Dr Oliver Mathematics
Base n

In this note, we will investigate base n.

1 Why do we need to know this?

Why to we need to? Well, is it not always base 10, is it?

(a) feet and inches: e.g., 4 feet 2 inches equals 50 inches,

(b) time (minutes and seconds): e.g., 2 minutes 5 seconds equals 125 seconds,

(c) time (days and hours): e.g., 5 days 11 hours equals 131 hours,

(d) time (weeks and days): e.g., 3 weeks 4 days equals 25 days,

(e) pounds and ounces: e.g., 6 pounds 10 ounces equals 106 ounces,

(f) stones and pounds: e.g., 11 stones 2 pounds equals 156 pounds,

(g) furlongs: 8 furlongs equals 1 mile,

and so on. It turns out that we need to do it constantly, even though it is (for most people) automatic.

2 Bases

The *base* is a non-negative number that is greater than or equal to 1. Bases, from 2 to 16, are recorded below.

Base	Measure
2	Binary
3	Ternary
4	Quaternary
5	Quinary
6	Senary
7	Septenary
8	Octal
9	Nonary
10	Decimal
11	Undecimal
12	Duodecimal
13	Tridecimal
14	Tetradecimal
15	Pentadecimal
16	Hexadecimal

(a) 0, 1, 2, ... 9 (as normal),

(b) "A", "B", "C", ..., "Z" as 10, 11, 12, ..., 35,

(c) "a", "b", "c", ..., "z" as 36, 37, 39, ..., 61,

and you can go on (and on and on ...).

In base n, we count

(a) $0, 1, 2, \ldots, (n-1)$,

(b) $1\,0, 1\,1, 1\,2, \ldots, 1\,(n-1)$,

(c) $2\,0, 2\,1, 2\,2, \ldots, 2\,(n-1)$,

(d) $3\,0, 3\,1, 3\,2, \ldots, 3\,(n-1)$,

and so on.

For example,

Base	Counting
2	0, 1, 10, 11, 100, 101, 110, 111, ...
3	0, 1, 2, 10, 11, 12, 20, 21, 22, 100, ...
4	0, 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, ...

3 Changing from base n to base 10

Example 1
Express $1\,0\,1\,1\,0_2$ in base 10.

Solution

$$\begin{aligned}1\,0\,1\,1\,0_2 &= \big[(1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (0 \times 2^0)\big]_{10} \\ &= [16 + 0 + 4 + 1 + 0]_{10} \\ &= \underline{\underline{2\,1_{10}}}.\end{aligned}$$

Example 2
Express $2\,0\,1\,2_3$ in base 10.

Solution

$$\begin{aligned}2\,0\,1\,2_3 &= \big[(2 \times 3^3) + (0 \times 3^2) + (1 \times 3^1) + (2 \times 3^0)\big]_{10} \\ &= [54 + 0 + 3 + 2]_{10} \\ &= \underline{\underline{5\,9_{10}}}.\end{aligned}$$

Example 3
Express $2\,\mathrm{C}\,5\,A_{14}$ in base 10.

Solution

$$\begin{aligned}2\,\mathrm{C}\,5\,A_{14} &= \big[(2 \times 14^3) + (12 \times 14^2) + (5 \times 14^1) + (10 \times 14^0)\big]_{10} \\ &= [5\,488 + 2\,352 + 70 + 10]_{10} \\ &= \underline{\underline{7\,9\,2\,0_{10}}}.\end{aligned}$$

4 Changing from base 10 to base n

Example 4
Express $2\,8\,7\,3_{10}$ in base 5.

Solution Well, it is quite hard to see what you need to do to. So, we take a logarithm to base 5:

$$\log_5 2\,873 = 4.947\ldots$$

and so you can be sure that 5^4 is less than $2\,873$. Now, we need to know what the quotient and remainder is on dividing $2\,873$ by 5^4:

$$2\,873 = 4 \times 5^4 + 373.$$

Next, we need to know what the quotient and remainder is on dividing $2\,873$ by 5^4:

$$373 = 2 \times 5^3 + 123,$$

and we keep going, from 5^2, 5^1, and 5^0:

$$\begin{aligned} 2\,873 &= 4 \times 5^4 + 373 \\ 373 &= 2 \times 5^3 + 123 \\ 123 &= 4 \times 5^2 + 23 \\ 23 &= 4 \times 5^1 + 3 \\ 3 &= 3 \times 5^0 + 0 \end{aligned}$$

and hence

$$2\,8\,7\,3_{10} = \underline{\underline{4\,2\,4\,4\,3_5}}.$$

What?! Explain, please!

