Dr Oliver Mathematics
Advance Level Mathematics
Core Mathematics 4: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. (a) Find the binomial series expansion of (5)

$$\sqrt{4-9x},\ |x| < \tfrac{4}{9},$$

in ascending powers of x, up to and including the term in x^2.
Give each coefficient in its simplest form.

Solution
We can use

$$(a+bx)^n \equiv a^n + na^{n-1}bx + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}(bx)^3 + \ldots$$

Now,

$$\begin{aligned}\sqrt{4-9x} &= (4-9x)^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} + \tfrac{1}{2}(4^{-\frac{1}{2}})(-9x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(4^{-\frac{3}{2}})(-9x)^2 + \ldots \\ &= \underline{\underline{2 - \tfrac{9}{4}x - \tfrac{81}{64}x^2 + \ldots}}\end{aligned}$$

(b) Use the expansion from part (a), with a suitable value of x, to find an approximate value for $\sqrt{310}$. (3)
Show all your working and give your answer to 3 decimal places.

Solution

$$\begin{aligned}\sqrt{310} &= \sqrt{100 \times 3.1} \\ &= 10\sqrt{4 - 9 \times 0.1} \\ &\approx 10[2 - \tfrac{9}{4}(0.1) - \tfrac{81}{64}(0.1)^2] \\ &= 17.623\,437\,5 \text{ (exact)} \\ &= \underline{\underline{17.623 \text{ (3 dp)}}}.\end{aligned}$$

2. (*Solutions based entirely on graphical or numerical methods are not acceptable.*)
The curve C has equation

$$x^2 + xy + y^2 - 4x - 5y + 1 = 0.$$

(a) Use implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. (5)

Solution

$$\begin{aligned}
& x^2 + xy + y^2 - 4x - 5y + 1 = 0 \\
\Rightarrow\quad & 2x + (y + x\frac{\mathrm{d}y}{\mathrm{d}x}) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 4 - 5\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\
\Rightarrow\quad & 2x + y - 4 = 5\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} \\
\Rightarrow\quad & 2x + y - 4 = (5 - x - 2y)\frac{\mathrm{d}y}{\mathrm{d}x} \\
\Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x + y - 4}{5 - x - 2y}}}.
\end{aligned}$$

(b) Find the x-coordinates of the two points on C where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. (5)
Give exact answers in their simplest form.

Solution

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{2x + y - 4}{5 - x - 2y} = 0 \\
&\Rightarrow 2x + y - 4 = 0 \\
&\Rightarrow y = -2x + 4.
\end{aligned}$$

Insert this into C:

$$\begin{aligned}
& x^2 + x(-2x + 4) + (-2x + 4)^2 - 4x - 5(-2x + 4) + 1 = 0 \\
\Rightarrow\quad & x^2 - 2x^2 + 4x + 4x^2 - 16x + 16 - 4x + 10x - 20 + 1 = 0 \\
\Rightarrow\quad & 3x^2 - 6x - 3 = 0 \\
\Rightarrow\quad & x^2 - 2x - 1 = 0 \\
\Rightarrow\quad & x^2 - 2x + 1 = 2 \\
\Rightarrow\quad & (x - 1)^2 = 2 \\
\Rightarrow\quad & x - 1 = \pm\sqrt{2} \\
\Rightarrow\quad & \underline{\underline{x = 1 \pm \sqrt{2}}}.
\end{aligned}$$

3. (a) Given that

$$\frac{13-4x}{(2x+1)^2(x+3)} \equiv \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2} + \frac{C}{(x+3)},$$

(i) find the values of the constants A, B, and C. (4)

