Further Pure Mathematics 1

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Further Mathematics

Complex numbers

A *complex number* is of the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$; we use $\mathbb{C}$ to denote the set of all complex numbers. The *real* and *imaginary parts* of $z = x + iy$ are defined to be

$$\text{Re}(z) = x \text{ and } \text{Im}(z) = y;$$

note that the imaginary part of a complex number is, in fact, real. For any $z \in \mathbb{C}$, we define the *complex conjugate*, denoted by z^*, by

$$(x + iy)^* = x - iy.$$

If $f(z)$ is a polynomial with real coefficients then any complex solutions appear as a complex conjugate pair. You need to be able to use this information in order to solve quadratic, cubic, and quartic equations given a single complex root, e.g., if you know that $2 - i$ is a root of the polynomial then $2 + i$ is also a root, $(x - (2 + i))$ and $(x - (2 - i))$ are linear factors and hence

$$(x - (2 + i))(x - (2 - i)) = x^2 + 4x + 5$$

is a quadratic factor.

Roots of a quadratic equation

If α and β are the roots of a quadratic then $(x - \alpha)$ and $(x - \beta)$ are factors and so, up to multiplication by a non-zero constant, the quadratic equation is

$$x^2 - (\alpha + \beta) + \alpha\beta = 0.$$

This is not essential for FP1 but it can save time and effort, particularly in an examination.

Series

You need to know and be able to apply the following results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$
$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$
$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Interval bisection

If $f(x) = x^3 - 2$ then there is a solution $1 < x < 2$ since $f(1) = -1$ and $f(2) = 6$ and hence there is a root since we have a change in sign of a continuous function. Since $f(1.5) = 1.375$ we now know that $1 < x < 1.5$.

Linear interpolation

If $(a, f(a))$ and $(b, f(b))$ are points between which we get a change in sign of a continuous function then linear interpolation gives

$$x = \frac{a\,|f(b)| + b\,|f(a)|}{|f(a)| + |f(b)|}$$

as an improved estimate for the root.

Newton-Raphson method

This is an iterative scheme for finding a root to an equation $f(x) = 0$ using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Matrices

An $n \times m$ matrix has n rows and m colums. If $\mathbf{A}$ is an $n \times m$ matrix and $\mathbf{B}$ is an $m \times p$ matrix then the product $\mathbf{AB}$ is an $n \times p$ matrix where the element in the i^{th} row and j^{th} column is formed by 'multiplying' the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$. Matrix multiplication is:

- associative, i.e., $\mathbf{ABC} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$;
- distributive, i.e., $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$;
- *not* commutative, i.e., $\mathbf{AB} \neq \mathbf{BA}$, even if both products can actually be formed.

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the *determinant* of $\mathbf{A}$, denoted by $\det(\mathbf{A})$, is given by $ad - bc$. $|\det(\mathbf{A})|$ is the area scale factor of the transformation associated with the 2×2 matrix. If $\det(\mathbf{A}) = 0$ then $\mathbf{A}$ is *singular* and it has no inverse; otherwise the inverse exists and $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. You must be able to use inverse matrices to solve two linear equations in two variables (although you have methods for Years 8 and 9 for solving such problems in other ways).

Parabola

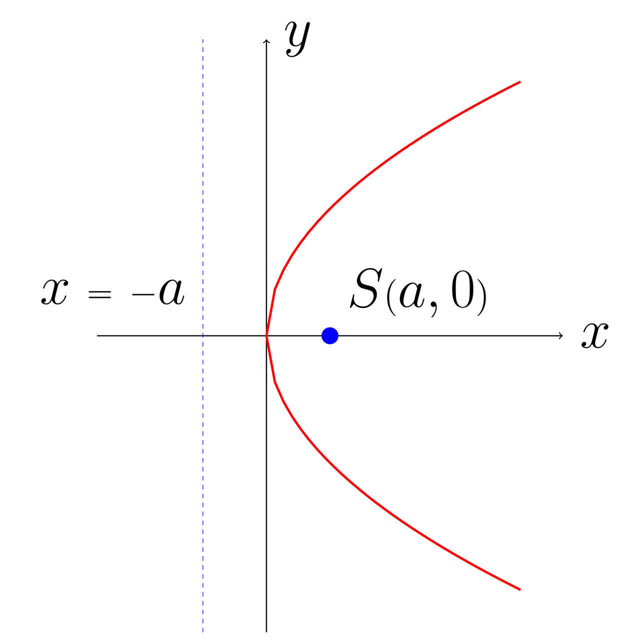
$y^2 = 4ax$, $a > 0$, is a **parabola**:

- eccentricity $e = 1$ and so the parabola is the locus of all points equidistant from the focus and the directrix,
- a focus at $S(a, 0)$ and a directrix $x = -a$,
- parametric equations $x = at^2$ and $y = 2at$,
- to find tangents and normals use

$$y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a \Rightarrow y = \frac{2a}{y}$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2at, \frac{\mathrm{d}y}{\mathrm{d}t} = 2a \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{t}.$$



Rectangular hyperbola

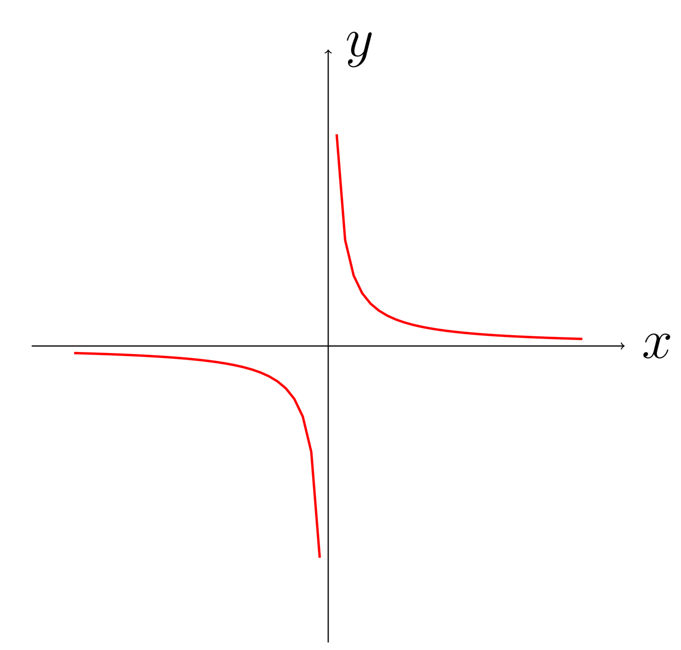
$xy = c^2$, $c > 0$, is a **rectangular hyperbola**:

- eccentricity $e = \sqrt{2}$,
- asymptotes with equations $x = 0$ and $y = 0$,
- parametric equations $x = ct$ and $y = \frac{c}{t}$, $t \neq 0$,
- to find tangents and normals use

$$xy = c^2 \Rightarrow y = \frac{c^2}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2}$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c, \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{c}{t^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}.$$



Linear transformations

A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is *linear* if

- $T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y})$ for all $\underline{x}, \underline{y} \in \mathbb{R}^m$;
- $T(\lambda\underline{x}) = \lambda T(\underline{x})$ for all $\lambda \in \mathbb{R}$ and $\underline{x} \in \mathbb{R}^m$.

A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if it can be represented by $n \times m$ matrix. You should be able to state and recognise the matrices for:

- a reflection in the coordinate axes,
- a reflection in the lines $y = \pm x$,
- an enlargement, centered at the origin, scale factor λ,
- a rotation through an angle θ anticlockwise about the origin.

Induction

You need to be able to use the technique of proof by induction in a variety of cases including:

- a sum of series, e.g,
$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1);$$
- divisibility, e.g., prove that $11^n - 1$ is divisible by 10 for all $n \in \mathbb{N}$;
- recurrence relations, e.g., given that $u_1 = 1$, $u_2 = 2$, and $u_{n+2} = 4u_{n+1} - 3u_n$, prove that $u_n = 3^n - 2$ for all $n \in \mathbb{N}$;
- matrix multiplication, e.g., if $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ prove that $\mathbf{A}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for all $n \in \mathbb{N}$;
- inequalities, e.g., $1 + 2 + \ldots + n > \frac{1}{2}n^2$ or all $n \in \mathbb{N}$.

You need to be able to lay out all of the steps correctly, namely

- establish the truth of a base case — typically this is $n = 1$ but it need not be;
- assume that the result is true for some $n = k$ — this is possible since there is at least one value (that established in the base case) for which the result is true;
- show that the result is now true for $n = k + 1$;
- hence, by mathematical induction, the result is true for all integers greater than or equal to that used in the base case — typically the result is now shown to be true for all $n \in \mathbb{N}$ but it need not be.