Certainly!

$$\begin{aligned} 2\,873 &= (4 \times 5^4) + 373 \\ &= (4 \times 5^4) + (2 \times 5^3) + 123 \\ &= (4 \times 5^4) + (2 \times 5^3) + (4 \times 5^2) + 23 \\ &= (4 \times 5^4) + (2 \times 5^3) + (4 \times 5^2) + (4 \times 5^1) + 3 \\ &= (4 \times 5^4) + (2 \times 5^3) + (4 \times 5^2) + (4 \times 5^1) + (3 \times 5^0) \\ &= \underline{\underline{4\,2\,4\,4\,3_5}}. \end{aligned}$$

Example 5
Express $2\,5\,0_{10}$ in base 9.

Solution

$$\log_9 250 = 2.512\ldots$$

and so we need 9^2, 9^1, and 9^0.

$$\begin{aligned} 250 &= 3 \times 9^2 + 7 \\ 7 &= 0 \times 9^1 + 7 \\ 7 &= 7 \times 9^0 + 0 \end{aligned}$$

and

$$250_{10} = \underline{\underline{307_9}}.$$

Is there another way? Yes, there is.

Example 6
Express 134_{10} in base 5.

Solution

$$\begin{aligned} 134 &= 26 \times 5 + 4 \\ 26 &= 5 \times 5 + 1 \\ 5 &= 1 \times 5 + 0 \\ 1 &= 0 \times 5 + 1 \end{aligned}$$

and we read upwards: hence,

$$134_{10} = \underline{\underline{1014_5}}.$$

Explain, please!

$$\begin{aligned} 134 &= 26 \times 5 + 4 \\ &= 5(5 \times 5 + 1) + 4 \\ &= (5^2 \times 5) + (5 \times 1) + 4 \end{aligned}$$

Well, there is a problem:

$$5^2 \times 5 \text{ is the same as } 5^3 \times 1$$

and so we convert the former to the latter:

$$\begin{aligned} &= (5^3 \times 1) + (5^2 \times 0) + (5 \times 1) + 4 \\ &= \underline{\underline{1014_5}}. \end{aligned}$$

Example 7
Express 247_{10} in base 5.

Solution

$$\begin{aligned} 247 &= 49 \times 5 + 2 \\ 49 &= 9 \times 5 + 4 \\ 9 &= 1 \times 5 + 4 \\ 1 &= 0 \times 5 + 1 \end{aligned}$$

and we read upwards: hence,

$$2\,4\,7_{10} = \underline{\underline{1\,4\,4\,2_5}}.$$

Example 8
Express $4\,9\,8_{10}$ in hexadecimal notation.

Solution

$$\begin{aligned} 498 &= 31 \times 16 + 2 \\ 31 &= 1 \times 16 + 15 \\ 1 &= 0 \times 16 + 1 \end{aligned}$$

and we read upwards: hence,

$$4\,9\,8_{10} = \underline{\underline{1\,\mathrm{F}\,2_{16}}},$$

On balance, I prefer the second approach (but it is up to you).

5 Changing from base n to base m

Example 9
Express $8\,1\,\mathrm{A}\,4_{15}$ in base 9.

Solution
First, change $8\,1\,\mathrm{A}\,4_{15}$ into base 10 and then convert this number into base 9.

$$\begin{aligned} 8\,1\,\mathrm{A}\,4_{15} &= \left[(8 \times 15^3) + (1 \times 15^2) + (10 \times 15^1) + (4 \times 15^0)\right]_{10} \\ &= [27\,000 + 225 + 15 + 4]_{10} \\ &= 2\,7\,2\,4\,4_{10} \end{aligned}$$

and

$$\begin{aligned} 27\,244 &= 3\,027 \times 9 + 1 \\ 3\,027 &= 336 \times 9 + 3 \\ 336 &= 37 \times 9 + 3 \\ 37 &= 4 \times 9 + 1 \\ 4 &= 0 \times 9 + 4 \end{aligned}$$

and we read upwards: hence,

$$8\,1\,\mathrm{A}\,4_{15} = \underline{\underline{4\,1\,3\,3\,1_9}}.$$

6 Problems

Here are some problems for you to try.

1. Express $2\,7\,1\,3_8$ in base 10.
2. Express $7\,8\,9\,5_{11}$ in base 9.
3. Express $5\,1\,7\,6\,1\,8_{10}$ in base 7.
4. Express $1\,2\,1\,1_3$ in base 2.