Solution

$$\begin{aligned}\frac{13-4x}{(2x+1)^2(x+3)} &\equiv \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2} + \frac{C}{(x+3)} \\ &\equiv \frac{A(2x+1)(x+3)+B(x+3)+C(2x+1)^2}{(2x+1)^2(x+3)}\end{aligned}$$

and so

$$13-4x \equiv A(2x+1)(x+3)+B(x+3)+C(2x+1)^2.$$

$\underline{x=-3}$:

$$25 = 25C \Rightarrow \underline{\underline{C=1}}.$$

$\underline{x=-\tfrac{1}{2}}$:

$$15 = \tfrac{5}{2}B \Rightarrow \underline{\underline{B=6}}.$$

$\underline{x=0}$:

$$13 = 3A+18+1 \Rightarrow \underline{\underline{A=-2}}.$$

(ii) Hence find (3)

$$\int \frac{13-4x}{(2x+1)^2(x+3)}\,\mathrm{d}x \; x > -\tfrac{1}{2}.$$

Solution

$$\begin{aligned}\int \frac{13-4x}{(2x+1)^2(x+3)}\,\mathrm{d}x &= \int \left[\frac{-2}{(2x+1)} + \frac{6}{(2x+1)^2} + \frac{1}{(x+3)}\right]\mathrm{d}x \\ &= \underline{\underline{-\ln|2x+1| - \frac{3}{2x+1} + \ln|x+3| + c}}.\end{aligned}$$

(b) Find (3)

$$\int (\mathrm{e}^x+1)^3\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int (e^x + 1)^3 \, dx &= \int (e^{3x} + 3e^{2x} + 3e^x + 1) \, dx \\ &= \underline{\underline{\tfrac{1}{3}e^{3x} + \tfrac{3}{2}e^{2x} + 3e^x + x + c.}}\end{aligned}$$

(c) Using the substitution $u^3 = x$, or otherwise, find (4)

$$\int \frac{1}{4x + 5x^{\frac{1}{3}}} \, dx, \; x > 0.$$

Solution

$$\begin{aligned}u = x^{\frac{1}{3}} &\Rightarrow \frac{du}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}} \\ &\Rightarrow 3 \, du = \frac{1}{x^{\frac{2}{3}}} \, dx \\ &\Rightarrow 3x^{\frac{2}{3}} \, du = dx \\ &\Rightarrow 3u^2 \, du = dx.\end{aligned}$$

Hence,

$$\begin{aligned}\int \frac{1}{4x + 5x^{\frac{1}{3}}} \, dx &= \int \frac{1}{4u^3 + 5u} \cdot 3u^2 \, du \\ &= \int \frac{3u}{4u^2 + 5} \, du \\ &= \tfrac{3}{8} \int \frac{8u}{4u^2 + 5} \, du \\ &= \tfrac{3}{8} \ln |4u^2 + 5| + c \\ &= \underline{\underline{\tfrac{3}{8} \ln |4x^{\frac{2}{3}} + 5| + c.}}\end{aligned}$$

4. A water container is made in the shape of a hollow inverted right circular cone with semi-vertical angle of 30°, as shown in Figure 1. The height of the container is 50 cm.

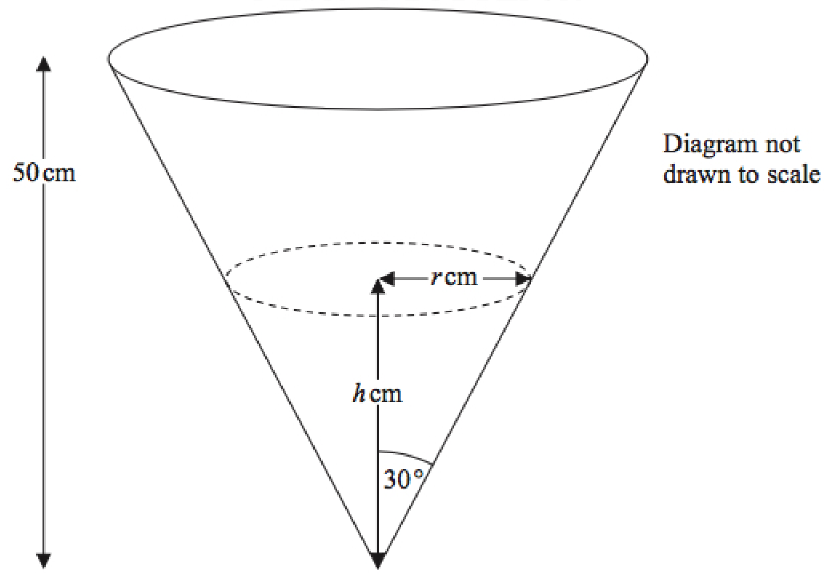

Figure 1: a water container

When the depth of the water in the container is h cm, the surface of the water has radius r cm, and the volume of water is V cm^3.

