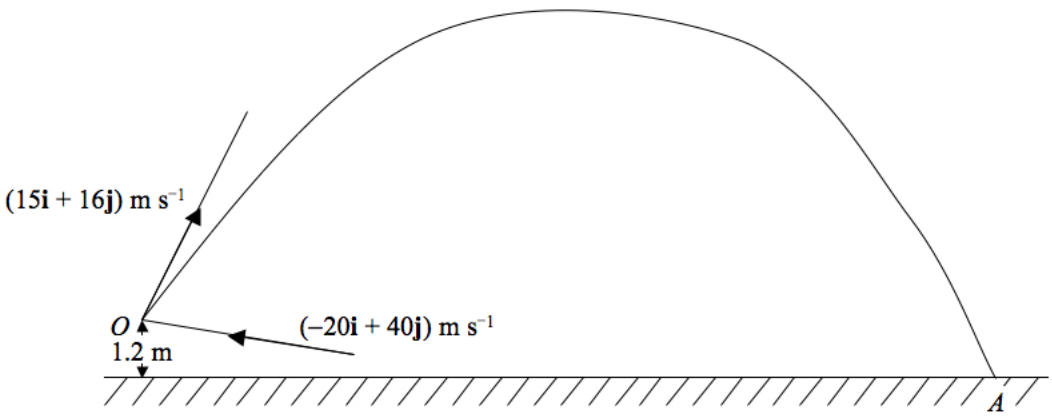

Figure 1: a ball B of mass 0.4 kg

The ball is modelled as a particle.

(a) Calculate the magnitude of the impulse exerted by the bat on B. (4)

Solution

$$\begin{aligned}\mathbf{I} &= 0.4\left[(15\mathbf{i} + 16\mathbf{j}) - (-20\mathbf{i} + 4\mathbf{j})\right] \\ &= 0.4(35\mathbf{i} + 12\mathbf{j}) \\ &= 14\mathbf{i} + 4.8\mathbf{j}\end{aligned}$$

and

$$|\mathbf{I}| = \sqrt{14^2 + 4.8^2} = \underline{\underline{14.8 \text{ Ns}}}.$$

(b) By using the principle of conservation of energy, or otherwise, find the speed of B when it reaches A. (6)

Solution

$s = -1.2$, $u =?$, $v =?$, $a = -9.8$, and $t =?$:

$$\begin{aligned}v_y^2 = u^2 + 2as \Rightarrow v_y^2 &= 16^2 + 2 \times (-9.8) \times (-1.2) \\ \Rightarrow v_y^2 &= 279.52\end{aligned}$$

and

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{15^2 + 279.52} \\ &= 22.461\,522\,66 \text{ (FCD)} \\ &= \underline{\underline{22 \text{ (2 sf)}}}. \end{aligned}$$

(c) Calculate the angle that the velocity of B makes with the ground when B reaches A. (4)

Solution

$$\begin{aligned} \text{Angle} &= \tan^{-1}\left(\frac{\sqrt{279.52}}{15}\right) \\ &= 48.101\,837\,16 \text{ (FCD)} \\ &= \underline{\underline{48^\circ \text{ (2 sf)}}}. \end{aligned}$$

(d) State two additional physical factors that could be taken into account in a refinement of the model of the situation which would make it more realistic. (2)

Solution

e.g, air resistance, wind, rotation of the ball.

3. A particle P of mass 0.75 kg is moving under the action of a single force F newtons. At time t seconds, the velocity t ms^{-1} of P is given by

$$\mathbf{v} = (t^2 + 2)\mathbf{i} - 6t\mathbf{j}.$$

(a) Find the magnitude of $\mathbf{F}$ when $t = 4$. (5)

Solution

$$\mathbf{v} = (t^2 + 2)\mathbf{i} - 6t\mathbf{j} \Rightarrow \mathbf{a} = 2t\mathbf{i} - 6\mathbf{j}$$

and

$$t = 4 \Rightarrow \mathbf{a} = 8\mathbf{i} - 6\mathbf{j}.$$

Finally,

$$\mathbf{F} = 0.75\sqrt{8^2 + 6^2} = \underline{\underline{7.5 \text{ N}}}.$$

When $t = 5$, the particle P receives an impulse of magnitude $9\sqrt{2}$ Ns in the direction of the vector $\mathbf{i} - \mathbf{j}$.

(b) Find the velocity of P immediately after the impulse. (4)

Solution

Now,

$$\mathbf{I} = 9\mathbf{i} - 9\mathbf{j}$$

and

$$\begin{aligned} 9\mathbf{i} - 9\mathbf{j} = 0.75[\mathbf{v} - (27\mathbf{i} - 30\mathbf{j})] &\Rightarrow 12\mathbf{i} - 12\mathbf{j} = \mathbf{v} - (27\mathbf{i} - 30\mathbf{j}) \\ &\Rightarrow \underline{\underline{\mathbf{v} = (39\mathbf{i} - 42\mathbf{j})\ \text{ms}^{-1}}}. \end{aligned}$$

4. A particle P is projected with velocity $(2u\mathbf{i} + 3u\mathbf{j})$ ms^{-1} from a point O on a horizontal plane, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively. The particle P strikes the plane at the point A which is 735 m from O.

(a) Show that $u = 24.5$. (6)

Solution

$$2ut = 735 \text{ and } 0 = 3ut - \tfrac{1}{2}gt^2.$$

Now,

$$\begin{aligned} 2ut = 735 &\Rightarrow t = \frac{735}{2u} \\ &\Rightarrow 0 = 3u\left(\frac{735}{2u}\right) - 4.9\left(\frac{735}{2u}\right)^2 \\ &\Rightarrow 1102.5 = \frac{661\,775\frac{5}{8}}{u^2} \\ &\Rightarrow u^2 = 600.25 \\ &\Rightarrow \underline{\underline{u = 24.5}}. \end{aligned}$$

(b) Find the time of flight from O to A. (2)

Solution

$$t = \frac{735}{2 \times 24.5} = \underline{\underline{15 \text{ s}}}.$$

The particle P passes through a point B with speed 65 ms^{-1}.

(c) Find the height of B above the horizontal plane. (4)

Solution
To begin with,
$$v^2 = (2u)^2 + (3u)^2 = 13u^2 = 7803.25$$
and
$$\begin{aligned}\tfrac{1}{2}mv^2 - \tfrac{1}{2}m(65^2) = mgh &\Rightarrow 3901.625 - 2112.5 = gh \\ &\Rightarrow h = 182.563\,775\,5 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{h = 180 \text{ m (2 sf)}}}.\end{aligned}$$

5. In this question **i** and **j** are perpendicular unit vectors in a horizontal plane. A ball has mass 0.2 kg. It is moving with velocity ($30\mathbf{i}$) ms^{-1} when it is struck by a bat. The bat exerts an impulse of $(-4\mathbf{i} + 4\mathbf{j})$ Ns on the ball. Find

(a) the velocity of the ball immediately after the impact, (3)

Solution
$$\begin{aligned}0.2(\mathbf{v} - 30\mathbf{i}) = -4\mathbf{i} + 4\mathbf{j} &\Rightarrow \mathbf{v} - 30\mathbf{i} = -20\mathbf{i} + 20\mathbf{j} \\ &\Rightarrow \underline{\underline{\mathbf{v} = 10\mathbf{i} + 20\mathbf{j}}}.\end{aligned}$$

(b) the angle through which the ball is deflected as a result of the impact, (2)

Solution
$$\tan^{-1}\tfrac{20}{10} = 63.434\,948\,82 \text{ (FCD)} = \underline{\underline{63^\circ \text{ (2 sf)}}}.$$

(c) the kinetic energy lost by the ball in the impact. (4)

Solution
$$\text{Initial KE} = \tfrac{1}{2} \times 0.2 \times 30^2 = 90$$
and
$$\text{final KE} = \tfrac{1}{2} \times 0.2 \times (10^2 + 20^2) = 50;$$
hence, it has lost $\underline{\underline{40 \text{ J}}}$.

6. At time t seconds, the velocity of a particle P is $[(4t-7)\mathbf{i}-5\mathbf{j}]$ ms^{-1}. When $t=0$, P is at the point with position vector $(3\mathbf{i}+5\mathbf{j})$ m relative to a fixed origin O.

(a) Find an expression for the position vector of P after t seconds, giving your answer in the form $(a\mathbf{i}+b\mathbf{j})$ m. (4)

Solution

$$s_P = \int [(4t-7)\mathbf{i}-5\mathbf{j}]\,\mathrm{d}t = (2t^2-7t)\mathbf{i}-5t\mathbf{j}+\mathbf{c}.$$

Now,

$$s_P = 3\mathbf{i}+5\mathbf{j}, t=0 : 3\mathbf{i}+5\mathbf{j} = \mathbf{0}+\mathbf{0}+\mathbf{c}$$

which gives

$$\underline{\underline{s_P = (2t^2-7t+3)\mathbf{i}+(5-5t)\mathbf{j}}}.$$

A second particle Q moves with constant velocity $(2\mathbf{i}-3\mathbf{j})$ ms^{-1}. When $t=0$, the position vector of Q is $(-7\mathbf{i})$ m.

(b) Prove that P and Q collide. (6)

Solution

$$s_Q = (2\mathbf{i}-3\mathbf{j})t-7\mathbf{i} = (2t-7)\mathbf{i}-3t\mathbf{j}$$

For $\mathbf{j}$:

$$\begin{aligned} 5-5t = -3t &\Rightarrow 2t = 5 \\ &\Rightarrow t = 2.5. \end{aligned}$$

For $\mathbf{i}$:

$$\begin{aligned} p_x &= 2\times 2.5^2 - 7\times 2.5 + 3 = -2 \\ p_y &= 2\times 2.5 - 7 = -2; \end{aligned}$$

hence, <u>there is collision</u>.

7. A particle P of mass 0.4 kg is moving under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity of P, $\mathbf{v}$ ms^{-1}, is given by

$$\mathbf{v} = (6t+4)\mathbf{i}+(t^2+3t)\mathbf{j}.$$

When $t=0$, P is at the point with position vector $(-3\mathbf{i}+4\mathbf{j})$ m. When $t=4$, P is at the point S.

(a) Calculate the magnitude of $\mathbf{F}$ when $t=4$. (4)

Solution

$$\mathbf{v} = (6t+4)\mathbf{i} + (t^2+3t)\mathbf{j} \Rightarrow \mathbf{a} = 6\mathbf{i} + (2t+3)\mathbf{j}$$

and

$$\begin{aligned}|\mathbf{F}| &= 0.4\sqrt{6^2 + (2\times 4+3)^2} \\ &= 5.011\,985\,634 \text{ (FCD)} \\ &= \underline{\underline{5.0 \text{ (2 sf)}}}.\end{aligned}$$

(b) Calculate the distance OS. (5)

Solution

$$\mathbf{v} = (6t+4)\mathbf{i} + (t^2+3t)\mathbf{j} \Rightarrow \mathbf{s} = (3t^2+4t)\mathbf{i} + (\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2)\mathbf{j} + \mathbf{c}.$$

Now,

$$t = 0, \mathbf{s} = -3\mathbf{i} + 4\mathbf{j} \Rightarrow \mathbf{s} = \mathbf{0} + \mathbf{0} - 3\mathbf{i} + 4\mathbf{j}$$

and

$$\mathbf{s} = (3t^2+4t-3)\mathbf{i} + (\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 + 4)\mathbf{j}.$$

For $t = 4$,

$$\mathbf{s} = 61\mathbf{i} + \tfrac{148}{3}\mathbf{j}$$

and

$$OS = \sqrt{61^2 + (\tfrac{148}{3})^2} = 78.452\,391\,79 \text{ (FCD)} = \underline{\underline{78 \text{ m (2 sf)}}}.$$

8. A particle P is projected from a point A with speed 32 ms^{-1} at an angle of elevation α, where $\sin\alpha = \frac{3}{5}$. The point O is on horizontal ground, with O vertically below A and $OA = 20$ m. The particle P moves freely under gravity and passes through a point B, which is 16 m above the ground, before reaching the ground at the point C, as shown in Figure 2.

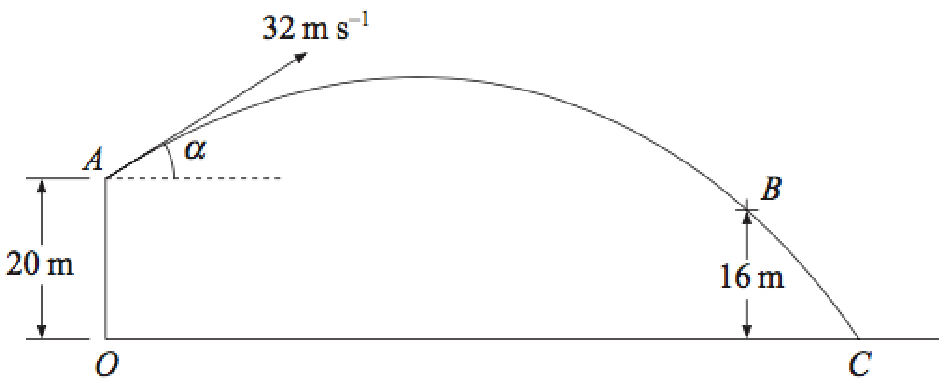

Figure 2: a particle P is projected from a point A

Calculate

(a) the time of flight from A to C, (5)

Solution
The vertical component of the velocity is

$$\tfrac{3}{5} \times 32 = 19.2.$$

$s = -20\,(\downarrow)$, $u = 19.2$, $v = ?$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} s = ut + \tfrac{1}{2}at^2 &\Rightarrow -20 = 19.2t - 4.9t^2 \\ &\Rightarrow 4.9t^2 - 19.2t - 20 = 0 \\ &\Rightarrow t = -0.855\,071\,702\,2 \text{ or } t = 4.773\,439\,049 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 4.8 \text{ (2 sf)}}}. \end{aligned}$$

(b) the distance OC, (3)

Solution
The horizontal component of the velocity is

$$\tfrac{4}{5} \times 32 = 25.6$$

and the distance is

$$25.6 \times 4.773\ldots = 122.200\,039\,7 \text{ (FCD)} = \underline{\underline{120 \text{ m (2 sf)}}}.$$

(c) the speed of P at B, (4)

Solution

$s = -4\,(\downarrow)$, $u = 19.2$, $v = ?$, $a = -9.8$, and $t = ?$:

$$v^2 = u^2 + 2as \Rightarrow v^2 = 19.2^2 + 2 \times (-9.8) \times (-4)$$
$$\Rightarrow v^2 = 447.04,$$

and

$$\text{speed} = \sqrt{25.6^2 + 447.04} = 32.202\,409\,55 \text{ (FCD)} = \underline{\underline{33 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

(d) the angle that the velocity of P at B makes with the horizontal. (3)

Solution

$$\tan^{-1} \frac{\sqrt{447.04}}{25.6} = 39.553\,673\,63 \text{ (FCD)} = \underline{\underline{40^\circ \text{ (2 sf)}}}.$$

9. A particle P moves in a horizontal plane. At time t seconds, the position vector of P is $\mathbf{r}$ metres relative to a fixed origin O, and $\mathbf{r}$ is given by

$$\mathbf{r} = (18t - 4t^3)\mathbf{i} + ct^2\mathbf{j},$$

where c is a positive constant. When $t = 1.5$, the speed of P is 15 ms^{-1}. Find

(a) the value of c, (6)

Solution

$$\mathbf{r} = (18t - 4t^3)\mathbf{i} + ct^2\mathbf{j} \Rightarrow \mathbf{v} = (18 - 12t^2)\mathbf{i} + 2ct\mathbf{j}$$

and

$$|\mathbf{v}| = 15 \Rightarrow 9^2 + (3c)^2 = 15^2$$
$$\Rightarrow 9c^2 = 144$$
$$\Rightarrow c^2 = 16$$
$$\Rightarrow \underline{\underline{c = 4}}$$

as c is a positive constant.

(b) the acceleration of P when $t = 1.5$. (3)

Solution

$$\mathbf{v} = (18 - 12t^2)\mathbf{i} + 8t\mathbf{j} \Rightarrow \mathbf{a} = -24t\mathbf{i} + 8\mathbf{j}$$

and $t = 1.5$ gives

$$\mathbf{a} = \underline{\underline{-36\mathbf{i} + 8\mathbf{j}}}.$$

10. A darts player throws darts at a dart board which hangs vertically. The motion of a dart is modelled as that of a particle moving freely under gravity. The darts move in a vertical plane which is perpendicular to the plane of the dart board. A dart is thrown horizontally with speed 12.6 ms^{-1}. It hits the board at a point which is 10 cm below the level from which it was thrown.

    (a) Find the horizontal distance from the point where the dart was thrown to the dart board. (4)

Solution

$$x = 12.6t \text{ and } 0.1 = 0^2 + \tfrac{1}{2}gt^2;$$

$$\begin{aligned} t = \frac{x}{12.6} &\Rightarrow 0.1 = \frac{4.9 \times x^2}{12.6^2} \\ &\Rightarrow x^2 = 3.24 \\ &\Rightarrow \underline{\underline{x = 1.8 \text{ m}}}. \end{aligned}$$

The darts player moves his position. He now throws a dart from a point which is at a horizontal distance of 2.5 m from the board. He throws the dart at an angle of elevation α to the horizontal, where $\tan\alpha = \frac{7}{24}$. This dart hits the board at a point which is at the same level as the point from which it was thrown.

