Dr Oliver Mathematics
Mathematics: Higher
2019 Paper 1: Non-Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

1. Find the x-coordinates of the stationary points on the curve with equation (4)

$$y = \tfrac{1}{2}x^4 - 2x^3 + 6.$$

Solution

$$\begin{aligned} y = \tfrac{1}{2}x^4 - 2x^3 + 6 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x^3 - 6x^2 \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x^2 - 12x \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2x^3 - 6x^2 = 0 \\ &\Rightarrow 2x^2(x-3) = 0 \\ &\Rightarrow \underline{\underline{x = 0 \text{ (double root) or } x = 3}}. \end{aligned}$$

2. The equation (3)

$$x^2 + (k-5)x + 1 = 0$$

has equal roots.

Determine the possible values of k.

Solution

$a = 1$, $b = k - 5$, and $c = 1$:

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (k-5)^2 - 4 \cdot 1 \cdot 1 = 0 \\ &\Rightarrow (k-5)^2 = 4 \\ &\Rightarrow k - 5 = \pm 2 \\ &\Rightarrow \underline{\underline{k = 3 \text{ or } k = 7}}. \end{aligned}$$

3. Circle C_1 has equation (2)

$$x^2 + y^2 - 6x - 2y - 26 = 0.$$

Circle C_2 has centre $(4, -2)$.
The radius of C_2 is equal to the radius of C_1.
Find the equation of circle C_2.

Solution

For C_1,

$$\begin{aligned} x^2 + y^2 - 6x - 2y - 26 = 0 &\Rightarrow x^2 - 6x + y^2 - 2y = 26 \\ &\Rightarrow (x^2 - 6x + 9) + (y^2 - 2y + 1) = 26 + 9 + 1 \\ &\Rightarrow (x-3)^2 + (y-1)^2 = 36; \end{aligned}$$

hence, C_1 has centre $(3, 1)$ and radius 6. So, the equation of circle C_2 is

$$\underline{\underline{(x-4)^2 + (y+2)^2 = 36}}.$$

4. A sequence is generated by the recurrence relation

$$u_{n+1} = mu_n + c,$$

where the first three terms of the sequence are 6, 9, and 11.

(a) Find the values of m and c. (3)

Solution

$$\begin{aligned} n = 2: &\quad 9 = 6m + c \quad (1) \\ n = 3: &\quad 11 = 9m + c \quad (2). \end{aligned}$$

Now, (2) − (1):

$$2 = 3m \Rightarrow \underline{\underline{m = \tfrac{2}{3}}}$$
$$\Rightarrow 9 = 4 + c$$
$$\Rightarrow \underline{\underline{c = 5}}.$$

(b) Hence, calculate the fourth term of the sequence. (1)

Solution

$$\begin{aligned} u_4 &= \tfrac{2}{3}(11) + 5 \\ &= 7\tfrac{1}{3} + 5 \\ &= \underline{\underline{12\tfrac{1}{3}}}. \end{aligned}$$

5. (a) Show that the points $A(1, 5, -3)$, $B(4, -1, 0)$, and $C(8, -9, 4)$ are collinear. (3)

Solution

$$\begin{aligned} \overrightarrow{AB} &= \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{BC} &= \begin{pmatrix} 8 \\ -9 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix} \\ &= \tfrac{4}{3} \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \\ &= \tfrac{4}{3}\overrightarrow{AB}. \end{aligned}$$

Now, B is in common which means that the points A, B, and C are collinear.