(a) Show that (2)

$$V = \tfrac{1}{9}\pi h^3.$$

[*You may assume the formula*

$$V = \tfrac{1}{3}\pi r^2 h$$

for the volume of a cone.]

Solution

$$\frac{r}{h} = \tan 30^\circ \Rightarrow r = \frac{1}{\sqrt{3}}h$$

and

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{3}\pi(\tfrac{1}{\sqrt{3}}h)^2 h \\ &= \underline{\underline{\tfrac{1}{9}\pi h^3}}, \end{aligned}$$

as required.

Given that the volume of water in the container increases at a constant rate of 200 cm^3 s^{-1},

(b) find the rate of change of the depth of the water, in cm s^{-1}, when $h = 15$. (4)
Give your answer in its simplest form in terms of π.

Solution

$$\frac{dV}{dh} = \frac{dV}{dt} \div \frac{dh}{dt} \Rightarrow \tfrac{1}{3}\pi h^2 = 200 \div \frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{600}{\pi \times 15^2}$$
$$\Rightarrow \underline{\underline{\frac{dh}{dt} = \frac{8}{3\pi}}}.$$

5. Figure 2 shows a sketch of the curve C with parametric equations

$$x = 1 + t - 5\sin t,\ y = 2 - 4\cos t,\ -\pi \leqslant t \leqslant \pi.$$

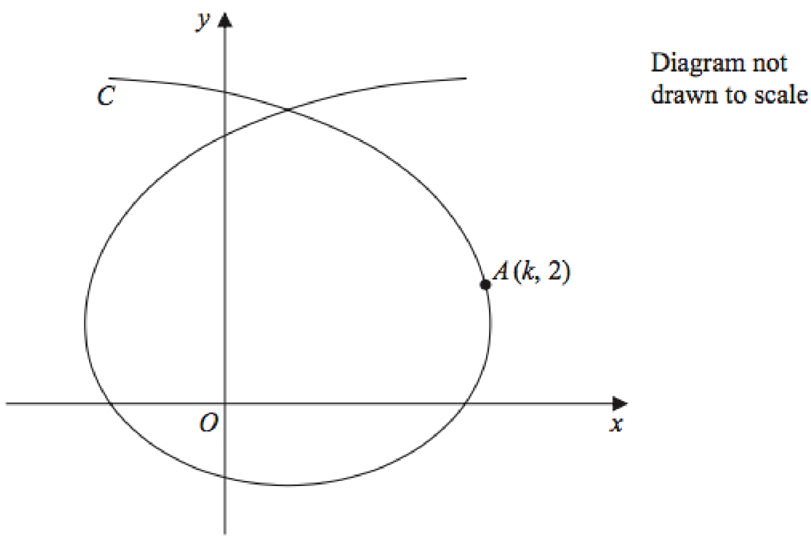

Figure 2: $x = 1 + t - 5\sin t$, $y = 2 - 4\cos t$

The point A lies on the curve C.
Given that the coordinates of A are $(k, 2)$, where $k > 0$,

(a) find the exact value of k, giving your answer in a fully simplified form. (2)

Solution

$$2 = 2 - 4\cos t \Rightarrow \cos t = 0$$
$$\Rightarrow t = -\tfrac{1}{2}\pi \text{ or } \tfrac{1}{2}\pi$$
$$\Rightarrow k = 1 + (-\tfrac{1}{2}\pi) - 5\sin(-\tfrac{1}{2}\pi) \text{ or } 1 + (\tfrac{1}{2}\pi) - 5\sin(\tfrac{1}{2}\pi)$$
$$\Rightarrow k = 1 - \tfrac{1}{2}\pi - 5(-1) \text{ or } 1 + \tfrac{1}{2}\pi - 5(1)$$
$$\Rightarrow k = 6 - \tfrac{1}{2}\pi \text{ or } -3 + \tfrac{1}{2}\pi;$$

as $k > 0$, $\underline{\underline{k = 6 - \tfrac{1}{2}\pi}}$.