    (b) Find the speed with which the dart is thrown. (6)

Solution

Now,

$$\sin\alpha = \tfrac{7}{25} \text{ and } \cos\alpha = \tfrac{24}{25}$$

and

$$ut\cos\alpha = 2.5 \text{ and } 0 = ut\sin\alpha - \tfrac{1}{2}gt^2 \Rightarrow u\sin\alpha = \tfrac{1}{2}gt;$$

$$t = \frac{62.5}{24u} \Rightarrow \frac{7u}{25} = 4.9t \times \frac{62.5}{24u}$$
$$\Rightarrow u^2 = 45\tfrac{55}{96}$$
$$\Rightarrow u = \tfrac{25\sqrt{42}}{24}$$
$$\Rightarrow \underline{\underline{u = 6.8 \text{ m (2 sf)}}}.$$

11. A particle P of mass 0.4 kg is moving so that its position vector $\mathbf{r}$ metres at time t seconds is given by

$$\mathbf{r} = (t^2 + 4t)\mathbf{i} + (3t - t^3)\mathbf{j}.$$

(a) Calculate the speed of P when $t = 3$. (5)

Solution

$$\mathbf{r} = (t^2 + 4t)\mathbf{i} + (3t - t^3)\mathbf{j} \Rightarrow \mathbf{v} = (2t + 4)\mathbf{i} + (3 - 3t^2)\mathbf{j}$$
$$\Rightarrow \mathbf{v}|_{t=3} = 10\mathbf{i} - 24\mathbf{j}$$
$$\Rightarrow |\mathbf{v}|_{t=3}| = \sqrt{10^2 + 24^2}$$
$$\Rightarrow |\mathbf{v}|_{t=3}| = \underline{\underline{26}}.$$

When $t = 3$, the particle P is given an impulse $(8\mathbf{i} - 12\mathbf{j})$ Ns.

(b) Find the velocity of P immediately after the impulse. (3)

Solution

$$0.4\left[\mathbf{v} - (10\mathbf{i} - 24\mathbf{j})\right] = 8\mathbf{i} - 12\mathbf{j}$$
$$\Rightarrow \quad \mathbf{v} - (10\mathbf{i} - 24\mathbf{j}) = 20\mathbf{i} - 30\mathbf{j}$$
$$\Rightarrow \quad \underline{\underline{\mathbf{v} = (30\mathbf{i} - 54\mathbf{j}) \text{ ms}^{-1}}}.$$

12. The object of a game is to throw a ball B from a point A to hit a target T which is placed at the top of a vertical pole, as shown in Figure 3.

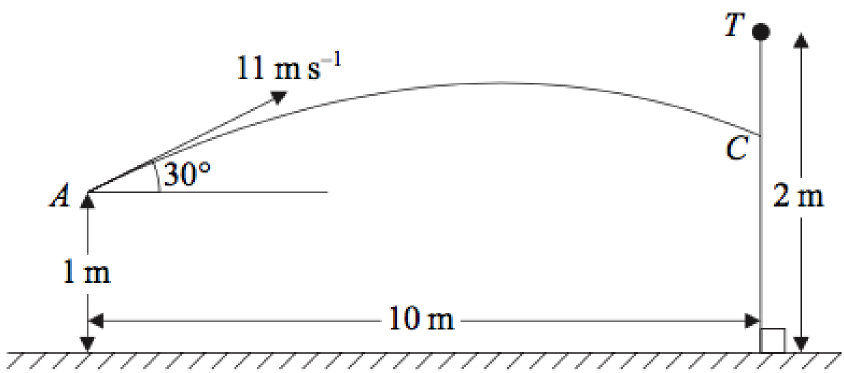

Figure 3: a ball B from a point A to hit a target T

The point A is 1 m above horizontal ground and the height of the pole is 2 m. The pole is at a horizontal distance of 10 m from A. The ball B is projected from A with a speed of 11 ms^{-1} at an angle of elevation of 30°. The ball hits the pole at the point C. The ball B and the target T are modelled as particles.

(a) Calculate, to 2 decimal places, the time taken for B to move from A to C. (3)

Solution

The horizontal speed of the ball is

$$11\cos 30^\circ = \frac{11\sqrt{3}}{2}$$

and

$$\text{time taken} = \frac{10}{\frac{11\sqrt{3}}{2}} = \frac{20\sqrt{3}}{33} = \underline{\underline{1.05 \text{ s}}}.$$

(b) Show that C is approximately 0.63 m below T. (4)

Solution

$s = ?\,(\uparrow)$, $u = 11\sin 30^\circ = 5.5$, $v = ?$, $a = -9.8$, and $t = \frac{20\sqrt{3}}{33}$:

$$\begin{aligned} s &= ut + \tfrac{1}{2}at^2 \\ &= 5.5 \times \frac{20\sqrt{3}}{33} - 4.9\left(\frac{20\sqrt{3}}{33}\right)^2 \\ &= 0.374\,053\,656\,1 \text{ (FCD)}; \end{aligned}$$

thus, it is

$$(2-1) - 0.37 = \underline{\underline{0.63 \text{ m below } T}}.$$

The ball is thrown again from A. The speed of projection of B is increased to V ms^{-1}, the angle of elevation remaining $30°$. This time B hits T.

(c) Calculate the value of V. (6)

Solution

$$tV\cos 30° = 10 \text{ and } 1 = tV\sin 30° - 4.9t^2;$$

$$\begin{aligned}
t = \frac{10}{V\cos 30°} &\Rightarrow 1 = V\sin 30°\left(\frac{10}{V\cos 30°}\right) - 4.9\left(\frac{10}{V\cos 30°}\right)^2 \\
&\Rightarrow 1 = 10\tan 30° - \frac{490}{V^2\cos^2 30°} \\
&\Rightarrow \frac{490}{V^2\cos^2 30°} = 10\tan 30° - 1 \\
&\Rightarrow \frac{V^2\cos^2 30°}{490} = \frac{1}{10\tan 30° - 1} \\
&\Rightarrow V^2\cos^2 30° = \frac{490}{10\tan 30° - 1} \\
&\Rightarrow V^2 = \frac{490}{\cos^2 30°(10\tan 30° - 1)} \\
&\Rightarrow V = \sqrt{\frac{490}{\cos^2 30°(10\tan 30° - 1)}} \\
&\Rightarrow V = 11.699\,002\,19 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{V = 12 \text{ (2 sf)}}}.
\end{aligned}$$

(d) Explain why, in practice, a range of values of V would result in B hitting the target. (1)

Solution

B and T are **not** particles.

13. A particle P moves on the x-axis. At time t seconds, its acceleration is $(5 - 2t)$ ms^{-2}, measured in the direction of x increasing. When $t = 0$, its velocity is 12 ms^{-1} measured in the direction of x increasing. Find the time when P is instantaneously at rest in the subsequent motion. (6)

Solution

$$a = 5 - 2t \Rightarrow v = 5t - t^2 + c$$

and

$$t = 0, v = 6 \Rightarrow 6 = 0 - 0 + c$$

which means that

$$v = 5t - t^2 + 6.$$

Now,

$$\begin{aligned} v = 0 &\Rightarrow 5t - t^2 + 6 = 0 \\ &\Rightarrow t^2 - 5t - 6 = 0 \\ &\Rightarrow (t-6)(t+1) = 0 \\ &\Rightarrow t - 6 = 0 \text{ or } t + 1 = 0 \\ &\Rightarrow \underline{\underline{t = 6 \text{ s (only)}}}. \end{aligned}$$

14. A cricket ball of mass 0.5 kg is struck by a bat. Immediately before being struck, the velocity of the ball is $(-30\mathbf{i})$ ms^{-1}. Immediately after being struck, the velocity of the ball is $(16\mathbf{i} + 20\mathbf{j})$ ms^{-1}.

(a) Find the magnitude of the impulse exerted on the ball by the bat. (4)

Solution

$$\begin{aligned} \text{Impulse} &= 0.5|(16\mathbf{i} + 20\mathbf{j}) - (-30\mathbf{i})| \\ &= 0.5|46\mathbf{i} + 20\mathbf{j}| \\ &= 0.5\sqrt{46^2 + 20^2} \\ &= 25.079\,872\,41 \text{ (FCD)} \\ &= \underline{\underline{25 \text{ Ns (2 sf)}}}. \end{aligned}$$

In the subsequent motion, the position vector of the ball is $\mathbf{r}$ metres at time t seconds. In a model of the situation, it is assumed that

$$\mathbf{r} = [16t\mathbf{i} + (20t - 5t^2)\mathbf{j}].$$

Using this model,

(b) find the speed of the ball when $t = 3$. (4)

Solution

$$\mathbf{r} = 16t\mathbf{i} + (20t - 5t^2)\mathbf{j} \Rightarrow \mathbf{v} = 16\mathbf{i} + (20 - 10t)\mathbf{j}$$
$$\Rightarrow \mathbf{v}|_{t=3} = 16\mathbf{i} - 10\mathbf{j}$$
$$\Rightarrow |\,\mathbf{v}|_{t=3}\,| = \sqrt{16^2 + 10^2}$$
$$\Rightarrow |\,\mathbf{v}|_{t=3}\,| = 18.867\,962\,26 \text{ (FCD)}$$
$$\Rightarrow |\,\mathbf{v}|_{t=3}\,| = \underline{\underline{19 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

15. A vertical cliff is 73.5 m high. Two stones A and B are projected simultaneously. Stone A is projected horizontally from the top of the cliff with speed 28 ms^{-1}. Stone B is projected from the bottom of the cliff with speed 35 ms^{-1} at an angle α above the horizontal. The stones move freely under gravity in the same vertical plane and collide in mid-air. By considering the horizontal motion of each stone,

(a) prove that $\cos\alpha = \frac{4}{5}$. (4)

Solution

$$x_A = 28t \text{ and } x_B = 35t\cos\alpha$$

and

$$28t = 35t\cos\alpha \Rightarrow \underline{\underline{\cos\alpha = \tfrac{4}{5}}}.$$

(b) Find the time which elapses between the instant when the stones are projected and the instant when they collide. (4)

Solution

$$y_A = 73.5 - 4.9t^2 \text{ and } y_B = 21t - 4.9t^2$$

and

$$73.5 - 4.9t^2 = 21t - 4.9t^2 \Rightarrow \underline{\underline{t = 3.5 \text{ s}}}.$$

16. A particle P of mass 0.5 kg is moving under the action of a single force $\mathbf{F}$ newtons. At time t seconds, $\mathbf{F} = (1.5t^2 - 3)\mathbf{i} + 2t\mathbf{j}$. When $t = 2$, the velocity of P is $(-4\mathbf{i} + 5\mathbf{j})$ ms^{-1}.

(a) Find the acceleration of P at time t seconds. (2)

Solution

$$\mathbf{F} = m\mathbf{a} \Rightarrow (1.5t^2 - 3)\mathbf{i} + 2t\mathbf{j} = 0.5\mathbf{a}$$
$$\Rightarrow \underline{\underline{\mathbf{a} = [(3t^2 - 6)\mathbf{i} + 4t\mathbf{j}] \text{ ms}^{-2}}}.$$

(b) Show that, when $t = 3$, the velocity of P is $(9\mathbf{i} + 15\mathbf{j})$ ms^{-1}. (5)

Solution

$$\mathbf{a} = (3t^2 - 6)\mathbf{i} + 4t\mathbf{j} \Rightarrow \mathbf{v} = (t^3 - 6t)\mathbf{i} + 2t^2\mathbf{j} + \mathbf{c}.$$

Now,

$$t = 2 \Rightarrow -4\mathbf{i} + 8\mathbf{j} + \mathbf{c} = -4\mathbf{i} + 5\mathbf{j} \Rightarrow \mathbf{c} = -3\mathbf{j}$$

and

$$\mathbf{v} = (t^3 - 6t)\mathbf{i} + (2t^2 - 3)\mathbf{j}.$$

Finally,

$$t = 3 \Rightarrow \mathbf{v} = \underline{\underline{(9\mathbf{i} + 15\mathbf{j})\ \text{ms}^{-1}}}.$$

When $t = 3$, the particle P receives an impulse $\mathbf{Q}$ Ns. Immediately after the impulse the velocity of P is $(-3\mathbf{i} + 20\mathbf{j})$ ms^{-1}. Find

(c) the magnitude of $\mathbf{Q}$, (3)

Solution

$$\begin{aligned}\mathbf{Q} &= 0.5[(-3\mathbf{i} + 20\mathbf{j}) - (9\mathbf{i} + 15\mathbf{j})] \\ &= 0.5(-12\mathbf{i} + 5\mathbf{j})\end{aligned}$$

and

$$|\mathbf{Q}| = 0.5 \times 13 = \underline{\underline{6.5\ \text{Ns}}}.$$

(d) the angle between $\mathbf{Q}$ and $\mathbf{i}$. (3)

Solution

$$\sin^{-1}\tfrac{5}{13} = 22.619\,864\,95 \text{ (FCD)}$$

and

$$180 - 22.619\ldots = 157.380\,013\,1 \text{ (FCD)}$$

and the required angle is $\underline{\underline{157^\circ \text{ (3 sf)}}}$.

17. A particle P is projected from a point A with speed u ms^{-1} at an angle of elevation θ, where $\cos\theta = \frac{4}{5}$. The point B, on horizontal ground, is vertically below A and $AB = 45$ m. After projection, P moves freely under gravity passing through a point C, 30 m above the ground, before striking the ground at the point D, as shown in Figure 4.

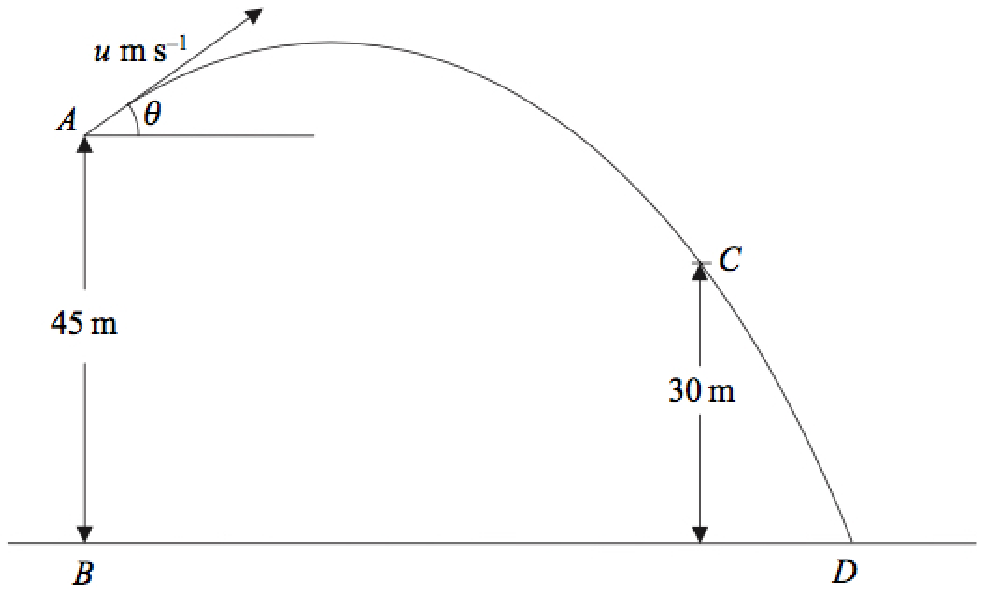

Figure 4: a particle P is projected from a point A

Given that P passes through C with speed 24.5 ms^{-1},

(a) using conservation of energy, or otherwise, show that $u = 17.5$, (4)

Solution
Using conservation of energy,

$$\tfrac{1}{2}m(24.5)^2 - \tfrac{1}{2}mu^2 = 15mg \Rightarrow 600.25 - u^2 = 294$$
$$\Rightarrow u^2 = 306.25$$
$$\Rightarrow \underline{\underline{u = 17.5}},$$

as required.

(b) find the size of the angle which the velocity of P makes with the horizontal as P passes through C, (3)

Solution
Now,

$$u\cos\theta = 17.5 \times \tfrac{4}{5} = 14$$

and

$$\cos^{-1}\tfrac{14}{24.5} = 55.150\,095\,42 \text{ (FCD)}$$
$$= \underline{\underline{55^\circ \text{ (2 sf)}}}.$$

(c) find the distance BD. (7)

Solution
First,

$$\cos\theta = \tfrac{4}{5} \Rightarrow \sin\theta = \tfrac{3}{5}$$

and

$$17.5 \times \tfrac{3}{5} = 10.5$$

is the initial vertical velocity.
Second, $s = -45$, $u = 10.5$, $v = ?$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} s = ut + \tfrac{1}{2}at^2 &\Rightarrow -45 = 10.5t - 4.9t^2 \\ &\Rightarrow 4.9t^2 - 10.5t - 45 = 0 \\ &\Rightarrow 49t^2 - 105t - 450 = 0 \\ &\Rightarrow (7t - 30)(7t + 15) = 0 \\ &\Rightarrow t = \tfrac{30}{7} \text{ (only)} \end{aligned}$$

which leaves

$$BD = 14 \times \tfrac{30}{7} = \underline{\underline{60 \text{ m}}}.$$

18. A particle P of mass 0.5 kg moves under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity v ms^{-1} of P is given by

$$\mathbf{v} = 3t^2\mathbf{i} + (1 - 4t)\mathbf{j}.$$

Find

(a) the acceleration of P at time t seconds, (2)

Solution

$$\mathbf{v} = 3t^2\mathbf{i} + (1 - 4t)\mathbf{j} \Rightarrow \underline{\underline{\mathbf{a} = 6t\mathbf{i} - 4\mathbf{j}}}.$$

(b) the magnitude of $\mathbf{F}$ when $t = 2$. (4)

Solution

$$\begin{aligned} t = 2 &\Rightarrow \mathbf{a} = 12\mathbf{i} - 4\mathbf{j} \\ &\Rightarrow |\mathbf{F}| = 0.5\sqrt{12^2 + 4^2} \\ &\Rightarrow |\mathbf{F}| = 2\sqrt{10} \\ &\Rightarrow |\mathbf{F}| = \underline{\underline{6.3 \text{ N (2 sf)}}}. \end{aligned}$$

19. A golf ball P is projected with speed 35 ms^{-1} from a point A on a cliff above horizontal ground. The angle of projection is α to the horizontal, where $\tan\alpha = \frac{4}{3}$. The ball moves freely under gravity and hits the ground at the point B, as shown in Figure 5.