(b) State the ratio in which B divides AC. (1)

Solution

$$1 : \tfrac{4}{3} = \underline{\underline{3 : 4}}.$$

6. Given that (3)

$$y = \frac{1}{(1-3x)^5},\ x \neq 1,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\begin{aligned} y = \frac{1}{(1-3x)^5} &\Rightarrow y = (1-3x)^{-5} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -5(1-3x)^{-6} \cdot (-3) \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 15(1-3x)^{-6}}}. \end{aligned}$$

7. The line, L, makes an angle of 30° with the positive direction of the x-axis. Find the equation of the line perpendicular to L, passing through $(0, -4)$. (4)

Solution

The line perpendicular to L makes an angle of

$$30 + 90 = 120^\circ$$

and the gradient is

$$\tan 120^\circ = -\sqrt{3}.$$

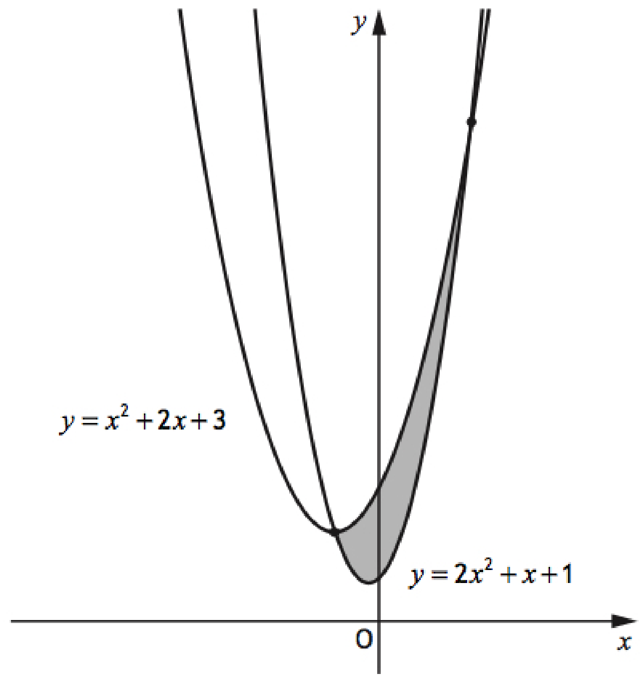
Hence, the equation of the line is

$$y + 4 = -\sqrt{3}(x - 0) \Rightarrow \underline{\underline{y = -\sqrt{3}\,x - 4}}.$$

8. The graphs of

$$y = x^2 + 2x + 3 \text{ and } y = 2x^2 + x + 1$$

are shown below.



The graphs intersect at the points where $x = -1$ and $x = 2$.

(a) Express the shaded area, enclosed between the curves, as an integral. (1)

Solution

$$\begin{aligned}\text{Shaded area} &= \int_{-1}^{2} \left[(x^2 + 2x + 3) - (2x^2 + x + 1)\right] \, \mathrm{d}x \\ &= \underline{\underline{\int_{-1}^{2} (-x^2 + x + 2) \, \mathrm{d}x.}}\end{aligned}$$

(b) Evaluate the shaded area. (3)

Solution

$$
\begin{aligned}
\text{Shaded area} &= \int_{-1}^{2} (-x^2 + x + 2)\,\mathrm{d}x \\
&= \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x\right]_{x=-1}^{2} \\
&= \left(-\tfrac{8}{3} + 2 + 4\right) - \left(\tfrac{1}{3} + \tfrac{1}{2} - 2\right) \\
&= 3\tfrac{1}{3} - (-1\tfrac{1}{6}) \\
&= \underline{\underline{4\tfrac{1}{2}}}.
\end{aligned}
$$

9. Vectors **u** and **v** have components

$$
\mathbf{u} = \begin{pmatrix} p \\ -2 \\ 4 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 2p+16 \\ -3 \\ 6 \end{pmatrix},\ p \in \mathbb{R}.
$$

(a) (i) Find an expression for **u**.**v**. (1)

Solution

$$
\begin{aligned}
\mathbf{u}.\mathbf{v} &= p(2p+16) + 6 + 24 \\
&= \underline{\underline{2p^2 + 16p + 30}}.
\end{aligned}
$$

(ii) Determine the values of p for which **u** and **v** are perpendicular. (3)

