Dr Oliver Mathematics
Disproof

In this note, we will examine disproof.

Normally, the question will tell you about a so-called "proof" and we simply come up with *one* counter-example.

Example 1
For all positive integer values of n, $(n^3 - n + 7)$ is prime.

Solution
E.g., we take $n = 7$:

$$7^3 - 7 + 7 = 343 = 7 \times 49,$$

and we have a counter-example. ■

This is easy: we take the constant and susbtitute it in (unless it is 1 and then you have to be sure).

Example 2
If x and y are irrational and $x \neq y$, then xy is irrational.

Solution
E.g., we take $x = \sqrt{2}$ and $y = 2\sqrt{2}$. Then $x \neq y$ but

$$xy = \sqrt{2} \times 2\sqrt{2} = 4,$$

and we have a counter-example. ■

Here are some examples for you to try.

1. $(3^n + 2)$ is prime for all positive integer values of n.

Solution

$n = 1$: $3^1 + 2 = 5$ which is prime.
$n = 2$: $3^2 + 2 = 11$ which is prime.
$n = 3$: $3^3 + 2 = 29$ which is prime.
$n = 4$: $3^4 + 2 = 83$ which is prime.
$n = 5$: $3^5 + 2 = 245 = 5 \times 49$ which is *not* prime.

2. For every $n \in \mathbb{N}$, the integer $n^2 - n + 11$ is prime.

Solution

E.g., we take $n = 11$:
$$11^2 - 11 + 11 = 121 = 11 \times 11.$$

3. For all real values of x,
$$\cos(90 - |x|)^\circ = \sin x^\circ.$$

Solution

E.g., we take $x = -90$:
$$\cos(90 - |-90|)^\circ = \cos 180^\circ = 0$$
but
$$\sin(-90)^\circ = -1.$$

4. If a and b are positive integers and $a \neq b$, then $\log_a b$ is irrational.

Solution

E.g., we take $x = 2$ and $y = 4$. Then
$$\log_2 4 = \log_2 2^2 = 2 \log_2 2 = 2.$$

5. $(n^2 + 3n + 13)$ is prime for all positive integer values of n.

Solution

E.g., we take $n = 13$:
$$13^2 + 3 \times 13 + 13 = 221 = 13 \times 17.$$

6. There exist positive integers, a and b, to $a^2 - b^2 = 6$.

Solution
$$a^2 - b^2 = 6 \Rightarrow (a + b)(a - b) = 6.$$

Now,

$$a + b > a - b \text{ and } a + b > 0.$$

Next,

$$a + b = 6 \text{ and } a - b = 1 \quad (1)$$

or

$$a + b = 3 \text{ and } a - b = 2 \quad (2).$$

Now, for (1),

$$2a = 7 \Rightarrow a = \tfrac{7}{2}$$

and, for (2),

$$2a = 5 \Rightarrow a = \tfrac{5}{2}.$$

Clearly, in neither case is a is a positive integer and we have no solutions.

7. if a is rational and b is irrational then $\log_a b$ is irrational.

Solution

E.g., we take $x = 2$ and $y = 2^{\frac{1}{2}}$. Then

$$\log_2 2^{\frac{1}{2}} = \tfrac{1}{2}\log_2 2 = \tfrac{1}{2}.$$

8. If $x, y \in \mathbb{R}$, then $|x + y| = |x| + |y|$.

Solution

E.g., we take $x = 1$ and $y = -1$: then $|x + y| = 0$ but $|x| + |y| = 2$.

9. For all $a, b, c \in \mathbb{N}$, if $a|bc$, then $a|b$ or $a|c$.

Solution

E.g., we take $a = 10$, $b = 4$ and $c = 5$: $a|bc$, then $a \nmid b$ or $a \nmid c$.

10. If $x, y \in \mathbb{R}$, and $|x + y| = |x - y|$, then $y = 0$.

Solution

E.g., we take $x = 0$ and $y = 1$: then $|x + y| = |x - y|$, then $y = 1$.

11. Samantha says that "all primes are odd". Is she correct?

> **Solution**
>
> No: 2 is prime and 2 is even.

12. The sum of two distinct square numbers is a square number.

> **Solution**
>
> E.g., take $1^2 = 1$ and $2^2 = 4$. Then
>
> $$1^2 + 2^2 = 5$$
>
> and so it is not a square number.

13. All positive cube numbers are either even or one less than a multiple of 3.

> **Solution**
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> No: $1^3 = 1$, it is not even, and it is not one less than a multiple of 3.

14. If the sum of two integers is even, then one of the summands is even.

> **Solution**
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> No: take 1 and 3; the sum of two integers is even but 1 and 3 are odd.

15. All natural numbers are either prime or have more than one factor.

> **Solution**
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> No: take 1: it is neither prime nor has more than factor.

16. If a and b are natural numbers, then so is their difference.

> **Solution**
>
> No: take $a = 1$ and $b = 2$. Then
>
> $$1 - 2 = -1.$$