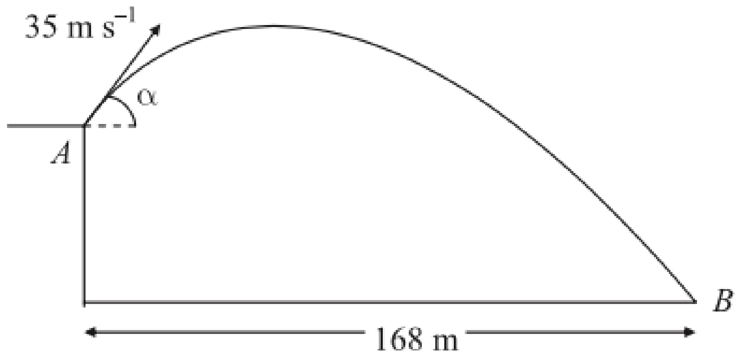

Figure 5: a golf ball P is projected with speed 35 ms^{-1}

(a) Find the greatest height of P above the level of A. (3)

Solution

$$\tan\alpha = \tfrac{4}{3} \Rightarrow \sin\alpha = \tfrac{4}{5}, \cos\alpha = \tfrac{3}{5}.$$

$s = ?\,(\uparrow)$, $u = 35 \times \frac{4}{5} = 28$, $v = 0$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} v^2 = u^2 + 2as &\Rightarrow 0 = 28^2 + 2 \times (-9.8) \times s \\ &\Rightarrow 19.6s = 784 \\ &\Rightarrow \underline{\underline{s = 40 \text{ m}}}. \end{aligned}$$

The horizontal distance from A to B is 168 m.

(b) Find the height of A above the ground. (6)

Solution

$$35\cos\alpha t = 168 \Rightarrow t = 8.$$

$s = ?\,(\downarrow)$, $u = -28$, $v = ?$, $a = 9.8$, and $t = 8$:

$$\begin{aligned} s &= ut + \tfrac{1}{2}at^2 \\ &= -28 \times 8 + 4.9 \times 8^2 \\ &= \underline{\underline{89.6 \text{ m}}}. \end{aligned}$$

By considering energy, or otherwise,

(c) find the speed of P as it hits the ground at B. (3)

Solution

$$\begin{aligned}\tfrac{1}{2}mv^2 - \tfrac{1}{2}m(35^2) = m \times 9.8 \times 89.6 &\Rightarrow v^2 - 1225 = 1756.16 \\ &\Rightarrow v^2 = 2981.16 \\ &\Rightarrow \underline{\underline{v = 54.6 \text{ ms}^{-1}}}.\end{aligned}$$

20. At time t seconds ($t \geqslant 0$), a particle P has position vector p metres, with respect to a fixed origin O, where

$$\mathbf{p} = (3t^2 - 6t + 4)\mathbf{i} + (3t^3 - 4t)\mathbf{j}.$$

Find

(a) the velocity of P at time t seconds, (2)

Solution

$$\mathbf{p} = (3t^2 - 6t + 4)\mathbf{i} + (3t^3 - 4t)\mathbf{j} \Rightarrow \underline{\underline{\mathbf{v} = (6t - 6)\mathbf{i} + (9t^2 - 4)\mathbf{j}}}.$$

(b) the value of t when P is moving parallel to the vector $\mathbf{i}$. (3)

Solution

$$\begin{aligned}\mathbf{j} = 0 &\Rightarrow 9t^2 - 4 = 0 \\ &\Rightarrow 9t^2 = 4 \\ &\Rightarrow t^2 = \tfrac{4}{9} \\ &\Rightarrow \underline{\underline{t = \tfrac{2}{3} \text{ s}}}.\end{aligned}$$

When $t = 1$, the particle P receives an impulse of $(2\mathbf{i} - 6\mathbf{j})$ Ns. Given that the mass of P is 0.5 kg,

(c) find the velocity of P immediately after the impulse. (4)

Solution

$$t = 1 \Rightarrow \mathbf{v} = 5\mathbf{j}$$

$$0.5(\mathbf{v} - 5\mathbf{j}) = 2\mathbf{i} - 6\mathbf{j} \Rightarrow \mathbf{v} - 5\mathbf{j}) = 4\mathbf{i} - 12\mathbf{j}$$
$$\Rightarrow \underline{\underline{\mathbf{v} = (4\mathbf{i} - 7\mathbf{j})\ \text{ms}^{-1}}}.$$

21. In this question, the unit vectors **i** and **j** are in a vertical plane, **i** being horizontal and **j** being vertical. A particle P is projected from the point A which has position vector $47.5\mathbf{j}$ metres with respect to a fixed origin O. The velocity of projection of P is $(2u\mathbf{i} + 5u\mathbf{j})$ ms^{-1}. The particle moves freely under gravity passing through the point B with position vector $30\mathbf{i}$ metres, as shown in Figure 6.

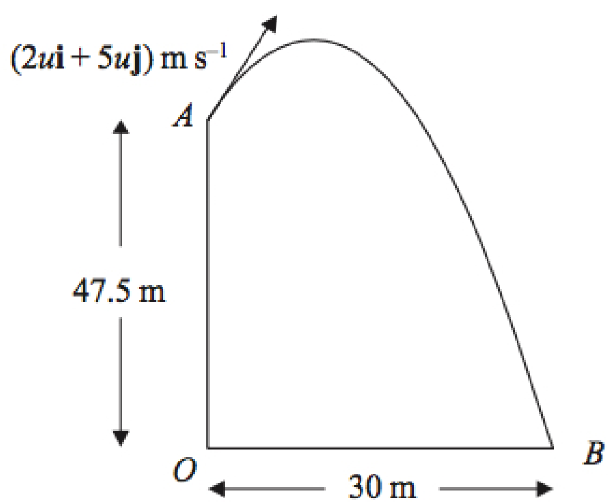

Figure 6: a particle P is projected from the point A

(a) Show that the time taken for P to move from A to B is 5 s. (6)

Solution

$$2ut = 30 \text{ and } -47.5 = 5ut - 4.9t^2;$$

$$u = \frac{30}{2t} \Rightarrow -47.5 = 5t \times \frac{30}{2t} - 4.9t^2$$
$$\Rightarrow -47.5 = 75 - 4.9t^2$$
$$\Rightarrow t^2 = 25$$
$$\Rightarrow \underline{\underline{t = 5 \text{ s}}}.$$

(b) Find the value of u. (2)

Solution

$$t = 5 \Rightarrow u = \frac{30}{2 \times 5} = \underline{\underline{3 \text{ ms}^{-1}}}.$$

(c) Find the speed of P at B. (5)

Solution

$$\begin{aligned} \tfrac{1}{2}mv^2 - \tfrac{1}{2}m(6^2 + 15^2) = m \times 9.8 \times 47.5 &\Rightarrow v^2 - 261 = 931 \\ &\Rightarrow v^2 = 1192 \\ &\Rightarrow v = 34.525\,353 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{v = 35 \text{ ms}^{-1} \text{ (2 sf)}}}. \end{aligned}$$

22. A particle P of mass 0.5 kg is moving under the action of a single force $\mathbf{F}$ newtons. At time t seconds,

$$\mathbf{F} = (6t - 5)\mathbf{i} + (t^2 - 2t)\mathbf{j}.$$

The velocity of P at time t seconds is v ms^{-1}. When $t = 0$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$.

(a) Find $\mathbf{v}$ at time t seconds. (6)

Solution

$$\begin{aligned} \mathbf{F} = m\mathbf{a} \Rightarrow \mathbf{a} &= (12t - 10)\mathbf{i} + (2t^2 - 4t)\mathbf{j} \\ \Rightarrow \mathbf{v} &= (6t^2 - 10t)\mathbf{i} + (\tfrac{2}{3}t^3 - 2t^2)\mathbf{j} + \mathbf{c}. \end{aligned}$$

Now,

$$t = 0, \mathbf{v} = \mathbf{i} - 4\mathbf{j} \Rightarrow \underline{\underline{\mathbf{v} = (6t^2 - 10t + 1)\mathbf{i} + (\tfrac{2}{3}t^3 - 2t^2 - 4)\mathbf{j}}}.$$

When $t = 3$, the particle P receives an impulse $(-5\mathbf{i} + 12\mathbf{j})$ Ns.

(b) Find the speed of P immediately after it receives the impulse. (6)

Solution

$$t = 3 \Rightarrow \mathbf{v} = 25\mathbf{i} - 4\mathbf{j}$$

and

$$0.5[\mathbf{v} - (25\mathbf{i} - 4\mathbf{j})] = -5\mathbf{i} + 12\mathbf{j} \Rightarrow \mathbf{v} - (25\mathbf{i} - 4\mathbf{j}) = -10\mathbf{i} + 24\mathbf{j}$$
$$\Rightarrow \mathbf{v} = 15\mathbf{i} + 20\mathbf{j}$$
$$\Rightarrow |\mathbf{v}| = \sqrt{15^2 + 20^2}$$
$$\Rightarrow |\mathbf{v}| = \underline{\underline{25 \text{ ms}^{-1}}}.$$

23. A ball is thrown from a point A at a target, which is on horizontal ground. The point A is 12 m above the point O on the ground. The ball is thrown from A with speed 25 ms^{-1} at an angle of 30° below the horizontal. The ball is modelled as a particle and the target as a point T. The distance OT is 15 m. The ball misses the target and hits the ground at the point B, where OTB is a straight line, as shown in Figure 7.

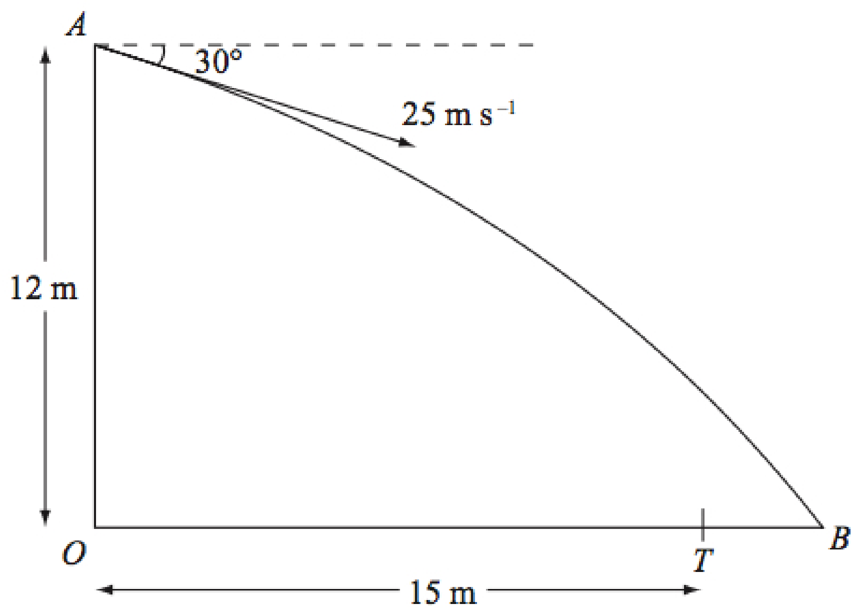

Figure 7: a ball is thrown from a point A at a target

Find

(a) the time taken by the ball to travel from A to B, (5)

Solution

$s = 12, (\downarrow)$, $u = 25\sin 30^\circ = 12.5$, $v = ?$, $a = 9.8$, and $t = ?$:

$$\begin{aligned}
s = ut + \tfrac{1}{2}at^2 &\Rightarrow 12 = 12.5t + 4.9t^2 \\
&\Rightarrow 4.9t^2 + 12.5t - 12 = 0 \\
&\Rightarrow 49t^2 + 125t - 120 = 0 \\
&\Rightarrow t = \frac{-125 \pm \sqrt{(-125)^2 - 4 \times 49 \times (-120)}}{98} \\
&\Rightarrow t = \frac{-125 \pm \sqrt{39\,145}}{98} \\
&\Rightarrow t = -3.294\,397\,49 \text{ or } t = 0.743\,377\,081\,7 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{t = 0.74 \text{ (2 sf)}}}.
\end{aligned}$$

(b) the distance TB. (4)

Solution

$$25\cos 30^\circ \times 0.743\ldots = 16.094\,585\,93 \text{ (FCD)};$$

finally,

$$16.094\ldots - 15 = 1.094\,585\,93 \text{ (FCD)} = \underline{\underline{1.1 \text{ m (2 sf)}}}.$$

The point X is on the path of the ball vertically above T.

(c) Find the speed of the ball at X. (5)

Solution

$$15 = u_x t \Rightarrow t = \frac{15}{u_x} = \frac{2\sqrt{3}}{5}$$

and

$$u_y = 12.5 + 9.8 \times \frac{2\sqrt{3}}{5} = \frac{625 + 196\sqrt{3}}{50}.$$

Finally,

$$\begin{aligned}
v &= \sqrt{u_x^2 + u_y^2} \\
&= 28.997\,244\,34 \text{ (FCD)} \\
&= \underline{\underline{29 \text{ ms}^{-1} \text{ (2 sf)}}}.
\end{aligned}$$

24. A particle P moves along the x-axis in a straight line so that, at time t seconds, the velocity of P is v ms^{-1}, where

$$v = \begin{cases} 10t - 2t^2, & 0 \leqslant t \leqslant 6, \\ -\dfrac{432}{t^2}, & t > 6. \end{cases}$$

At $t = 0$, P is at the origin O. Find the displacement of P from O when

(a) $t = 6$, (3)

Solution

$$v = 10t - 2t^2 \Rightarrow s = 5t^2 - \tfrac{2}{3}t^3 + c$$

and

$$t = 0 \Rightarrow 0 = 0 - 0 + c$$

and this means

$$s = 5t^2 - \tfrac{2}{3}t^3.$$

Finally,

$$t = 6 \Rightarrow \underline{\underline{s = 36 \text{ m}}}.$$

(b) $t = 10$. (5)

Solution

$$v = -\frac{432}{t^2} \Rightarrow s = \frac{432}{t} + d$$

and

$$t = 6 \Rightarrow 36 = \frac{432}{6} + d \Rightarrow d = -36$$

and this means

$$s = \frac{432}{t} - 36.$$

Finally,

$$t = 10 \Rightarrow \underline{\underline{s = 7.2 \text{ m}}}.$$

25. A cricket ball is hit from a point A with velocity of $(p\mathbf{i} + q\mathbf{j})$ ms^{-1}, at an angle α above the horizontal. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are respectively horizontal and vertically upwards. The point A is 0.9 m vertically above the point O, which is on horizontal ground. The ball takes 3 seconds to travel from A to B, where B is on the ground and $OB = 57.6$ m, as shown in Figure 8.

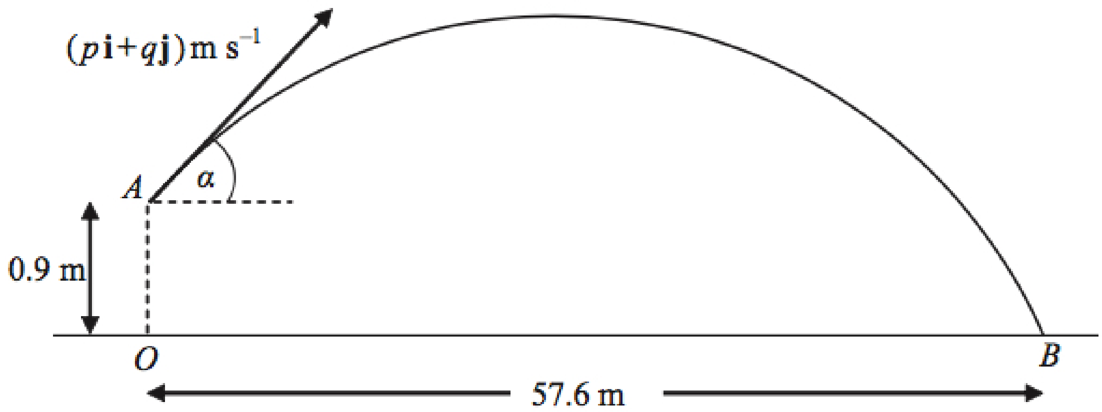

Figure 8: a cricket ball is hit from a point A

By modelling the motion of the cricket ball as that of a particle moving freely under gravity,

(a) find the value of p, (2)

Solution

$$p = \frac{57.6}{3} = \underline{\underline{19.2}}.$$

(b) show that $q = 14.4$, (3)

Solution

$s = -0.9$, $u = q$, $v = ?$, $a = -9.8$, and $t = 3$:

$$\begin{aligned} s = ut + \tfrac{1}{2}at^2 &\Rightarrow -0.9 = 3q - 44.1 \\ &\Rightarrow 3q = 43.2 \\ &\Rightarrow \underline{\underline{q = 14.4}}. \end{aligned}$$

(c) find the initial speed of the cricket ball, (2)

Solution

$$\text{Speed} = \sqrt{19.2^2 + 14.4^2} = \underline{\underline{24 \text{ ms}^{-1}}}.$$

(d) find the exact value of $\tan\alpha$. (1)

Solution

$$\tan\alpha = \frac{14.4}{19.2} = \underline{\underline{\frac{3}{4}}}.$$

(e) Find the length of time for which the cricket ball is at least 4 m above the ground. (6)

Solution

$s = 3.1$, $u = 14.4$, $v =?$, $a = -9.8$, and $t =?$:

$$\begin{aligned} s = ut + \tfrac{1}{2}at^2 &\Rightarrow 3.1 = 14.4t - 4.9t^2 \\ &\Rightarrow 4.9t^2 - 14.4t + 3.1 = 0 \\ &\Rightarrow t = \frac{14.4 \pm \sqrt{14.4^2 - 4 \times 4.9 \times 3.1}}{9.8} \\ &\Rightarrow t = 0.233\,892\,990\,4 \text{ or } 2.704\,882\,52 \text{ (FCD)} \end{aligned}$$

and the time taken is

$$2.704\ldots - 0.233\ldots = \underline{\underline{2.5 \text{ s (2 sf)}}}.$$

(f) State an additional physical factor which may be taken into account in a refinement of the above model to make it more realistic. (1)

Solution

e.g., air resistance, wind, swing of the ball, ball is not a particle

26. A particle of mass 0.25 kg is moving with velocity $(3\mathbf{i} + 7\mathbf{j})$ ms^{-1} when it receives the impulse $(5\mathbf{i} - 3\mathbf{j})$ Ns. (5)

Find the speed of the particle immediately after the impulse.