Solution

$$
\begin{aligned}
\mathbf{u}.\mathbf{v} = 0 &\Rightarrow 2p^2 + 16p + 30 = 0 \\
&\Rightarrow 2(p^2 + 8p + 15) = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll} \text{add to:} & +8 \\ \text{multiply to:} & +15 \end{array}\right\} +3,\ +5
$$

$$
\begin{aligned}
&\Rightarrow 2(p+3)(p+5) = 0 \\
&\Rightarrow \underline{\underline{p = -5 \text{ or } p = -3}}.
\end{aligned}
$$

(b) Determine the value of p for which **u** and **v** are parallel. (2)

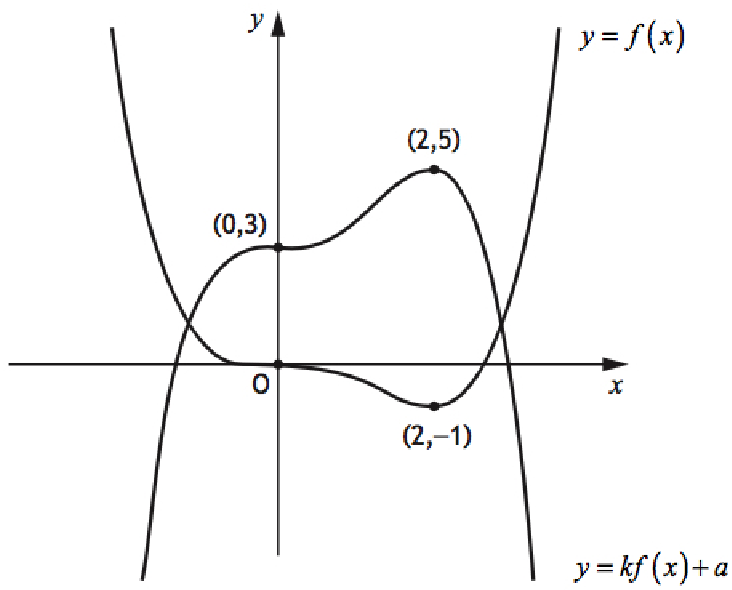
Solution

They are parallel when $\lambda \mathbf{u} = \mathbf{v}$ for some λ. Then,

$$\frac{3}{2}p = 2p + 16 \Rightarrow -\frac{1}{2}p = 16$$
$$\Rightarrow \underline{\underline{p = -32}}.$$

10. The diagram shows the graphs with equations

$$y = \mathrm{f}(x) \text{ and } y = k\,\mathrm{f}(x) + a.$$



(a) State the value of a. (1)

Solution

$\underline{\underline{a = 3}}$. (It comes from $(0, 0)$ being mapped to $(0, 3)$.)

(b) Find the value of k. (1)

Solution

$(2, -1)$ is a point on the graph $y = \mathrm{f}(x)$ and so we do

$$\begin{aligned} 5 = (-1)k + 3 &\Rightarrow -k = 2 \\ &\Rightarrow \underline{\underline{k = -2}}. \end{aligned}$$

11. Evaluate (4)

$$\int_0^{\frac{1}{9}\pi} \cos(3x - \tfrac{1}{6}\pi)\,\mathrm{d}x.$$

Solution

$$\begin{aligned} \int_0^{\frac{1}{9}\pi} \cos(3x - \tfrac{1}{6}\pi)\,\mathrm{d}x &= \left[\tfrac{1}{3}\sin(3x - \tfrac{1}{6}\pi)\right]_{x=0}^{\frac{1}{9}\pi} \\ &= \tfrac{1}{3}\sin(\tfrac{1}{3}\pi - \tfrac{1}{6}\pi) + \tfrac{1}{3}\sin(0 - \tfrac{1}{6}\pi) \\ &= \tfrac{1}{3}\sin(\tfrac{1}{6}\pi) - \tfrac{1}{3}\sin(\tfrac{1}{6}\pi) \\ &= \tfrac{2}{3}\sin(\tfrac{1}{6}\pi) \\ &= \tfrac{2}{3} \cdot \tfrac{1}{2} \\ &= \underline{\underline{\tfrac{1}{3}}}. \end{aligned}$$

12. Functions f and g are defined by

- $\mathrm{f}(x) = \dfrac{1}{\sqrt{x}}$, where $x > 0$ and
- $\mathrm{g}(x) = 5 - x$, where $x \in \mathbb{R}$.