(b) Find the equation of the tangent to C at the point A. (5)
Give your answer in the form $y = px + q$, where p and q are exact real values.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - 5\cos t, \frac{\mathrm{d}y}{\mathrm{d}t} = 4\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sin t}{1 - 5\cos t}.$$

When $t = -\tfrac{1}{2}\pi$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sin(-\tfrac{1}{2}\pi)}{1 - 5\cos(-\tfrac{1}{2}\pi)} = -4.$$

Finally, the equation of the tangent is

$$y - 2 = -4[x - (6 - \tfrac{1}{2}\pi)] \Rightarrow y - 2 = -4x + 24 - 2\pi$$
$$\Rightarrow \underline{\underline{y = -4x + 26 - 2\pi}};$$

hence, $\underline{\underline{p = -4}}$ and $\underline{\underline{q = 26 - 2\pi}}$.

6. Given that $y = 2$ when $x = -\tfrac{1}{8}\pi$, solve the differential equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{3\cos^2 2x}, -\tfrac{1}{2} < x < \tfrac{1}{2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{3\cos^2 2x} &\Rightarrow \frac{1}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3\cos^2 2x}\,\mathrm{d}x \\
&\Rightarrow \frac{1}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{3}\sec^2 2x\,\mathrm{d}x \\
&\Rightarrow \int \frac{1}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \int \tfrac{1}{3}\sec^2 2x\,\mathrm{d}x \\
&\Rightarrow -\frac{1}{y} = \tfrac{1}{6}\tan 2x + c \\
&\Rightarrow -y = \frac{1}{\frac{1}{6}\tan 2x + c} \\
&\Rightarrow y = \frac{-1}{\frac{1}{6}\tan 2x + c}.
\end{aligned}$$

Now,

$$\begin{aligned}
2 = \frac{-1}{\frac{1}{6}\tan(-\frac{1}{8}\pi) + c} &\Rightarrow 2 = \frac{-1}{-\frac{1}{6} + c} \\
&\Rightarrow -\tfrac{1}{6} + c = -\tfrac{1}{2} \\
&\Rightarrow c = -\tfrac{1}{3}
\end{aligned}$$

and so

$$\underline{\underline{y = \frac{-1}{\frac{1}{6}\tan 2x - \frac{1}{3}}}}.$$

7. The point A with coordinates $(-3, 7, 2)$ lies on a line l_1.
The point B also lies on the line l_1.
Given that $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$,

(a) find the coordinates of point B. (2)

Solution

$$\begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

and so the coordinates are $\underline{\underline{(1, 1, 4)}}$.

The point P has coordinates $(9, 1, 8)$.

(b) Find the cosine of the angle PAB, giving your answer as a simplified surd. (3)

Solution

Now, $\overrightarrow{AP} = \begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix}$ and so

$$\begin{aligned} \cos\ PAB &= \frac{\overrightarrow{AP} \,.\, \overrightarrow{AB}}{|\overrightarrow{PA}||\overrightarrow{AB}|} \\ &= \frac{48 + 36 + 12}{\sqrt{(-12)^2 + 6^2 + (-6)^2}\sqrt{4^2 + (-6)^2 + 2^2}} \\ &= \frac{96}{6\sqrt{6} \times 2\sqrt{14}} \\ &= \underline{\underline{\frac{4\sqrt{21}}{21}}}. \end{aligned}$$

(c) Find the exact area of triangle PAB, giving your answer in its simplest form. (3)

Solution

We know that

$$\cos\ \angle PAB = \frac{4\sqrt{21}}{21}$$

but what is $\sin\ \angle PAB$? Well,

$$\begin{aligned} \sin\ \angle PAB &= \sqrt{1 - \left(\frac{4\sqrt{21}}{21}\right)^2} \\ &= \frac{\sqrt{105}}{21}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area} &= \tfrac{1}{2} \times 6\sqrt{6} \times 2\sqrt{14} \times \frac{\sqrt{105}}{21} \\ &= \underline{\underline{12\sqrt{5}}}. \end{aligned}$$

The line l_2 passes through the point P and is parallel to the line l_1.