Solution

$$
\begin{aligned}
0.25[\mathbf{v} - (3\mathbf{i} + 7\mathbf{j})] = 5\mathbf{i} - 3\mathbf{j} &\Rightarrow \mathbf{v} - (3\mathbf{i} + 7\mathbf{j}) = 20\mathbf{i} - 12\mathbf{j} \\
&\Rightarrow \mathbf{v} = 23\mathbf{i} - 5\mathbf{j} \\
&\Rightarrow |\mathbf{v}| = \sqrt{23^2 + 5^2} \\
&\Rightarrow |\mathbf{v}| = 23.537\,204\,59 \text{ (FCD)} \\
&\Rightarrow |\mathbf{v}| = \underline{\underline{24 \text{ ms}^{-1} \text{ (2 sf)}}}.
\end{aligned}
$$

27. At time $t = 0$ a particle P leaves the origin O and moves along the x-axis. At time t seconds the velocity of P is v ms^{-1}, where

$$v = 8t - t^2.$$

(a) Find the maximum value of v. (4)

Solution

$$
\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} = 0 &\Rightarrow 8 - 2t = 0 \\
&\Rightarrow t = 4 \\
&\Rightarrow \underline{\underline{v = 16 \text{ ms}^{-1}}}.
\end{aligned}
$$

(b) Find the time taken for P to return to O. (5)

Solution

$$v = 8t - t^2 \Rightarrow s = 4t^2 - \tfrac{1}{3}t^3 + c;$$

now,

$$s = 0, t = 0 \Rightarrow c = 0 \Rightarrow s = 4t^2 - \tfrac{1}{3}t^3.$$

Finally,

$$
\begin{aligned}
s = 0 &\Rightarrow 4t^2 - \tfrac{1}{3}t^3 = 0 \\
&\Rightarrow \tfrac{4}{3}t^2(12 - t) = 0 \\
&\Rightarrow t = 0 \text{ or } \underline{\underline{t = 12 \text{ s}}}.
\end{aligned}
$$

28. A child playing cricket on horizontal ground hits the ball towards a fence 10 m away. The ball moves in a vertical plane which is perpendicular to the fence. The ball just passes over the top of the fence, which is 2 m above the ground, as shown in Figure 9.

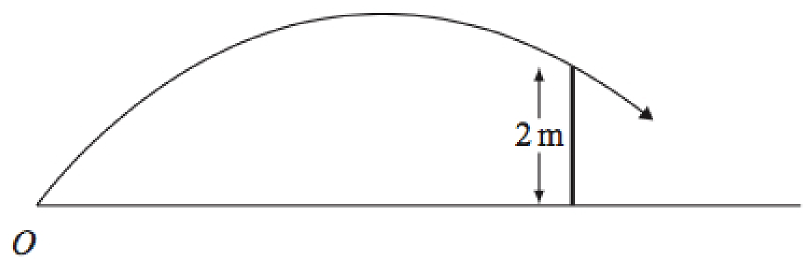

Figure 9: a child playing cricket on horizontal ground

The ball is modelled as a particle projected with initial speed u ms^{-1} from point O on the ground at an angle α to the ground.

(a) By writing down expressions for the horizontal and vertical distances, from O of the ball t seconds after it was hit, show that (6)

$$2 = 10\tan\alpha - \frac{50g}{u^2\cos^2\alpha}.$$

Solution

$$ut\cos\alpha = 10 \text{ and } 2 = ut\sin\alpha - \tfrac{1}{2}gt^2.$$

Now,

$$t = \frac{10}{u\cos\alpha} \Rightarrow 2 = u\sin\alpha \times \frac{10}{u\cos\alpha} - \tfrac{1}{2}g \times \left(\frac{10}{u\cos\alpha}\right)^2$$
$$\Rightarrow \underline{\underline{2 = 10\tan\alpha - \frac{50g}{u^2\cos^2\alpha}}},$$

as required.

Given that $\alpha = 45°$,

(b) find the speed of the ball as it passes over the fence. (6)

Solution

$$\alpha = 45° \Rightarrow 2 = 10 - \frac{100g}{u^2}$$
$$\Rightarrow 8 = \frac{100g}{u^2}$$
$$\Rightarrow u^2 = 12.5g.$$

Finally,

$$\tfrac{1}{2}m(12.5g) - \tfrac{1}{2}mv^2 = m \times 9.8 \times 2 \Rightarrow 122.5 - v^2 = 39.2$$
$$\Rightarrow v^2 = 83.3$$
$$\Rightarrow v = 9.126\,883\,367 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{v = 9.1 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

29. A particle P moves along the x-axis. At time t seconds the velocity of P is v ms^{-1} in the positive x-direction, where (8)

$$v = 3t^2 - 4t + 3.$$

When $t = 0$, P is at the origin O. Find the distance of P from O when P is moving with minimum velocity.

Solution

$$v = 3t^2 - 4t + 3 \Rightarrow s = t^3 - 2t^2 + 3t + c;$$

now,

$$t = 0, s = 0 \Rightarrow 0 = 0 - 0 + 0 + c$$

and so

$$s = t^3 - 2t^2 + 3t.$$

Second,

$$v = 3t^2 - 4t + 3 \Rightarrow a = 6t - 4$$

and

$$a = 0 \Rightarrow t = \tfrac{2}{3}.$$

Finally,

$$t = \tfrac{2}{3} \Rightarrow \underline{\underline{s = 1\tfrac{11}{27} \text{ m}}}.$$

30. The points A, B, and C lie in a horizontal plane. A batsman strikes a ball of mass 0.25 kg. Immediately before being struck, the ball is moving along the horizontal line AB with speed 30 ms^{-1}. Immediately after being struck, the ball moves along the horizontal line BC with speed 40 ms^{-1}. The line BC makes an angle of 60° with the original direction of motion AB, as shown in Figure 10. (8)

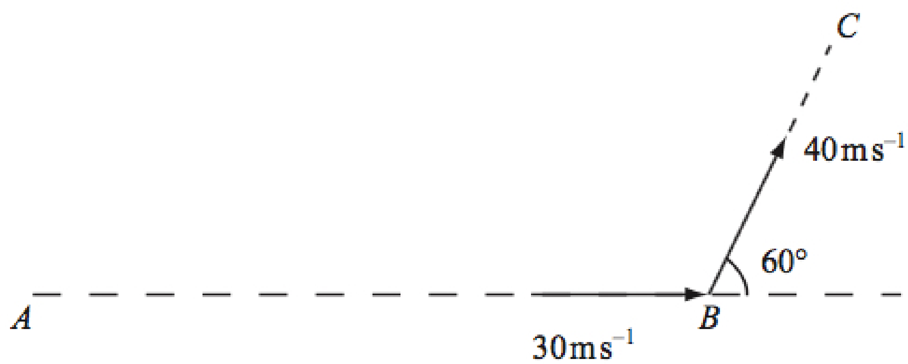

Figure 10: a batsman strikes a ball of mass 0.25 kg

Find, to 3 significant figures,

(a) the magnitude of the impulse given to the ball,

Solution

$$\begin{aligned}\text{Impulse} &= 0.25[(20\mathbf{i} + 20\sqrt{3}\mathbf{j}) - 30\mathbf{i}] \\ &= 0.25(-10\mathbf{i} + 20\sqrt{3}\mathbf{j}) \\ &= -2.5\mathbf{i} + 5\sqrt{3}\mathbf{j}\end{aligned}$$

and so

$$\begin{aligned}|\text{impulse}| &= \sqrt{2.5^2 + (5\sqrt{3})^2} \\ &= 9.013\,878\,189 \text{ (FCD)} \\ &= \underline{\underline{9.0 \text{ Ns (2 sf)}}}.\end{aligned}$$

(b) the size of the angle that the direction of this impulse makes with the original direction of motion AB.

Solution

$$\tan^{-1}\theta = \frac{5\sqrt{3}}{2.5} \Rightarrow \theta = 73.897\,886\,25 \text{ (FCD)};$$

now,

$$180 - \theta = 106.102\,113\,8 \text{ (FCD)} = \underline{\underline{110^\circ \text{ (2 sf)}}} \text{ (why?)}.$$

31. (In this question **i** and **j** are unit vectors in a horizontal and upward vertical direction respectively) A particle P is projected from a fixed point O on horizontal ground with

velocity $u(\mathbf{i} + c\mathbf{j})$ ms^{-1}, where c and u are positive constants. The particle moves freely under gravity until it strikes the ground at A, where it immediately comes to rest. Relative to O, the position vector of a point on the path of P is $(x\mathbf{i} + y\mathbf{j})$ m.

(a) Show that (5)

$$y = cx - \frac{4.9x^2}{u^2}.$$

Solution

$$x = ut \text{ and } y = cut - 4.9t^2;$$

now,

$$t = \frac{x}{u} \Rightarrow y = cu\left(\frac{x}{u}\right) - 4.9\left(\frac{x}{u}\right)^2$$
$$\Rightarrow \underline{\underline{y = cx - \frac{4.9x^2}{u^2}}},$$

as required.

Given that $u = 7$, $OA = R$ m, and the maximum vertical height of P above the ground is H m,

(b) using the result in part (a), or otherwise, find, in terms of c, (6)

(i) R,

Solution
The initial velocity is $(7\mathbf{i} + 7c\mathbf{j})$ ms^{-1}.

$$0 = cx - \frac{4.9x^2}{u^2} \Rightarrow 0 = x\left(c - \frac{4.9x}{u^2}\right)$$
$$\Rightarrow (x = 0 \text{ or})\, x = \frac{cu^2}{4.9}$$
$$\Rightarrow x = \frac{49c}{4.9}$$
$$\Rightarrow \underline{\underline{x = 10c}}.$$

(ii) H.

Solution
So, $x = 5c \Rightarrow y = H$:

$$(5c)c - \frac{(5c)^2}{10} = \underline{\underline{2.5c^2}}.$$

Given also that when P is at the point Q, the velocity of P is at right angles to its initial velocity,

(c) find, in terms of c, the value of x at Q. (6)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = c - \frac{9.8x}{u^2} = c - \frac{9.8x}{49} = c - \tfrac{1}{5}x.$$

When $x = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = c$ so

$$c - \tfrac{1}{5}x = -\tfrac{1}{c} \Rightarrow \tfrac{1}{5}x = c + \tfrac{1}{c} \Rightarrow \underline{\underline{x = 5(c + \tfrac{1}{c})}}.$$

32. A particle P moves on the x-axis. The acceleration of P at time t seconds, $t \geqslant 0$, is $(3t + 5)$ ms^{-2} in the positive x-direction. When $t = 0$, the velocity of P is 2 ms^{-1} in the positive x-direction. When $t = T$, the velocity of P is 6 ms^{-1} in the positive x-direction. Find the value of T. (6)

Solution

$$a = 3t + 5 \Rightarrow v = \tfrac{3}{2}t^2 + 5t + c;$$

now,

$$t = 0, v = 2 \Rightarrow v = \tfrac{3}{2}t^2 + 5t + 2.$$

Finally,

$$\begin{aligned} v = 6 &\Rightarrow \tfrac{3}{2}t^2 + 5t + 2 = 6 \\ &\Rightarrow \tfrac{3}{2}t^2 + 5t - 4 = 0 \\ &\Rightarrow 3t^2 + 10t - 8 = 0 \\ &\Rightarrow (3t - 2)(t + 4) = 0 \\ &\Rightarrow (t = -4 \text{ or}) \underline{\underline{t = \tfrac{2}{3}}}. \end{aligned}$$

33. (In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors in a horizontal plane.) A ball of mass 0.5 kg is moving with velocity $(10\mathbf{i} + 24\mathbf{j})$ ms^{-1} when it is struck by a bat. Immediately after the impact the ball is moving with velocity $20\mathbf{i}$ ms^{-1}. Find

(a) the magnitude of the impulse of the bat on the ball, (4)

Solution

$$\begin{aligned}\text{Impulse} &= 0.5[20\mathbf{i} - (10\mathbf{i} + 24\mathbf{j})] \\ &= -5\mathbf{i} + 12\mathbf{j},\end{aligned}$$

and so

$$|\text{impulse}| = \sqrt{5^2 + 12^2} = \underline{\underline{13 \text{ Ns}}}.$$

(b) the size of the angle between the vector **i** and the impulse exerted by the bat on the ball, (2)

Solution

$$\tan^{-1}\frac{12}{5} = 67.380\,135\,05 \text{ (FCD)} = \underline{\underline{67^\circ \text{ (2 sf)}}}.$$

(c) the kinetic energy lost by the ball in the impact. (3)

Solution

$$\text{Initial KE} = \tfrac{1}{2} \times 0.5 \times (10^2 + 24^2) = 169$$

and

$$\text{final KE} = \tfrac{1}{2} \times 0.5 \times 20^2 = 100;$$

hence,

$$\text{energy lost} = \underline{\underline{69 \text{ J}}}.$$

34. A ball is projected with speed 40 ms^{-1} from a point P on a cliff above horizontal ground. The point O on the ground is vertically below P and OP is 36 m. The ball is projected at an angle θ° to the horizontal. The point Q is the highest point of the path of the ball and is 12 m above the level of P. The ball moves freely under gravity and hits the ground at the point R, as shown in Figure 11.

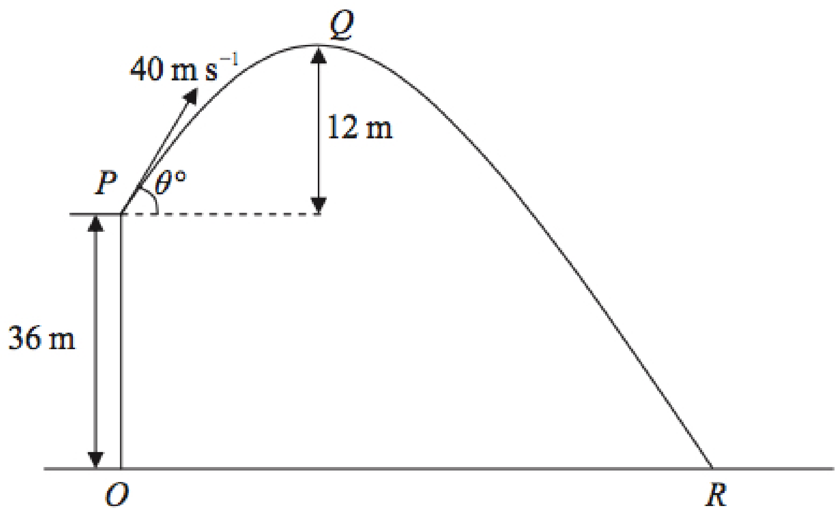

Figure 11: a ball is projected with speed 40 ms^{-1}

Find

(a) the value of θ, (3)

Solution

$s = 12\,(\uparrow)$, $u = 40\sin\theta°$, $v = 0$, $a = -9.8$, and $t =$?:

$$\begin{aligned} v^2 = u^2 + 2as &\Rightarrow 0 = (40\sin\theta°)^2 + 2 \times 9.8 \times 12 \\ &\Rightarrow \sin^2\theta° = 0.147 \\ &\Rightarrow \sin\theta° = 0.383\,405\,790\,3 \text{ (FCD)} \\ &\Rightarrow \theta = 22.544\,805\,37 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 23 \text{ (2 sf)}}}. \end{aligned}$$

(b) the distance OR, (6)

Solution

$$40t\cos\theta° = OR \text{ and } -36 = 40t\sin\theta° - 4.9t^2;$$

now,

$$t = \frac{OR}{40\cos\theta^\circ} \Rightarrow -36 = 40\sin\theta^\circ\left(\frac{OR}{40\cos\theta^\circ}\right) - 4.9\left(\frac{OR}{40\cos\theta^\circ}\right)^2$$
$$\Rightarrow -36 = OR\tan\theta^\circ - \frac{4.9OR^2}{1600\cos^2\theta^\circ}$$
$$\Rightarrow \frac{4.9OR^2}{1600\cos^2\theta^\circ} - OR\tan\theta^\circ - 36 = 0$$
$$\Rightarrow OR = \frac{\tan\theta^\circ \pm \sqrt{\tan^2\theta^\circ - 4\times\frac{4.9}{1600\cos^2\theta^\circ}\times(-36)}}{2\times\frac{4.9}{1600\cos^2\theta^\circ}}$$
$$\Rightarrow OR = -57.813\,210\,83 \text{ or } OR = 173.439\,632\,5 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{OR = 170 \text{ (2 sf)}}}.$$

(c) the speed of the ball as it hits the ground at R. (3)

Solution

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}m(40^2) = m\times 9.8\times 36 \Rightarrow v^2 - 1600 = 705.6$$
$$\Rightarrow v^2 = 2305.6$$
$$\Rightarrow v = 48.016\,663\,77 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{v = 48 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

35. A particle of mass 2 kg is moving with velocity $(5\mathbf{i}+\mathbf{j})$ ms^{-1} when it receives an impulse of $(-6\mathbf{i}+8\mathbf{j})$ Ns. Find the kinetic energy of the particle immediately after receiving the impulse. (5)

Solution

$$2[\mathbf{v} - (5\mathbf{i}+\mathbf{j})] = -6\mathbf{i}+8\mathbf{j} \Rightarrow \mathbf{v} - (5\mathbf{i}+\mathbf{j}) = -3\mathbf{i}+4\mathbf{j}$$
$$\Rightarrow \mathbf{v} = 2\mathbf{i}+5\mathbf{j}$$
$$\Rightarrow \text{kinetic energy} = \tfrac{1}{2}\times 2\times(2^2+5^2)$$
$$\Rightarrow \text{kinetic energy} = \underline{\underline{29 \text{ J}}}.$$

36. A particle moves along the x-axis. At time $t = 0$ the particle passes through the origin with speed 8 ms^{-1} in the positive x-direction. The acceleration of the particle at time t seconds, $t \geqslant 0$, is $(4t^3 - 12t)$ ms^{-2} in the positive x-direction. Find

(a) the velocity of the particle at time t seconds, (3)

Solution

$$a = 4t^3 - 12t \Rightarrow \underline{\underline{v = (t^4 - 6t^2 + 8)\ \text{ms}^{-1}}}.$$

(b) the displacement of the particle from the origin at time t seconds, (2)

Solution

$$v = t^4 - 6t^2 + 8 \Rightarrow \underline{\underline{s = \tfrac{1}{5}t^5 - 2t^3 + 8t}}$$

because $t = 0 \Rightarrow s = 0$.