(a) Determine an expression for $\mathrm{f}(\mathrm{g}(x))$. (2)

Solution

$$\begin{aligned} \mathrm{f}(\mathrm{g}(x)) &= \mathrm{f}(5 - x) \\ &= \underline{\underline{\frac{1}{\sqrt{5 - x}}}}. \end{aligned}$$

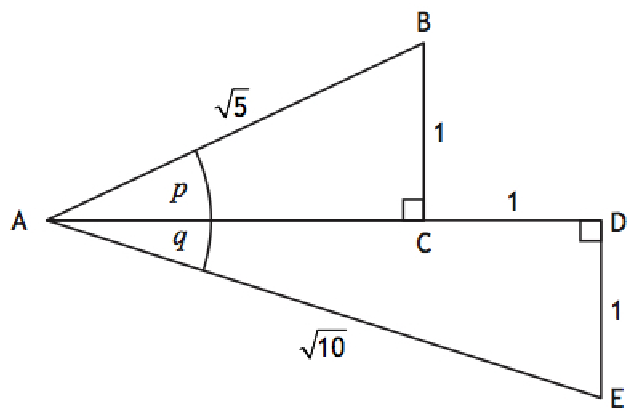
(b) State the range of values of x for which $\mathrm{f}(\mathrm{g}(x))$ is undefined. (1)

Solution

$$\begin{aligned}\sqrt{5-x} > 0 &\Rightarrow 5 - x > 0 \\ &\Rightarrow x < 5\end{aligned}$$

and so, if $\mathrm{f}(\mathrm{g}(x))$ is undefined, we have $\underline{\underline{x \geqslant 5}}$.

13. Triangles ABC and ADE are both right-angled.
Angles p and q are as shown in the diagram.



(a) Determine the value of

(i) $\cos p$ and (1)

Solution

$$\begin{aligned}AC &= \sqrt{(\sqrt{5})^2 - 1} \\ &= \sqrt{5-1} \\ &= \sqrt{4} \\ &= 2\end{aligned}$$

and $\cos p = \underline{\underline{\frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}}}$.

(ii) $\cos q$. (1)

Solution

$\cos q = \underline{\underline{\frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}}}.$

(b) Hence determine the value of $\sin(p+q)$. (3)

Solution

$$\begin{aligned}
\sin(p+q) &= \sin p \cos q + \sin q \cos p \\
&= (\tfrac{\sqrt{5}}{5} \times \tfrac{3\sqrt{10}}{10}) + (\tfrac{\sqrt{10}}{10} \times \tfrac{2\sqrt{5}}{5}) \\
&= \tfrac{3\sqrt{50}}{50} + \tfrac{2\sqrt{50}}{50} \\
&= \tfrac{5\sqrt{50}}{50} \\
&= \tfrac{\sqrt{25\times 2}}{10} \\
&= \tfrac{\sqrt{25}\times\sqrt{2}}{10} \\
&= \tfrac{5\sqrt{2}}{10} \\
&= \underline{\underline{\tfrac{\sqrt{2}}{2}}}.
\end{aligned}$$

14. (a) Evaluate (3)

$$\log_{10} 4 + 2\log_{10} 5.$$

Solution

$$\begin{aligned}
\log_{10} 4 + 2\log_{10} 5 &= \log_{10} 4 + \log_{10} 5^2 \\
&= \log_{10} 4 + \log_{10} 25 \\
&= \log_{10}(4 \times 25) \\
&= \log_{10} 100 \\
&= \log_{10} 10^2 \\
&= \underline{\underline{2}}.
\end{aligned}$$

(b) Solve (3)

$$\log_2(7x-2) - \log_2 3 = 5,\ x \geqslant 1.$$

Solution

$$\begin{aligned}\log_2(7x-2)-\log_2 3 = 5 &\Rightarrow \log_2\left(\frac{7x-2}{3}\right)=5\\ &\Rightarrow \frac{7x-2}{3}=2^5\\ &\Rightarrow \frac{7x-2}{3}=32\\ &\Rightarrow 7x-2=96\\ &\Rightarrow 7x=98\\ &\Rightarrow \underline{\underline{x=14}}.\end{aligned}$$

15. (a) Solve the equation (4)

$$\sin 2x^\circ + 6\cos x^\circ = 0$$

for $0 \leqslant x < 360$.