(d) Find a vector equation for the line l_2. (2)

Solution

$$\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix} + t\begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}.$$

The point Q lies on the line l_2.
Given that the line segment AP is perpendicular to the line segment BQ,

(e) find the coordinates of the point Q. (5)

Solution

$$\overrightarrow{BQ} = \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} + \begin{pmatrix} 9+4t \\ 1-6t \\ 8+2t \end{pmatrix} = \begin{pmatrix} 4t+8 \\ -6t \\ 2t+4 \end{pmatrix}.$$

Now,

$$\begin{aligned} \overrightarrow{BQ}\,.\,\overrightarrow{AP} = 0 &\Rightarrow 12(4t+8) - 6(-6t) + 6(2t+4) = 0 \\ &\Rightarrow 48t + 96 + 36t + 12t + 24 = 0 \\ &\Rightarrow 96t = -120 \\ &\Rightarrow t = -\tfrac{5}{4} \end{aligned}$$

and

$$\overrightarrow{BQ} = \begin{pmatrix} 3 \\ 7\frac{1}{2} \\ 1\frac{1}{2} \end{pmatrix};$$

hence, $Q(4, 8\frac{1}{2}, 5\frac{1}{2})$.

8. (*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(a) Find (3)

$$\int x\cos 4x\,\mathrm{d}x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 4x \Rightarrow v = \tfrac{1}{4}\sin 4x.$$

Now,

$$\int x\cos 4x \, \mathrm{d}x = \tfrac{1}{4}x\sin 4x - \int \left(1 \times \tfrac{1}{4}\sin 4x\right) \mathrm{d}x$$
$$= \underline{\underline{\tfrac{1}{4}x\sin 4x + \tfrac{1}{16}\cos 4x + c}}.$$

Figure 3 shows part of the curve with equation

$$y = \sqrt{x}\sin 2x, \; x \geqslant 0.$$

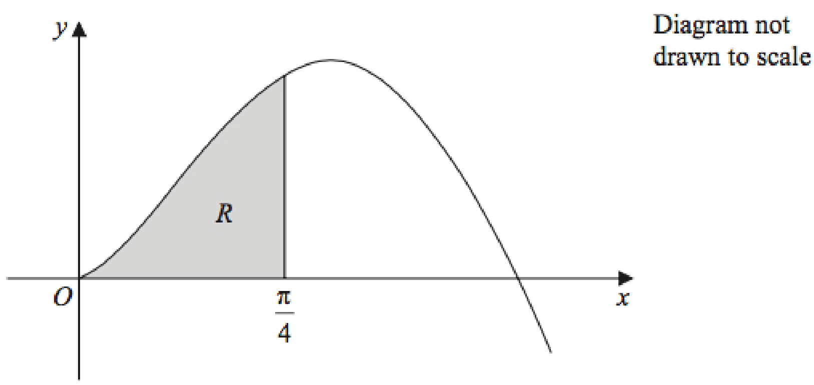

Figure 3: $y = \sqrt{x}\sin 2x$

The finite region R, shown shaded in Figure 3, is bounded by the curve, the x-axis, and the line with equation $x = \frac{1}{4}\pi$.

The region R is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact value of the volume of this solid of revolution, giving your answer in its simplest form. (6)

Solution

$$
\begin{aligned}
\text{Volume} &= \int_0^{\frac{1}{4}\pi} \pi(\sqrt{x}\sin 2x)^2 \,\mathrm{d}x \\
&= \pi \int_0^{\frac{1}{4}\pi} x \sin^2 2x \,\mathrm{d}x \\
&= \pi \int_0^{\frac{1}{4}\pi} x(\tfrac{1}{2} - \tfrac{1}{2}\cos 4x) \,\mathrm{d}x \\
&= \pi \left[\tfrac{1}{4}x^2 - \tfrac{1}{8}x\sin 4x - \tfrac{1}{32}\cos 4x\right]_{x=0}^{\frac{1}{4}\pi} \\
&= \pi \left[(\tfrac{1}{64}\pi^2 - 0 + \tfrac{1}{32}) - (0 - 0 - \tfrac{1}{32})\right] \\
&= \pi(\tfrac{1}{64}\pi^2 + \tfrac{1}{16}) \\
&= \underline{\underline{\tfrac{1}{64}\pi(\pi^2 + 4)}}.
\end{aligned}
$$