(c) the values of t at which the particle is instantaneously at rest. (3)

Solution

$$\begin{aligned} v = 0 &\Rightarrow t^4 - 6t^2 + 8 = 0 \\ &\Rightarrow (t^2 - 2)(t^2 - 4) = 0 \\ &\Rightarrow t^2 = 2 \text{ or } t^2 = 4 \\ &\Rightarrow \underline{\underline{t = \sqrt{2}\text{ s}}} \text{ or } \underline{\underline{t = 2\text{ s}}} \end{aligned}$$

37. (In this question, the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are in a vertical plane, $\mathbf{i}$ being horizontal and $\mathbf{j}$ being vertically upwards.) At time $t = 0$, a particle P is projected from the point A which has position vector $10\mathbf{j}$ metres with respect to a fixed origin O at ground level. The ground is horizontal. The velocity of projection of P is $(3\mathbf{i} + 5\mathbf{j})$ ms^{-1}, as shown in Figure 12.

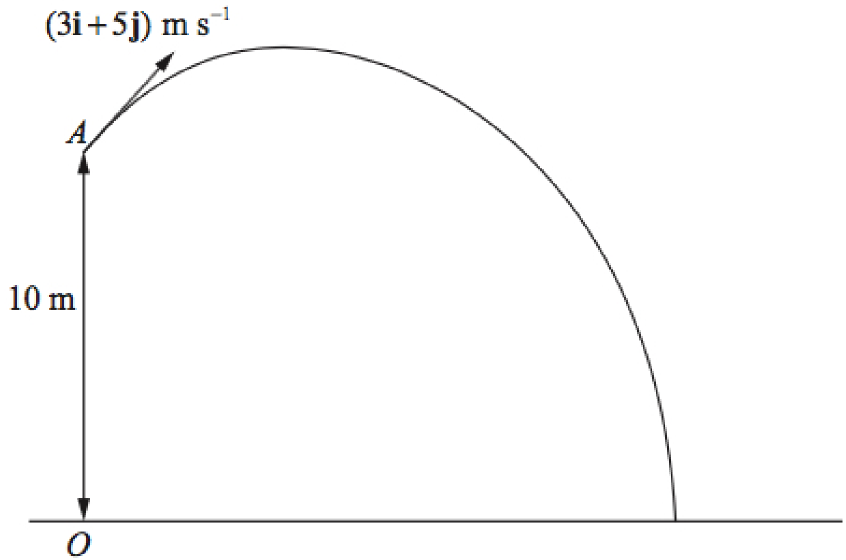

Figure 12: a particle P is projected from the point A

The particle moves freely under gravity and reaches the ground after T seconds.

(a) For $0 \leqslant t \leqslant T$, show that, with respect to O, the position vector, $\mathbf{r}$ metres, of P at time t seconds is given by (3)

$$\mathbf{r} = 3t\mathbf{i} + (10 + 5t - 4.9t^2)\mathbf{j}.$$

Solution

The x-coordinate is

$$r_x = 3t$$

and the y-coordinate is

$$r_y = 10 + 5u + \tfrac{1}{2} \times (-9.8) \times t^2 = 10 + 5t - 4.9t^2;$$

hence,

$$\underline{\underline{\mathbf{r} = 3t\mathbf{i} + (10 + 5t - 4.9t^2)\mathbf{j}}}.$$

(b) Find the value of T. (3)

Solution

$$
\begin{aligned}
10 + 5t - 4.9t^2 = 0 \Rightarrow 4.9t^2 - 5t - 10 = 0 \\
\Rightarrow t &= \frac{5 \pm \sqrt{5^2 - 4 \times 4.9 \times (-10)}}{9.8} \\
\Rightarrow t &= -1.006\,741\,09 \text{ or } t = 2.027\,149\,872 \text{ (FCD)} \\
\Rightarrow t &= 2.0 \text{ s (2 sf)}.
\end{aligned}
$$

(c) Find the velocity of P at time t seconds ($0 \leqslant t \leqslant T$). (2)

Solution

$$\mathbf{r} = 3t\mathbf{i} + (10 + 5t - 4.9t^2)\mathbf{j} \Rightarrow \mathbf{v} = [3\mathbf{i} + (5 - 9.8t)\mathbf{j}] \text{ ms}^{-1}.$$

When P is at the point B, the direction of motion of P is $45°$ below the horizontal.

(d) Find the time taken for P to move from A to B. (2)

Solution

$$
\begin{aligned}
\frac{5 - 9.8t}{3} = -1 &\Rightarrow 5 - 9.8t = -3 \\
&\Rightarrow 9.8t = 8 \\
&\Rightarrow t = \tfrac{40}{49} \text{ or } 0.82 \text{ s (2 sf)}.
\end{aligned}
$$

(e) Find the speed of P as it passes through B. (2)

Solution

The velocity is

$$\mathbf{v} = 3\mathbf{i} - 3\mathbf{j} \text{ (why?)}$$

and so the

$$\text{speed} = \sqrt{3^2 + 3^2} = 4.242\,640\,687 \text{ (FCD)} = 4.2 \text{ ms}^{-1} \text{ (2 sf)}.$$

38. A ball of mass 0.5 kg is moving with velocity $12\mathbf{i}$ ms^{-1} when it is struck by a bat. The impulse received by the ball is $(-4\mathbf{i} + 7\mathbf{j})$ Ns. By modelling the ball as a particle, find

(a) the speed of the ball immediately after the impact, (4)

Solution

$$\begin{aligned}0.5(\mathbf{v} - 12\mathbf{i}) = -4\mathbf{i} + 7\mathbf{j} &\Rightarrow \mathbf{v} - 12\mathbf{i} = -8\mathbf{i} + 14\mathbf{j}\\ &\Rightarrow \mathbf{v} = 4\mathbf{i} + 14\mathbf{j}\\ &\Rightarrow |\mathbf{v}| = \sqrt{4^2 + 14^2}\\ &\Rightarrow |\mathbf{v}| = 14.560\,219\,78 \text{ (FCD)}\\ &\Rightarrow |\mathbf{v}| = \underline{\underline{15 \text{ ms}^{-1} \text{ (2 sf)}}}.\end{aligned}$$

(b) the angle, in degrees, between the velocity of the ball immediately after the impact and the vector **i**, (2)

Solution

$$\begin{aligned}\tan^{-1}\frac{14}{4} &= 74.054\,604\,1 \text{ (FCD)}\\ &= \underline{\underline{74^\circ \text{ (2 sf)}}}.\end{aligned}$$

(c) the kinetic energy gained by the ball as a result of the impact. (2)

Solution

$$\begin{aligned}\text{Gain in KE} &= \tfrac{1}{2} \times 0.5 \times (4^2 + 14^2) - \tfrac{1}{2} \times 0.5 \times 12^2\\ &= 53 - 36\\ &= \underline{\underline{17 \text{ J}}}.\end{aligned}$$

39. A particle P moves on the x-axis. The acceleration of P at time t seconds is $(t - 4)$ ms^{-2} in the positive x-direction. The velocity of P at time t seconds is v ms^{-1}. When $t = 0$, $v = 6$. Find

(a) v in terms of t, (4)

Solution

$$a = t - 4 \Rightarrow v = \tfrac{1}{2}t^2 - 4t + c;$$

now,

$$t = 0, v = 6 \Rightarrow 6 = 0 - 0 + c \Rightarrow \underline{\underline{v = \tfrac{1}{2}t^2 - 4t + 6}}.$$

(b) the values of t when P is instantaneously at rest, (3)

Solution

$$\begin{aligned} v = 0 &\Rightarrow \tfrac{1}{2}t^2 - 4t + 6 = 0 \\ &\Rightarrow t^2 - 8t + 12 = 0 \\ &\Rightarrow (t-2)(t-6) = 0 \\ &\Rightarrow \underline{\underline{t = 2 \text{ s}}} \text{ or } \underline{\underline{t = 6 \text{ s}}}. \end{aligned}$$

(c) the distance between the two points at which P is instantaneously at rest. (4)

Solution

$$v = \tfrac{1}{2}t^2 - 4t + 6 \Rightarrow s = \tfrac{1}{6}t^3 - 2t^2 + 6t + c.$$

Now,

$$s(2) = 5\tfrac{1}{3} + c$$

and

$$s(6) = c$$

and

$$\text{distance} = (5\tfrac{1}{3} + c) - c = \underline{\underline{5\tfrac{1}{3} \text{ m}}}.$$

40. A particle is projected from a point O with speed u at an angle of elevation α above the horizontal and moves freely under gravity. When the particle has moved a horizontal distance x, its height above O is y.

(a) Show that (4)

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}.$$

Solution

$$x = ut\cos\alpha \text{ and } y = ut\sin\alpha - \tfrac{1}{2}gt^2;$$

now,

$$\begin{aligned} t = \frac{x}{u\cos\alpha} &\Rightarrow y = u\sin\alpha\left(\frac{x}{u\cos\alpha}\right) - \tfrac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2 \\ &\Rightarrow \underline{\underline{y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}}}. \end{aligned}$$

A girl throws a ball from a point A at the top of a cliff. The point A is 8 m above a horizontal beach. The ball is projected with speed 7 ms^{-1} at an angle of elevation of $45°$. By modelling the ball as a particle moving freely under gravity,

(b) find the horizontal distance of the ball from A when the ball is 1 m above the beach. (5)

Solution

$$\begin{aligned} y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} &\Rightarrow -7 = x\tan 45^\circ - \frac{gx^2}{2\times 7^2\times\cos^2 45^\circ} \\ &\Rightarrow -7 = x - \tfrac{1}{5}x^2 \\ &\Rightarrow \tfrac{1}{5}x^2 - x - 7 = 0 \\ &\Rightarrow x^2 - 5x - 35 = 0 \\ &\Rightarrow x = \frac{5\pm\sqrt{5^2 - 4\times 1\times(-35)}}{2} \\ &\Rightarrow x = \frac{5\pm\sqrt{165}}{2} \\ &\Rightarrow x = -3.922\,616\,289 \text{ or } x = 8.922\,616\,289 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 8.9 \text{ m (2 sf)}}}. \end{aligned}$$

A boy is standing on the beach at the point B vertically below A. He starts to run in a straight line with speed v ms^{-1}, leaving B 0.4 seconds after the ball is thrown. He catches the ball when it is 1 m above the beach.

(c) Find the value of v. (4)

Solution

$$\text{Time to run} = \frac{8.922\ldots}{7\cos 45^\circ} = 1.802\,640\,71 \text{ (FCD)}$$

and therefore

$$v = \frac{8.922\ldots}{1.402\,640\,71} = 6.361\,298\,533 \text{ (FCD)} = \underline{\underline{6.4 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

41. A tennis ball of mass 0.1 kg is hit by a racquet. Immediately before being hit, the ball has velocity $30\mathbf{i}$ ms^{-1}. The racquet exerts an impulse of $(-2\mathbf{i} - 4\mathbf{j})$ Ns on the ball. By modelling the ball as a particle, find the velocity of the ball immediately after being hit. (4)

Solution

$$\begin{aligned}0.1(\mathbf{v} - 30\mathbf{i}) = -2\mathbf{i} - 4\mathbf{j} &\Rightarrow \mathbf{v} - 30\mathbf{i} = -20\mathbf{i} - 40\mathbf{j}\\ &\Rightarrow \mathbf{v} = \underline{\underline{(10\mathbf{i} - 40\mathbf{j})\ \mathrm{ms}^{-1}}}.\end{aligned}$$

42. A particle P is moving in a plane. At time t seconds, P is moving with velocity $\mathbf{v}$ ms^{-1}, where

$$\mathbf{v} = 2t\mathbf{i} - 3t^2\mathbf{j}.$$

Find

(a) the speed of P when $t = 4$, (2)

Solution

$$\begin{aligned}t = 4 &\Rightarrow \mathbf{v} = 8\mathbf{i} - 48\mathbf{j}\\ &\Rightarrow |\mathbf{v}| = \sqrt{8^2 + 48^2}\\ &\Rightarrow |\mathbf{v}| = 48.662\,100\,24 \text{ (FCD)}\\ &\Rightarrow |\mathbf{v}| = \underline{\underline{49\ \mathrm{ms}^{-1}\ (2\text{ sf})}}.\end{aligned}$$

(b) the acceleration of P when $t = 4$. (3)

Solution

$$\mathbf{a} = 2\mathbf{i} - 6t\mathbf{j}$$

and

$$t = 4 \Rightarrow \mathbf{a} = \underline{\underline{(2\mathbf{i} - 24\mathbf{j})\ \mathrm{ms}^{-2}}}.$$

Given that P is at the point with position vector $(-4\mathbf{i} + \mathbf{j})$ m when $t = 1$,

(c) find the position vector of P when $t = 4$. (5)

Solution

$$\mathbf{v} = 2t\mathbf{i} - 3t^2\mathbf{j} \Rightarrow \mathbf{s} = t^2\mathbf{i} - t^3\mathbf{j} + \mathbf{c};$$

now,

$$t = 1, \mathbf{s} = -4\mathbf{i} + \mathbf{j} \Rightarrow -4\mathbf{i} + \mathbf{j} = \mathbf{i} - \mathbf{j} + \mathbf{c}$$
$$\Rightarrow \mathbf{c} = -5\mathbf{i} + 2\mathbf{j}$$
$$\Rightarrow \mathbf{s} = (t^2 - 5)\mathbf{i} + (2 - t^3)\mathbf{j}.$$

Finally,

$$t = 4 \Rightarrow \mathbf{s} = \underline{\underline{(11\mathbf{i} - 62\mathbf{j})}} \text{ m}.$$

43. (In this question, the unit vectors **i** and **j** are horizontal and vertical respectively.) The point O is a fixed point on a horizontal plane. A ball is projected from O with velocity $(6\mathbf{i} + 12\mathbf{j})$ ms^{-1}, and passes through the point A at time t seconds after projection. The point B is on the horizontal plane vertically below A, as shown in Figure 13.

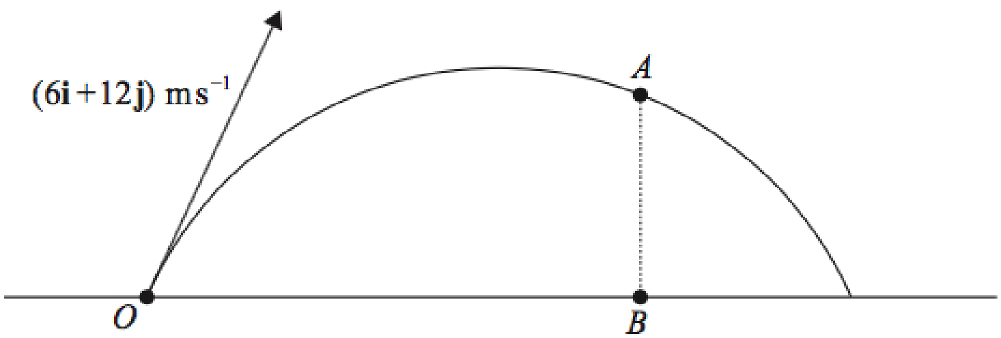

Figure 13: a ball is projected from O with velocity $(6\mathbf{i} + 12\mathbf{j})$ ms^{-1}

It is given that $OB = 2AB$. Find

(a) the value of t, (7)

Solution

$$x = 6t \text{ and } y = 12t - 4.9t^2;$$

now,

$$6t = 2(12t - 4.9t^2) \Rightarrow 6t = 24t - 9.8t^2$$
$$\Rightarrow 9.8t^2 - 18t = 0$$
$$\Rightarrow t(9.8t - 18) = 0$$
$$\Rightarrow t = 0 \text{ or } t = 1\tfrac{41}{49}$$
$$\Rightarrow \underline{\underline{t = 1.8 \text{ s (2 sf)}}}.$$

(b) the speed, V ms^{-1}, of the ball at the instant when it passes through A. (5)

Solution

$s =?\,(\uparrow)$, $u = 12$, $v =?$, $a = -9.8$, and $t = 1\frac{41}{49}$:

$$\begin{aligned} v &= u + at \\ &= 12 - 9.8 \times 1\tfrac{41}{49} \\ &= -6, \end{aligned}$$

and so the

$$\text{speed} = \sqrt{6^2 + 6^2} = 6\sqrt{2} = \underline{\underline{8.5 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

At another point C on the path the speed of the ball is also V ms^{-1}.