Solution

$$\begin{aligned}\sin 2x^\circ + 6\cos x^\circ = 0 &\Rightarrow 2\sin x^\circ \cos x^\circ + 6\cos x^\circ = 0\\ &\Rightarrow 2\cos x^\circ(\sin x^\circ + 3) = 0\\ &\Rightarrow \cos x^\circ = 0 \text{ (as } \sin x^\circ \neq -3)\\ &\Rightarrow \underline{\underline{x = 90, 270}}.\end{aligned}$$

(b) Hence solve (1)

$$\sin 4x^\circ + 6\cos 2x^\circ = 0$$

for $0 \leqslant x < 360$.

Solution

$$\begin{aligned}\sin 4x^\circ + 6\cos 2x^\circ = 0 &\Rightarrow 2\sin 2x^\circ \cos 2x^\circ + 6\cos 2x^\circ = 0\\ &\Rightarrow 2\cos 2x^\circ(\sin 2x^\circ + 3) = 0\\ &\Rightarrow \cos 2x^\circ = 0 \text{ (as } \sin 2x^\circ \neq -3)\\ &\Rightarrow 2x = 90, 270, 450, 630\\ &\Rightarrow \underline{\underline{x = 45, 135, 225, 315}}.\end{aligned}$$

16. The point P has coordinates $(4, k)$.
C is the centre of the circle with equation

$$(x-1)^2 + (y+2)^2 = 25.$$

(a) Show that the distance between the points P and C is given by (2)

$$\sqrt{k^2 + 4k + 13}.$$

Solution
The radius of C is $\sqrt{25} = 5$ and the centre is $(1, -2)$. So, the distance is

$$\begin{aligned}\sqrt{(4-1)^2 + (k+2)^2} &= \sqrt{9 + (k^2 + 4k + 4)} \\ &= \underline{\underline{\sqrt{k^2 + 4k + 13}}},\end{aligned}$$

as required.

(b) Hence, or otherwise, find the range of values of k such that P lies outside the circle. (4)

Solution

$$\begin{aligned}\sqrt{k^2 + 4k + 13} > 5 &\Rightarrow k^2 + 4k + 13 > 25 \\ &\Rightarrow k^2 + 4k + 4 > 16 \\ &\Rightarrow (k+2)^2 > 16 \\ &\Rightarrow k + 2 < -4 \text{ or } k + 2 > 4 \\ &\Rightarrow \underline{\underline{k < -6 \text{ or } k > 2}}.\end{aligned}$$

17. (a) Express (3)

$$(\sin x - \cos x)^2$$

in the form

$$p + q \sin rx,$$

where p, q, and r are integers.

Solution

$$\begin{aligned}(\sin x - \cos x)^2 &= \sin^2 x - 2\sin x - \cos x + \cos^2 x \\ &= (\sin^2 x + \cos^2 x) - 2\sin x - \cos x \\ &= \underline{\underline{1 - \sin 2x}};\end{aligned}$$

so, $\underline{\underline{p = 1}}$, $\underline{\underline{q = -1}}$, and $\underline{\underline{r = 2}}$.

(b) Hence, find (2)

$$\int (\sin x - \cos x)^2 \, \mathrm{d}x.$$

Solution

$$\begin{aligned}\int (\sin x - \cos x)^2 \, \mathrm{d}x &= \int (1 - \sin 2x) \, \mathrm{d}x \\ &= \underline{\underline{x + \tfrac{1}{2}\cos 2x + c}}.\end{aligned}$$