(c) Find the time taken for the ball to travel from O to C. (3)

Solution

$$12t - 4.9t^2 = 0 \Rightarrow t(12 - 4.9t) = 0 \Rightarrow t = 0 \text{ or } t = 2\tfrac{22}{49}$$

and

$$\text{time taken} = 2\tfrac{22}{49} - 1\tfrac{41}{49} = \tfrac{30}{49} = \underline{\underline{0.61 \text{ s (2 sf)}}}.$$

44. (In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors in a horizontal plane.) A particle P moves in such a way that its velocity v ms^{-1} at time t seconds is given by

$$\mathbf{v} = (3t^2 - 1)\mathbf{i} + (4t - t^2)\mathbf{j}.$$

(a) Find the magnitude of the acceleration of P when $t = 1$. (5)

Solution

$$\mathbf{v} = (3t^2 - 1)\mathbf{i} + (4t - t^2)\mathbf{j} \Rightarrow \mathbf{a} = 6t\mathbf{i} + (4 - 2t)\mathbf{j}$$

and

$$\begin{aligned} t = 2 \Rightarrow \mathbf{a} &= 6\mathbf{i} + 2\mathbf{j} \\ \Rightarrow |\mathbf{a}| &= \sqrt{6^2 + 2^2} \\ \Rightarrow |\mathbf{a}| &= 6.324\,555\,32 \text{ (FCD)} \\ \Rightarrow |\mathbf{a}| &= \underline{\underline{6.3 \text{ ms}^{-2} \text{ (2 sf)}}}. \end{aligned}$$

Given that, when $t = 0$, the position vector of P is $\mathbf{i}$ metres,

(b) find the position vector of P when $t = 3$. (5)

Solution

$$\mathbf{v} = (3t^2 - 1)\mathbf{i} + (4t - t^2)\mathbf{j} \Rightarrow \mathbf{r} = (t^3 - t)\mathbf{i} + (2t^2 - \tfrac{1}{3}t^3)\mathbf{j} + \mathbf{c}.$$

Now,

$$t = 0, \mathbf{r} = \mathbf{i} \Rightarrow \mathbf{r} = (t^3 - t + 1)\mathbf{i} + (2t^2 - \tfrac{1}{3}t^3)\mathbf{j}.$$

Finally,

$$t = 3 \Rightarrow \underline{\underline{\mathbf{r} = (25\mathbf{i} + 9\mathbf{j}) \text{ m}}}.$$

45. A small ball B of mass 0.25 kg is moving in a straight line with speed 30 ms^{-1} on a smooth horizontal plane when it is given an impulse. The impulse has magnitude 12.5 Ns and is applied in a horizontal direction making an angle of $(90^\circ + \alpha)$, where $\tan\alpha = \frac{3}{4}$, with the initial direction of motion of the ball, as shown in Figure 14. (6)

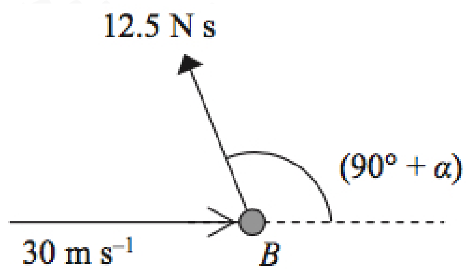

Figure 14: a small ball B of mass 0.25 kg

(a) Find the speed of B immediately after the impulse is applied.

Solution

$$\tan\alpha = \tfrac{3}{4} \Rightarrow \sin\alpha = \tfrac{3}{5}, \cos\alpha = \tfrac{4}{5}$$

and

$$\begin{aligned} 0.25(\mathbf{v} - 30\mathbf{i}) = 12.5(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}) &\Rightarrow \mathbf{v} - 30\mathbf{i} = 50(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}) \\ &\Rightarrow \mathbf{v} - 30\mathbf{i} = -30\mathbf{i} + 40\mathbf{j} \\ &\Rightarrow \mathbf{v} = 40\mathbf{j}; \end{aligned}$$

the speed is $\underline{\underline{40 \text{ ms}^{-1}}}$ ...

(b) Find the direction of motion of B immediately after the impulse is applied.

Solution

... vertically upwards.

46. A small stone is projected from a point O at the top of a vertical cliff OA. The point O is 52.5 m above the sea. The stone rises to a maximum height of 10 m above the level of O before hitting the sea at the point B, where $AB = 50$ m, as shown in Figure 15.

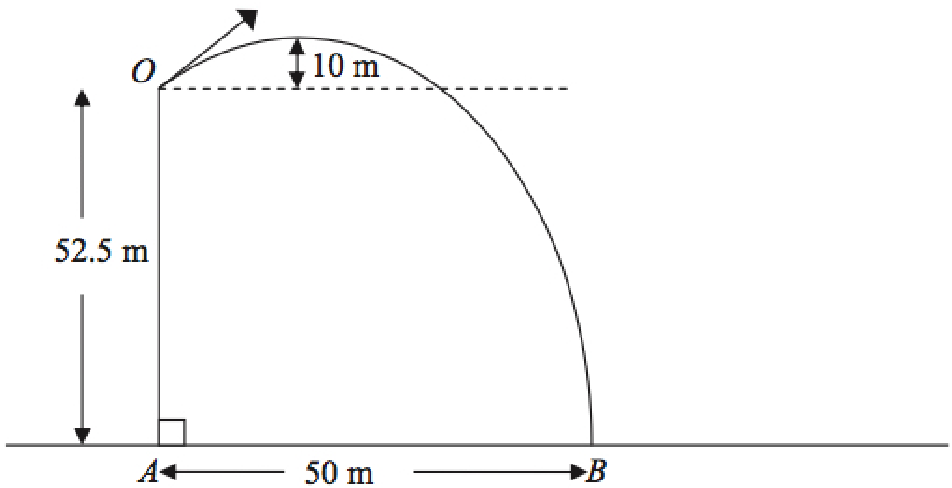

Figure 15: a small stone is projected from a point O

The stone is modelled as a particle moving freely under gravity.

(a) Show that the vertical component of the velocity of projection of the stone is 14 ms^{-1}. (3)

Solution

$s = 10\,(\uparrow)$, $u = ?$, $v = 0$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} v^2 = u^2 + 2as &\Rightarrow 0 = u^2 + 2 \times (-9.8) \times 10 \\ &\Rightarrow u^2 = 196 \\ &\Rightarrow \underline{\underline{u = 14 \text{ ms}^{-1}}}. \end{aligned}$$

(b) Find the speed of projection. (9)

Solution

$s = -52.5\,(\uparrow)$, $u = 14$, $v = ?$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} s = ut + \tfrac{1}{2}at^2 &\Rightarrow -52.5 = 14t - 4.9t^2 \\ &\Rightarrow 4.9t^2 - 14t - 52.5 = 0 \\ &\Rightarrow 49t^2 - 140t - 525 = 0 \\ &\Rightarrow (49t + 105)(t - 5) = 0 \\ &\Rightarrow t = -\tfrac{15}{7} \text{ or } t = 5. \end{aligned}$$

So,

$$5x = 50 \Rightarrow x = 10$$

and

$$\begin{aligned} \text{speed} &= \sqrt{10^2 + 14^2} \\ &= 17.204\,650\,53 \text{ (FCD)} \\ &= \underline{\underline{17 \text{ ms}^{-1} \text{ (2 sf)}}}. \end{aligned}$$

(c) Find the time after projection when the stone is moving parallel to OB. (5)

Solution

$$\tan \angle OBA = \frac{52.5}{50} = 1.05$$

and

$$u_v = 1.05 \times 10 = 10.5.$$

Now, $s = ?\,(\uparrow)$, $u = 14$, $v = -10.5$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} v = u + at &\Rightarrow -10.5 = 14 - 9.8t \\ &\Rightarrow 9.8t = 24.5 \\ &\Rightarrow \underline{\underline{u = 2.5 \text{ s}}}. \end{aligned}$$

47. At time t seconds the velocity of a particle P is $[(4t - 5)\mathbf{i} + 3\mathbf{j}]$ ms^{-1}. When $t = 0$, the position vector of P is $(2\mathbf{i} + 5\mathbf{j})$ m, relative to a fixed origin O.

(a) Find the value of t when the velocity of P is parallel to the vector $\mathbf{j}$. (1)

Solution

$$4t - 5 = 0 \Rightarrow 4t = 5 \Rightarrow \underline{\underline{t = 1.25 \text{ s}}}.$$

(b) Find an expression for the position vector of P at time t seconds. (4)

Solution

$$\mathbf{v} = (4t - 5)\mathbf{i} + 3\mathbf{j} \Rightarrow \mathbf{r} = (2t^2 - 5t)\mathbf{i} + 3t\mathbf{j} + \mathbf{c};$$

now,

$$t = 0, \mathbf{r} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \mathbf{r} = \underline{\underline{[(2t^2 - 5t + 2)\mathbf{i} + (3t + 5)\mathbf{j}] \text{ m}}}.$$

A second particle Q moves with constant velocity $(-2\mathbf{i} + c\mathbf{j})$ ms^{-1}. When $t = 0$, the position vector of Q is $(11\mathbf{i} + 2\mathbf{j})$ m. The particles P and Q collide at the point with position vector $(d\mathbf{i} + 14\mathbf{j})$ m.

(c) Find (5)

(i) the value of c,

Solution

$$\mathbf{r}_Q = (11 - 2t)\mathbf{i} + (2 + ct)\mathbf{j}$$

and

$$\begin{aligned} 3t + 5 = 14 &\Rightarrow 3t = 9 \\ &\Rightarrow t = 3 \end{aligned}$$

but

$$2 + c \times 3 = 14 \Rightarrow 3c = 12 \Rightarrow \underline{\underline{c = 4}}.$$

(ii) the value of d.

Solution

$$d = 11 - 2 \times 3 = \underline{\underline{5}}.$$

48. A ball is thrown from a point O, which is 6 m above horizontal ground. The ball is projected with speed u ms^{-1} at an angle θ above the horizontal. There is a thin vertical post which is 4 m high and 8 m horizontally away from the vertical through O, as shown in Figure 16.

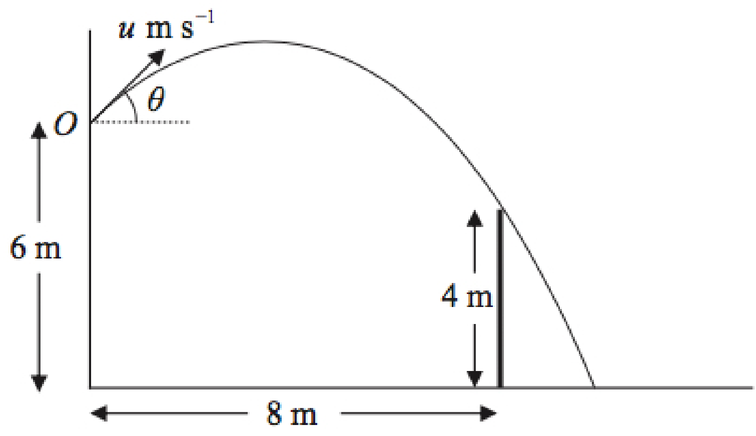

Figure 16: a ball is thrown from a point O

The ball passes just above the top of the post 2 s after projection. The ball is modelled as a particle.

(a) Show that $\tan\theta = 2.2$. (5)

Solution
For $t = 2$,
$$2u\cos\theta = 8 \Rightarrow \cos\theta = \frac{4}{u}$$
and
$$-2 = 2u\sin\theta - 19.6 \Rightarrow \sin\theta = \frac{8.8}{u}$$
and divide:
$$\tan\theta = \underline{\underline{2.2}}.$$

(b) Find the value of u. (2)

Solution
$$\begin{aligned} u &= \frac{4}{\cos\theta} \\ &= 9.666\,436\,779 \text{ (FCD)} \\ &= \underline{\underline{9.7 \text{ ms}^{-1} \text{ (2 sf)}}}. \end{aligned}$$

The ball hits the ground T seconds after projection.

(c) Find the value of T. (3)

Solution

$s = -6\,(\uparrow)$, $u = 9.666\ldots\sin\theta = 8.8$, $v = ?$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} s = ut + \tfrac{1}{2}at^2 &\Rightarrow -6 = 8.8t - 4.9t^2 \\ &\Rightarrow 4.9t^2 - 8.8t - 6 = 0 \\ &\Rightarrow t = \frac{8.8 \pm \sqrt{8.8^2 - 4 \times 4.9 \times (-6)}}{2 \times 4.9} \\ &\Rightarrow t = \frac{8.8 \pm \sqrt{195.04}}{9.8} \\ &\Rightarrow t = -0.527\,109\,408\,2 \text{ or } t = 2.323\,027\,776 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 2.3 \text{ s (2 sf)}}}. \end{aligned}$$

Immediately before the ball hits the ground the direction of motion of the ball makes an angle α with the horizontal.

(d) Find α. (5)

Solution

$$v = u + at = 8.8 - 9.8 \times 2.323\ldots = -13.965\,672\,2 \text{ (FCD)}$$

and the speed of projection horizontally is

$$\frac{8}{2} = 4;$$

now,

$$\begin{aligned} \tan\alpha = \frac{13.965\ldots}{4} &\Rightarrow \alpha = 74.017\,409\,64 \text{ (FCD)} \\ &\Rightarrow \alpha = \underline{\underline{74^\circ \text{ (2 sf)}}}. \end{aligned}$$

49. A particle P of mass 2 kg is moving with velocity $(\mathbf{i} - 4\mathbf{j})$ ms^{-1} when it receives an impulse of $(3\mathbf{i} + 6\mathbf{j})$ Ns. Find the speed of P immediately after the impulse is applied. (5)

Solution

$$
\begin{aligned}
2[\mathbf{v} - (\mathbf{i} - 4\mathbf{j})] = 3\mathbf{i} + 6\mathbf{j} &\Rightarrow \mathbf{v} - (\mathbf{i} - 4\mathbf{j}) = \tfrac{3}{2}\mathbf{i} + 3\mathbf{j} \\
&\Rightarrow \mathbf{v} = \tfrac{5}{2}\mathbf{i} + \mathbf{j} \\
&\Rightarrow |\mathbf{v}| = \sqrt{(\tfrac{5}{2})^2 + 1^2} \\
&\Rightarrow |\mathbf{v}| = 2.692\,582\,404 \text{ (FCD)} \\
&\Rightarrow |\mathbf{v}| = \underline{\underline{2.7 \text{ ms}^{-1} \text{ (2 sf)}}}.
\end{aligned}
$$

50. A particle P moves on the x-axis. At time t seconds the velocity of P is v ms^{-1} in the direction of x increasing, where

$$v = 2t^2 - 14t + 20, \; t \geqslant 0.$$

Find

(a) the times when P is instantaneously at rest, (3)

Solution

$$
\begin{aligned}
v = 0 &\Rightarrow 2t^2 - 14t + 20 = 0 \\
&\Rightarrow 2(t^2 - 7t + 10) = 0 \\
&\Rightarrow 2(t - 2)(t - 5) = 0 \\
&\Rightarrow \underline{\underline{t = 2 \text{ s}}} \text{ or } \underline{\underline{t = 5 \text{ s}}}.
\end{aligned}
$$

(b) the greatest speed of P in the interval $0 \leqslant t \leqslant 4$, (5)

Solution

$$t = 0, v = 0 - 0 + 20 = 20$$

and

$$
\begin{aligned}
a = 0 &\Rightarrow 4t - 14 = 0 \\
&\Rightarrow t = 3.5 \\
&\Rightarrow v = 2 \times 3.5^2 - 14 \times 3.5 + 20 \\
&\Rightarrow v = -4.5;
\end{aligned}
$$

hence, the greatest speed of P is $\underline{\underline{20 \text{ ms}^{-1}}}$.

(c) the total distance travelled by P in the interval $0 \leqslant t \leqslant 4$. (5)

Solution

$$v = 2t^2 - 14t + 20 \Rightarrow r = \tfrac{2}{3}t^3 - 7t^2 + 20t + c.$$

Now, $t = 2$ splits in whole interval into two regions:

$$\begin{aligned}[\tfrac{2}{3}t^3 - 7t^2 + 20t]_{t=2}^{4} &= (\tfrac{128}{3} - 112 + 80) - (\tfrac{16}{3} - 28 + 40) \\ &= \tfrac{32}{3} - \tfrac{52}{3} \\ &= -\tfrac{20}{3}\end{aligned}$$

and

$$\begin{aligned}[\tfrac{2}{3}t^3 - 7t^2 + 20t]_{t=0}^{2} &= (\tfrac{16}{3} - 28 + 40) - (0 - 0 + 0) \\ &= \tfrac{52}{3} - 0 \\ &= \tfrac{52}{3}.\end{aligned}$$

Hence,

$$\text{total distance} = \tfrac{20}{3} + \tfrac{52}{3} = \underline{\underline{24 \text{ m}}}.$$

51. A ball is projected from a point A which is 8 m above horizontal ground as shown in Figure 17.

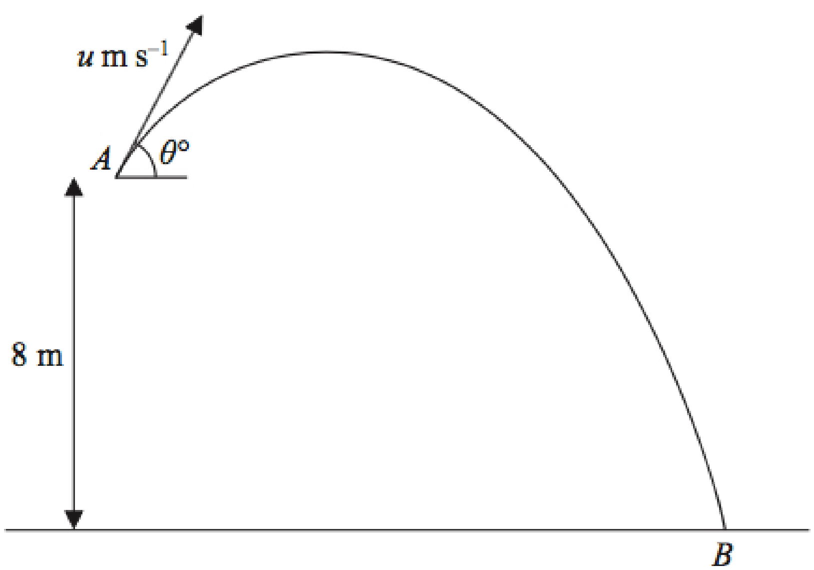

Figure 17: a ball is projected from a point A

The ball is projected with speed u ms^{-1} at an angle θ° above the horizontal. The ball moves freely under gravity and hits the ground at the point B. The speed of the ball immediately before it hits the ground is $2u$ ms^{-1}.

(a) By considering energy, find the value of u. (5)

Solution

$$\begin{aligned}\tfrac{1}{2}m(2u)^2 - \tfrac{1}{2}mu^2 = mg \times 8 &\Rightarrow 3u^2 = 156.8\\ &\Rightarrow u^2 = 52\tfrac{4}{15}\\ &\Rightarrow u = 7.229\,568\,913 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{u = 7.2 \text{ ms}^{-1} \text{ (2 sf)}}}.\end{aligned}$$

The time taken for the ball to move from A to B is 2 seconds. Find

(b) the value of θ, (4)

Solution

$$\begin{aligned}-8 = 2u\sin\theta^\circ - 19.6 &\Rightarrow 2u\sin\theta^\circ - = 11.6\\ &\Rightarrow \sin\theta^\circ - = \frac{5.8}{u}\\ &\Rightarrow \theta = 53.346\,541\,88 \text{ (FCD)}\\ &\Rightarrow \theta = \underline{\underline{53 \text{ (nearest degree)}}}.\end{aligned}$$

(c) the minimum speed of the ball on its path from A to B. (2)

Solution

At its maximum height, the vertical speed is zero and

$$u\cos\theta^\circ = 4.315\,862\,216 \text{ (FCD)} = \underline{\underline{4.3 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

52. A ball of mass 0.2 kg is projected vertically upwards from a point O with speed 20 ms^{-1}. The non-gravitational resistance acting on the ball is modelled as a force of constant magnitude 1.24 N and the ball is modelled as a particle. Find, using the work-energy principle, the speed of the ball when it first reaches the point which is 8 m vertically above O. (6)

Solution

$$\begin{aligned} & 1.24 \times 8 = \tfrac{1}{2} \times 0.2 \times 20^2 - \tfrac{1}{2} \times 0.2 \times v^2 - 0.2 \times 9.8 \times 8 \\ \Rightarrow\ & 9.92 = 40 - 0.1v^2 - 15.68 \\ \Rightarrow\ & 0.1v^2 = 14.4 \\ \Rightarrow\ & v^2 = 144 \\ \Rightarrow\ & v = \underline{\underline{12 \text{ ms}^{-1}}}. \end{aligned}$$

53. A particle P moves along a straight line in such a way that at time t seconds its velocity v ms^{-1} is given by

$$v = \tfrac{1}{2}t^2 - 3t + 4.$$

Find

(a) the times when P is at rest, (4)

Solution

$$\begin{aligned} v = 0 &\Rightarrow \tfrac{1}{2}t^2 - 3t + 4 = 0 \\ &\Rightarrow t^2 - 6t + 8 = 0 \\ &\Rightarrow (t-2)(t-4) = 0 \\ &\Rightarrow \underline{\underline{t = 2 \text{ s}}} \text{ or } \underline{\underline{t = 4 \text{ s}}}. \end{aligned}$$

(b) the total distance travelled by P between $t = 0$ and $t = 4$. (5)

Solution

$$v = \tfrac{1}{2}t^2 - 3t + 4 \Rightarrow r = \tfrac{1}{6}t^3 - \tfrac{3}{2}t^2 + 4t + c;$$

now,

$$\begin{aligned} [\tfrac{1}{6}t^3 - \tfrac{3}{2}t^2 + 4t]_{t=2}^4 &= (\tfrac{32}{3} - 24 + 16) - (\tfrac{4}{3} - 6 + 8) \\ &= \tfrac{8}{3} - \tfrac{10}{3} \\ &= -\tfrac{2}{3} \end{aligned}$$

and

$$\begin{aligned} [\tfrac{1}{6}t^3 - \tfrac{3}{2}t^2 + 4t]_{t=0}^2 &= (\tfrac{4}{3} - 6 + 8) - (0 - 0 + 0) \\ &= \tfrac{10}{3} - 0 \\ &= \tfrac{10}{3}. \end{aligned}$$

Hence,

$$\text{total distance} = \tfrac{2}{3} + \tfrac{10}{3} = \underline{\underline{4\text{ m}}}.$$

54. A small ball is projected from a fixed point O so as to hit a target T which is at a horizontal distance $9a$ from O and at a height $6a$ above the level of O. The ball is projected with speed $\sqrt{27ag}$ at an angle θ to the horizontal, as shown in Figure 18.

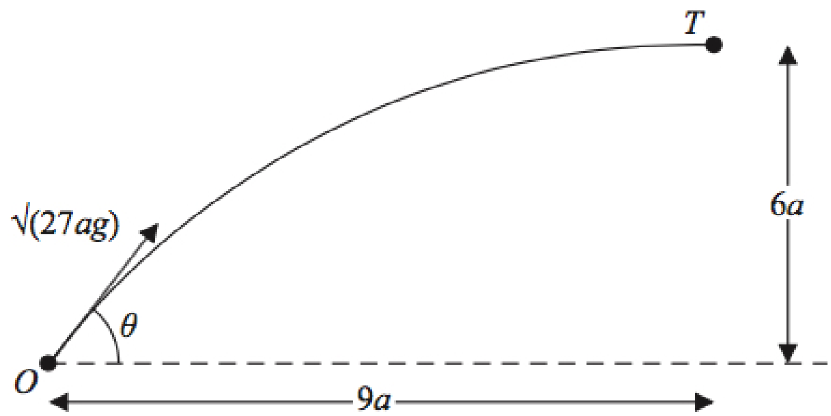

Figure 18: a small ball is projected from a fixed point O

The ball is modelled as a particle moving freely under gravity.

(a) Show that $\tan^2\theta - 6\tan\theta + 5 = 0$. (7)

Solution

$$9a = t\sqrt{27ag}\cos\theta \text{ and } 6a = t\sin\theta\sqrt{27ag} - 4.9t^2;$$

now,

$$\begin{aligned}
t = \frac{9a}{\sqrt{27ag}\cos\theta} &\Rightarrow 6a = \sin\theta\sqrt{27ag}\left(\frac{9a}{\sqrt{27ag}\cos\theta}\right) - 4.9\left(\frac{9a}{\sqrt{27ag}\cos\theta}\right)^2 \\
&\Rightarrow 6a = 9a\tan\theta - \frac{396.9a}{27g\cos^2\theta} \\
&\Rightarrow 6g = 9g\tan\theta - 14.7\sec^2\theta \\
&\Rightarrow 58.8 = 88.2\tan\theta - 14.7(\tan^2\theta + 1) \\
&\Rightarrow 58.8 = 88.2\tan\theta - 14.7\tan^2\theta - 14.7 \\
&\Rightarrow 14.7\tan^2\theta - 88.2\tan\theta + 73.5 = 0 \\
&\Rightarrow \underline{\underline{\tan^2\theta - 6\tan\theta + 5 = 0}},
\end{aligned}$$

as required.

The two possible angles of projection are θ_1 and θ_2, where $\theta_1 > \theta_2$.

(b) Find $\tan\theta_1$ and $\tan\theta_2$. (3)

Solution

$$\tan^2\theta - 6\tan\theta + 5 = 0 \Rightarrow (\tan\theta - 1)(\tan\theta - 5) = 0$$
$$\Rightarrow \underline{\underline{\tan\theta_2 = 1}} \text{ or } \underline{\underline{\tan\theta_1 = 5}}.$$

The particle is projected at the larger angle θ_1.

(c) Show that the time of flight from O to T is $\sqrt{\dfrac{78a}{g}}$. (3)

Solution

$$\tan\theta = 5 \Rightarrow \cos\theta = \tfrac{1}{\sqrt{26}}$$

and

$$\begin{aligned} t &= \frac{9a}{\sqrt{27ag}\cos\theta} \\ &= \frac{9a}{\sqrt{27ag} \times \frac{1}{\sqrt{26}}} \\ &= \frac{9\sqrt{26}a}{\sqrt{27ag}} \\ &= \sqrt{\frac{2106a^2}{27ag}} \\ &= \underline{\underline{\sqrt{\frac{78a}{g}}}}, \end{aligned}$$

as required.

(d) Find the speed of the particle immediately before it hits T. (3)

Solution

$$\tfrac{1}{2}m(27ag) - \tfrac{1}{2}mu^2 = 6amg \Rightarrow 27ag - u^2 = 12ag$$
$$\Rightarrow u^2 = 15ag$$
$$\Rightarrow \underline{\underline{u = \sqrt{15ag}}}.$$

55. At time t seconds, where $t \geqslant 0$, a particle P is moving on a horizontal plane with acceleration $[(3t^2 - 4t)\mathbf{i} + (6t - 5)\mathbf{j}]$ ms^{-2}. When $t = 3$ the velocity of P is $(11\mathbf{i} + 10\mathbf{j})$ ms^{-1}. Find

(a) the velocity of P at time t seconds, (5)

Solution

$$\mathbf{a} = (3t^2 - 4t)\mathbf{i} + (6t - 5)\mathbf{j} \Rightarrow \mathbf{v} = (t^3 - 2t^2)\mathbf{i} + (3t^2 - 5t)\mathbf{j} + \mathbf{c};$$

now,

$$\begin{aligned} t = 3 &\Rightarrow 11\mathbf{i} + 10\mathbf{j} = 9\mathbf{i} + 12\mathbf{j} + \mathbf{c} \\ &\Rightarrow \mathbf{c} = 2\mathbf{i} - 2\mathbf{j} \\ &\Rightarrow \underline{\underline{\mathbf{v} = [(t^3 - 2t^2 + 2)\mathbf{i} + (3t^2 - 5t - 2)\mathbf{j}] \text{ ms}^{-1}}}. \end{aligned}$$

(b) the speed of P when it is moving parallel to the vector $\mathbf{i}$. (4)

Solution

$$\begin{aligned} \mathbf{j} = 0 &\Rightarrow 3t^2 - 5t - 2 = 0 \\ &\Rightarrow (3t + 1)(t - 2) = 0 \\ &\Rightarrow t = 2, \end{aligned}$$

as $t \geqslant 0$, and

$$\mathbf{v} = 2\mathbf{i} \Rightarrow |\mathbf{v}| = \underline{\underline{2 \text{ ms}^{-1}}}.$$

56. A small ball is projected with speed 14 ms^{-1} from a point A on horizontal ground. The angle of projection is α above the horizontal. A horizontal platform is at height h metres above the ground. The ball moves freely under gravity until it hits the platform at the point B, as shown in Figure 19.

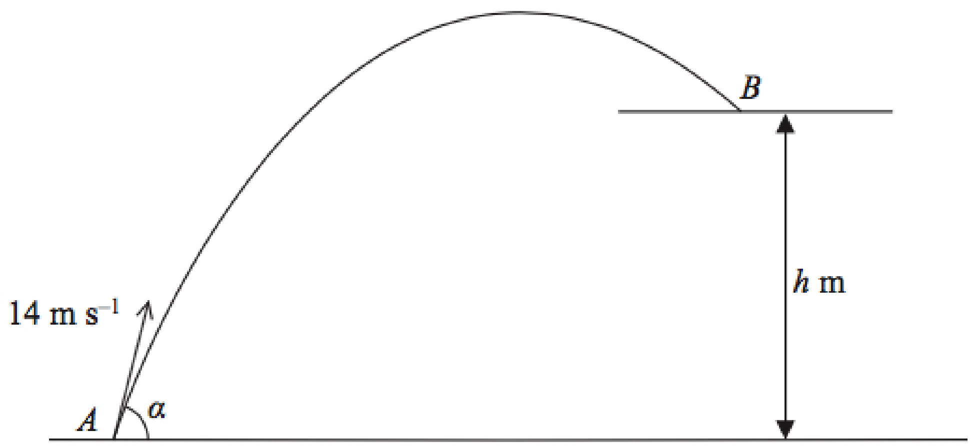

Figure 19: a small ball is projected with speed 14 ms^{-1}

The speed of the ball immediately before it hits the platform at B is 10 ms^{-1}.

(a) Find the value of h. (4)

Solution

$$\tfrac{1}{2}m(14^2) = \tfrac{1}{2}m(10^2) + 9.8mh \Rightarrow 196 = 100 + 19.6h$$
$$\Rightarrow 19.6h = 96$$
$$\Rightarrow h = 4\tfrac{44}{49}$$
$$\Rightarrow \underline{\underline{h = 4.9 \text{ m (2 sf)}}}.$$

Given that $\sin\alpha = 0.85$,

(b) find the horizontal distance from A to B. (8)

Solution

$$x = 14t\cos\alpha \text{ and } 4\tfrac{44}{49} = 14t\sin\alpha - 4.9t^2;$$

now,

$$4\tfrac{44}{49} = 11.9t - 4.9t^2 \Rightarrow 4.9t^2 - 11.9t + 4\tfrac{44}{49} = 0$$
$$\Rightarrow t = 0.525\,151\,265\,1 \text{ or } t = 1.903\,420\,163 \text{ (FCD)}.$$

But $t = 1.903\ldots$ is the one we need ($t = 0.525\ldots$ is going up) and

$$\begin{aligned} x &= 14t\cos\alpha \\ &= 14.037\,643\,05 \text{ (FCD)} \\ &= \underline{\underline{14 \text{ m (2 sf)}}}. \end{aligned}$$

57. A ball of mass 0.4 kg is moving in a horizontal plane when it is struck by a bat. The bat exerts an impulse $(-5\mathbf{i} + 3\mathbf{j})$ Ns on the ball. Immediately after receiving the impulse the ball has velocity $(12\mathbf{i} + 15\mathbf{j})$ ms^{-1}. Find

(a) the speed of the ball immediately before the impact, (4)

Solution

$$\begin{aligned} 0.4[(12\mathbf{i} + 15\mathbf{j}) - \mathbf{v}] = -5\mathbf{i} + 3\mathbf{j} &\Rightarrow (12\mathbf{i} + 15\mathbf{j}) - \mathbf{v} = -12.5\mathbf{i} + 7.5\mathbf{j} \\ &\Rightarrow \mathbf{v} = 24.5\mathbf{i} + 7.5\mathbf{j} \\ &\Rightarrow |\mathbf{v}| = \sqrt{24.5^2 + 7.5^2} \\ &\Rightarrow |\mathbf{v}| = 25.622\,255\,95 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{|\mathbf{v}| = 26 \text{ ms}^{-1} \text{ (2 sf)}}}. \end{aligned}$$

(b) the size of the angle through which the direction of motion of the ball is deflected by the impact. (3)

Solution

$$\tan^{-1}\frac{15}{12} = 51.340\,191\,75 \text{ (FCD)}$$

and

$$\tan^{-1}\frac{7.5}{24.5} = 17.020\,525\,61 \text{ (FCD)}$$

and subtract:

$$34.319\,666\,13 \text{ (FCD)} = \underline{\underline{34^\circ \text{ (2 sf)}}}.$$

58. A particle P is projected from a point A with speed 25 ms^{-1} at an angle of elevation α, where $\sin\alpha = \frac{4}{5}$. The point A is 10 m vertically above the point O which is on horizontal ground, as shown in Figure 20.

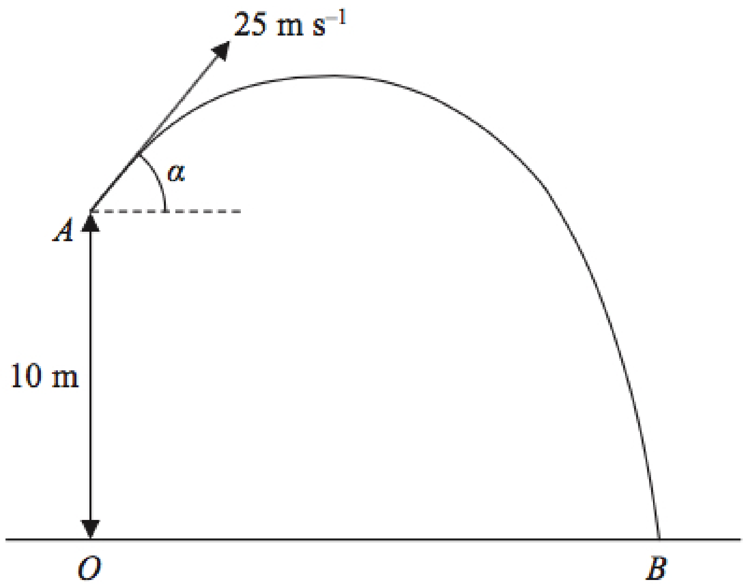

Figure 20: a particle P is projected from a point A with speed 25 ms^{-1}

The particle P moves freely under gravity and reaches the ground at the point B. Calculate

(a) the greatest height above the ground of P, as it moves from A to B, (3)

Solution

$s = ?\,(\uparrow)$, $u = 25 \times \frac{4}{5} = 20$, $v = 0$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} v^2 = u^2 + 2as \Rightarrow 0 &= 20^2 + 2 \times (-9.8) \times s \\ \Rightarrow 19.6s &= 400 \\ \Rightarrow s &= 20\tfrac{20}{49}; \end{aligned}$$

hence,

$$\text{greatest height} = 10 + 20\tfrac{20}{49} = \underline{\underline{30 \text{ m (2 sf)}}}.$$

(b) the distance OB. (6)

Solution

$$OB = 15t \text{ and } -10 = 20t - 4.9t^2;$$

now,

$$4.9t^2 - 20t - 10 = 0 \Rightarrow t = \frac{20 \pm \sqrt{20^2 - 4 \times 4.9 \times (-10)}}{9.8}$$
$$\Rightarrow t = -0.450\,317\,472\,6 \text{ or } t = 4.531\,950\,126 \text{ (FCD)}$$

and

$$\begin{aligned} OB &= 15 \times 4.531\ldots \\ &= 67.979251\,88 \text{ (FCD)} \\ &= \underline{\underline{68 \text{ m (2 sf)}}}. \end{aligned}$$

The point C lies on the path of P. The direction of motion of P at C is perpendicular to the direction of motion of P at A.

(c) Find the time taken by P to move from A to C. (4)

Solution

At C, the horizontal speed is 15 ms^{-1} (see part (b)) and the vertical speed is

$$\frac{15}{\tan\alpha} = \frac{15}{\frac{4}{3}} = 11.25 \text{ ms}^{-1}.$$

$s = ?\,(\downarrow)$, $u = -20$, $v = 11.25$, $a = 9.8$, and $t = ?$:

$$\begin{aligned} v = u + at &\Rightarrow 11.25 = -20 + 9.8t \\ &\Rightarrow 9.8t = 31.25 \\ &\Rightarrow t = 3\tfrac{37}{196} \\ &\Rightarrow \underline{\underline{t = 3.2 \text{ s (2 sf)}}}. \end{aligned}$$

59. A particle P of mass 0.75 kg is moving with velocity $4\mathbf{i}$ ms^{-1} when it receives an impulse $(6\mathbf{i} + 6\mathbf{j})$ Ns. The angle between the velocity of P before the impulse and the velocity of P after the impulse is θ°. Find

(a) the value of θ, (5)

Solution

$$0.75[\mathbf{v} - 4\mathbf{i}] = 6\mathbf{i} + 6\mathbf{j} \Rightarrow \mathbf{v} - 4\mathbf{i} = 8\mathbf{i} + 8\mathbf{j}$$
$$\Rightarrow \mathbf{v} = 12\mathbf{i} + 8\mathbf{j}$$
$$\Rightarrow \theta = \tan^{-1}\frac{8}{12}$$
$$\Rightarrow \theta = 33.690\,067\,53 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 34 \text{ (2 sf)}}}.$$

(b) the kinetic energy gained by P as a result of the impulse. (3)

Solution

$$\text{Kinetic energy gained} = \tfrac{1}{2} \times 0.75 \times (12^8 + 8^2) - \tfrac{1}{2} \times 0.75 \times 4^2$$
$$= 78 - 6$$
$$= \underline{\underline{72 \text{ J}}}.$$

60. A particle P moves on the positive x-axis. The velocity of P at time t seconds is $(2t^2 - 9t + 4)$ ms^{-1}. When $t = 0$, P is 15 m from the origin O. Find

(a) the values of t when P is instantaneously at rest, (3)

Solution

$$v = 0 \Rightarrow 2t^2 - 9t + 4 = 0$$
$$\Rightarrow (2t - 1)(t - 4) = 0$$
$$\Rightarrow \underline{\underline{t = 0.5 \text{ s}}} \text{ or } \underline{\underline{t = 4 \text{ s}}}$$

(b) the acceleration of P when $t = 5$, (3)

Solution

$$v = 2t^2 - 9t + 4 \Rightarrow a = 4t - 9,$$

and

$$t = 5 \Rightarrow \underline{\underline{a = 11 \text{ ms}^{-2}}}.$$

(c) the total distance travelled by P in the interval $0 \leqslant t \leqslant 5$. (5)

Solution

$$\begin{aligned}[\tfrac{2}{3}t^3 - \tfrac{9}{2}t^2 + 4t]_{t=4}^{5} &= (\tfrac{250}{3} - \tfrac{225}{2} + 20) - (\tfrac{128}{3} - 72 + 16)\\ &= -\tfrac{55}{6} - (-\tfrac{40}{3})\\ &= \tfrac{25}{6};\\ [\tfrac{2}{3}t^3 - \tfrac{9}{2}t^2 + 4t]_{t=0.5}^{4} &= (\tfrac{128}{3} - 72 + 16) - (\tfrac{1}{12} - \tfrac{9}{8} + 2)\\ &= -\tfrac{40}{3} - \tfrac{23}{24}\\ &= -\tfrac{343}{24};\\ [\tfrac{2}{3}t^3 - \tfrac{9}{2}t^2 + 4t]_{t=0}^{0.5} &= (\tfrac{1}{12} - \tfrac{9}{8} + 2) - (0 - 0 + 0)\\ &= \tfrac{23}{24} - 0\\ &= \tfrac{23}{24};\end{aligned}$$

hence,

$$\begin{aligned}\text{total distance} &= \tfrac{25}{6} + \tfrac{343}{24} + \tfrac{23}{24}\\ &= 19\tfrac{5}{12}\\ &= \underline{\underline{19 \text{ m (2 sf)}}}.\end{aligned}$$

61. At time $t = 0$, a particle is projected from a fixed point O on horizontal ground with speed u ms^{-1} at an angle θ° to the horizontal. The particle moves freely under gravity and passes through the point A when $t = 4$ s. As it passes through A, the particle is moving upwards at 20° to the horizontal with speed 15 ms^{-1}, as shown in Figure 21.

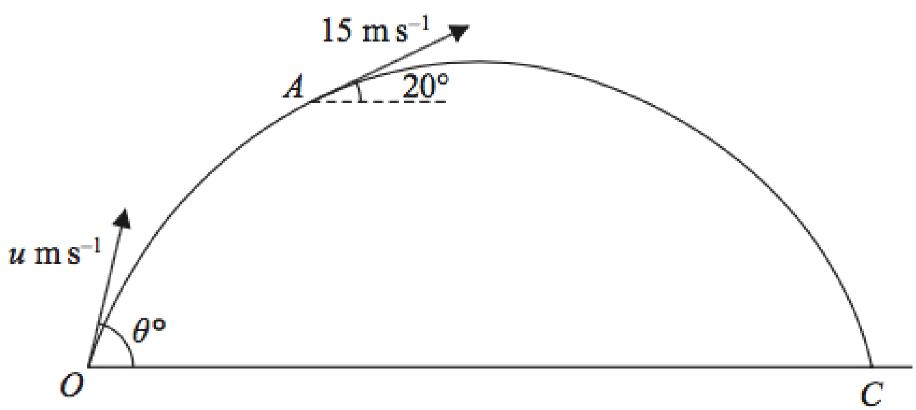

Figure 21: a particle is projected from a fixed point O

(a) Find the value of u and the value of θ. (7)

Solution

$s = ?\,(\rightarrow)$, $u = u\cos\theta^\circ$, $v = 15\cos 20^\circ$, $a = 0$, and $t = 4$:

$$u\cos\theta^\circ = 15\cos 20^\circ;$$

$s = ?\,(\uparrow)$, $u = u\sin\theta^\circ$, $v = 15\sin 20^\circ$, $a = -9.8$, and $t = 4$:

$$v = u + at \Rightarrow 15\sin 20^\circ = u\sin\theta^\circ - 39.2.$$

Now,

$$\begin{aligned} u = \frac{15\cos 20^\circ}{\cos\theta^\circ} &\Rightarrow 15\sin 20^\circ = 15\cos 20^\circ\tan\theta^\circ - 39.2 \\ &\Rightarrow 15\cos 20^\circ\tan\theta^\circ = 15\sin 20^\circ + 39.2 \\ &\Rightarrow \tan\theta^\circ = \frac{15\sin 20^\circ + 39.2}{15\cos 20^\circ} \\ &\Rightarrow \theta = 72.361\,268\,96 \text{ (FCD)} \\ &\Rightarrow u = 46.517\,262\,26 \text{ (FCD)}; \end{aligned}$$

hence,

$$\underline{\underline{u = 47 \text{ ms}^{-1} \text{ (2 sf)}}} \text{ and } \underline{\underline{\theta = 72^\circ \text{ (2 sf)}}}.$$

At the point B on its path the particle is moving downwards at 20° to the horizontal with speed 15 ms^{-1}.

(b) Find the time taken for the particle to move from A to B. (2)

Solution

$s = ?\,(\uparrow)$, $u = 15\sin\theta^\circ$, $v = -15\sin 20^\circ$, $a = -9.8$, and $t = ?$:

$$\begin{aligned} v = u + at &\Rightarrow -15\sin 20^\circ = 15\sin\theta^\circ - 9.8t \\ &\Rightarrow 9.8t = 30\sin\theta^\circ \\ &\Rightarrow t = 1.047\,000,439 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 1.0 \text{ s (2 sf)}}}. \end{aligned}$$

The particle reaches the ground at the point C.

(c) Find the distance OC. (3)

Solution

Now,

$$\text{total time} = 4 + 1.047\ldots + 4 = 9.047\,000,439 \text{ (FCD)}$$

and

$$\begin{aligned} OC &= 46.517\cos(72.361\ldots) \times 9.047\ldots \\ &= 127.520\,993\,3 \text{ (FCD)} \\ &= \underline{\underline{130 \text{ m (2 sf)}}}. \end{aligned}$$

62. A particle P moves along a straight line. The speed of P at time t seconds ($t \geqslant 0$) is v ms^{-1}, where

$$v = pt^2 + qt + r,$$

and p, q, and r are constants. When $t = 2$ the speed of P has its minimum value. When $t = 0$, $v = 11$ and, when $t = 2$, $v = 3$. Find

(a) the acceleration of P when $t = 3$, (8)

Solution

$$t = 0, v = 11 \Rightarrow v = pt^2 + qt + 11,$$

and

$$t = 2, v = 3 \Rightarrow 3 = 4p + 2q + 11 \Rightarrow 4p + 2q = -8.$$

Now,

$$v = pt^2 + qt + 11 \Rightarrow a = 2pt + q$$

and

$$\begin{aligned} t = 2, a = 0 &\Rightarrow 4p + q = 0 \\ &\Rightarrow (-8 - 2q) + q = 0 \\ &\Rightarrow q = -8 \\ &\Rightarrow p = 2. \end{aligned}$$

Hence,

$$v = 2t^2 - 8t + 11 \text{ and } a = 4t - 8$$

and

$$t = 3 \Rightarrow a = 4 \times 3 - 8 = \underline{\underline{4 \text{ ms}^{-2}}}.$$

(b) the distance travelled by P in the third second of the motion. (5)

Solution

$$\begin{aligned}[\tfrac{2}{3}t^3 - 4t^2 + 11t]_{t=2}^{3} &= (18 - 36 + 33) - (\tfrac{16}{3} - 16 + 22)\\ &= 15 - 11\tfrac{1}{3}\\ &= \underline{\underline{3\tfrac{2}{3}\text{ m}}}.\end{aligned}$$

63. A particle of mass 0.6 kg is moving with constant velocity $(c\mathbf{i} + 2c\mathbf{j})$ ms^{-1}, where c is a positive constant. The particle receives an impulse of magnitude $2\sqrt{10}$ Ns. Immediately after receiving the impulse the particle has velocity $(2c\mathbf{i} - c\mathbf{j})$ ms^{-1}. Find the value of c. (6)

Solution

$$0.6[(2c\mathbf{i} - c\mathbf{j}) - (c\mathbf{i} + 2c\mathbf{j})] = 0.6[c\mathbf{i} - 3c\mathbf{j}]$$

and

$$\begin{aligned}0.6\sqrt{c + (3c)^2} = 2\sqrt{10} &\Rightarrow 0.6 \times \sqrt{10c^2} = 2\sqrt{10}\\ &\Rightarrow \underline{\underline{c = 3\tfrac{1}{3}}}.\end{aligned}$$

64. (In this question, $\mathbf{i}$ is a horizontal unit vector and $\mathbf{j}$ is an upward vertical unit vector.) A particle P is projected from a fixed origin O with velocity $(3\mathbf{i} + 4\mathbf{j})$ ms^{-1}. The particle moves freely under gravity and passes through the point A with position vector $\lambda(\mathbf{i} - \mathbf{j})$ m, where λ is a positive constant.

(a) Find the value of λ. (6)

Solution

$$x = 3t \text{ and } y = 4t - 4.9t^2;$$

now,

$$\begin{aligned}t = \frac{x}{3} &\Rightarrow y = 4 \times \frac{x}{3} - 4.9 \times \left(\frac{x}{3}\right)^2\\ &\Rightarrow y = \frac{4x}{3} - \frac{4.9x^2}{9}.\end{aligned}$$

In addition, $(\mathbf{i}-\mathbf{j})$ lies on A:

$$\begin{aligned}\Rightarrow \lambda &= -\left[\frac{4\lambda}{3} - \frac{4.9\lambda^2}{9}\right] \\ \Rightarrow \lambda &= -\frac{4\lambda}{3} + \frac{4.9\lambda^2}{9} \\ \Rightarrow \frac{7\lambda}{3} &= \frac{4.9\lambda^2}{9} \\ \Rightarrow \frac{7}{3} &= \frac{4.9\lambda}{9} \\ \Rightarrow \underline{\underline{\lambda = 4\tfrac{2}{7}}},\end{aligned}$$

as $\lambda > 0$.

(b) Find (7)

(i) the speed of P at the instant when it passes through A,

Solution
Horizontally, $u_x = 3$ and, vertically, $s = ?\,(\uparrow)$, $u = 4$, $v = ?$, $a = -9.8$, and $t = ?$:

$$v = u + at = 4 - 9.8 \times \frac{\lambda}{3} = -10.$$

Now,

$$\begin{aligned}\text{speed} &= \sqrt{3^2 + 10^2} \\ &= \sqrt{109} \\ &= \underline{\underline{10 \text{ ms}^{-1} \text{ (2 sf)}}}.\end{aligned}$$

(ii) the direction of motion of P at the instant when it passes through A.

Solution
The direction is

$$\begin{aligned}\tan^{-1}\frac{10}{3} &= 73.300\,755\,77 \text{ (FCD)} \\ &= \underline{\underline{73^\circ \text{ (2 sf) below the horizontal}}}.\end{aligned}$$

65. A particle P of mass 0.5 kg is moving with velocity $4\mathbf{j}$ ms^{-1} when it receives an impulse $\mathbf{I}$ Ns. Immediately after P receives the impulse, the velocity of P is $(2\mathbf{i} + 3\mathbf{j})$ ms^{-1}. Find

(a) the magnitude of $\mathbf{I}$, (4)

Solution

$$\begin{aligned}\mathbf{I} = 0.5[(2\mathbf{i} + 3\mathbf{j}) - 4\mathbf{j}] &\Rightarrow \mathbf{I} = 0.5[2\mathbf{i} - \mathbf{j}] \\ &\Rightarrow |\mathbf{I}| = 0.5\sqrt{2^2 + 1^2} \\ &\Rightarrow |\mathbf{I}| = 1.118\,033\,989 \text{ (FCD)} \\ &\Rightarrow |\mathbf{I}| = \underline{\underline{1.1 \text{ Ns (2 sf)}}}.\end{aligned}$$

(b) the angle between $\mathbf{I}$ and $\mathbf{j}$. (2)

Solution

$$\begin{aligned}\tan^{-1}\frac{2}{1} = 63.434\,948\,82 \text{ (FCD)} &\Rightarrow \text{angle} = 180 - 63.434\ldots \\ &\Rightarrow \text{angle} = 116.565\,051 \text{ (FCD)} \\ &\Rightarrow \text{angle} = \underline{\underline{117^\circ \text{ (3 sf)}}}.\end{aligned}$$

66. At time $t = 0$ a particle P leaves the origin O and moves along the x-axis. At time t seconds, the velocity of P is v ms^{-1} in the positive x direction, where

$$v = 3t^2 - 16t + 21.$$

The particle is instantaneously at rest when $t = t_1$ and when $t = t_2$ ($t_1 < t_2$).

(a) Find the value of t_1 and the value of t_2. (2)

Solution

$$\begin{aligned}v = 0 &\Rightarrow 3t^2 - 16t + 21 = 0 \\ &\Rightarrow (3t - 7)(t - 3) = 0 \\ &\Rightarrow \underline{\underline{t_1 = 2\tfrac{1}{3} \text{ s}}} \text{ or } \underline{\underline{t_2 = 3 \text{ s}}}\end{aligned}$$

(b) Find the magnitude of the acceleration of P at the instant when $t = t_1$. (3)

Solution

$$v = 3t^2 - 16t + 21 \Rightarrow a = 6t - 16$$

and

$$t = 2\tfrac{1}{3} \Rightarrow a = -2 \Rightarrow |a| = \underline{\underline{2\text{ ms}^{-2}}}.$$

(c) Find the distance travelled by P in the interval $t_1 \leqslant t \leqslant t_2$. (4)

Solution

$$\begin{aligned} \left[t^3 - 8t^2 + 21t\right]_{t=2\frac{1}{3}}^{3} &= (27 - 72 + 63) - (12\tfrac{19}{27} - 43\tfrac{5}{9} + 49) \\ &= 18 - 18\tfrac{4}{27} \\ &= -\tfrac{4}{27} \end{aligned}$$

and hence

$$\text{distance} = \underline{\underline{0.15\text{ m (2 sf)}}}.$$

(d) Show that P does not return to O. (3)

Solution

$$s = t^3 - 8t^2 + 21t = t(t^2 - 8t + 21) = t[(t-4)^2 + 5];$$

the particle travels away until $t_1 = 2\frac{1}{3}$ s, then backwards until $t_2 = 3$ s, and then forwards again; $\underline{\underline{P \text{ does not return to } O}}$.

67. The points A and B lie 40 m apart on horizontal ground. At time $t = 0$ the particles P and Q are projected in the vertical plane containing AB and move freely under gravity. Particle P is projected from A with speed v ms^{-1} at 60° to AB and particle Q is projected from B with speed q ms^{-1} at angle θ to BA, as shown in Figure 22.

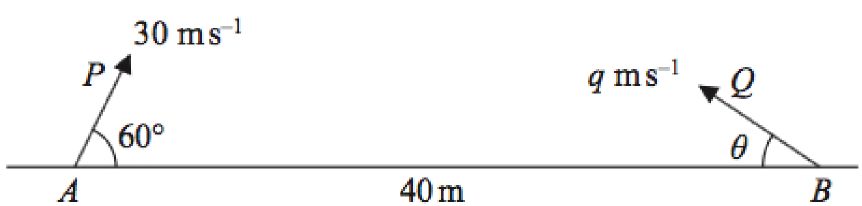

Figure 22: the points A and B lie 40 m apart on horizontal ground

At $t = 2$ seconds, P and Q collide.

(a) Find (6)

(i) the size of angle θ,

Solution
Horizontally,

$$2 \times 30 \cos 60^\circ + 2 \times q \cos \theta^\circ = 40 \Rightarrow q \cos \theta^\circ = 5,$$

and, vertically,

$$2 \times 30 \sin 60^\circ - 4.9 \times 2^2 = 2 \times q \sin \theta^\circ - 4.9 \times 2^2 \Rightarrow q \sin \theta^\circ = 15\sqrt{3}.$$

Divide:

$$\tan \theta^\circ = 3\sqrt{3} \Rightarrow \theta = 79.106\,605\,35 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 79 \text{ (2 sf)}}}.$$

(ii) the value of q.

Solution

$$q = \frac{5}{\cos 79.106\ldots} = 26.457\,513\,11 \text{ (FCD)} = \underline{\underline{26 \text{ ms}^{-1} \text{ (2 sf)}}}.$$

(b) Find the speed of P at the instant before it collides with Q. (5)

Solution
Horizontally,

$$30 \cos 60^\circ = 15$$

and, vertically,

$$30 \sin 60^\circ - 9.8 \times 2 = \frac{-98 + 75\sqrt{3}}{5};$$

hence,

$$\text{speed} = \sqrt{15^2 + \left(\frac{-98 + 75\sqrt{3}}{5}\right)^2}$$
$$= 16.300\,740\,02 \text{ (FCD)}$$
$$= \underline{\underline{16 \text{ ms}^{-1} \text{ (2 sf)}}}.$$