Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2011 June Paper 2 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. (a) Given that (1)

$$y = \sin 3x,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = \sin 3x \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos 3x}}.$$

(b) Hence, find the approximate increase in y as x increases from $\frac{1}{9}\pi$ to $\frac{1}{9}\pi + p$, where p is small (2)

Solution

Well,

$$x = \tfrac{1}{9}\pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}$$

and the approximate increase in y is

$$\begin{aligned} \delta y &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x \\ &= \underline{\underline{\tfrac{3}{2}p}}. \end{aligned}$$

2. (a) An outdoor club has three sections, walking, biking, and rock-climbing.

Using to denote $\mathscr{E}$ the set of all members of the club and W, B, and R to denote the members of the walking, biking, and rock-climbing sections respectively, write each of the following statements using set notation.

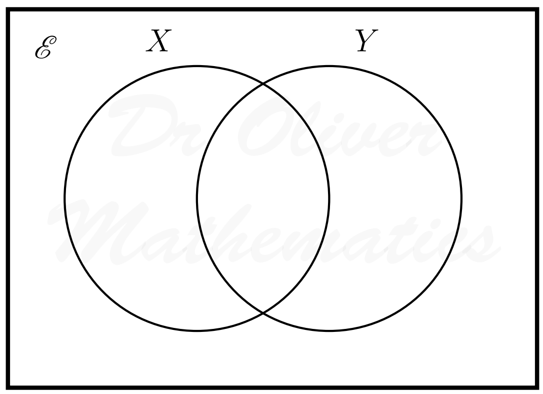
(i) There are 72 members in the club. (1)

Solution
$\underline{\underline{\mathrm{n}(W \cup B \cup R) = 72}}$.

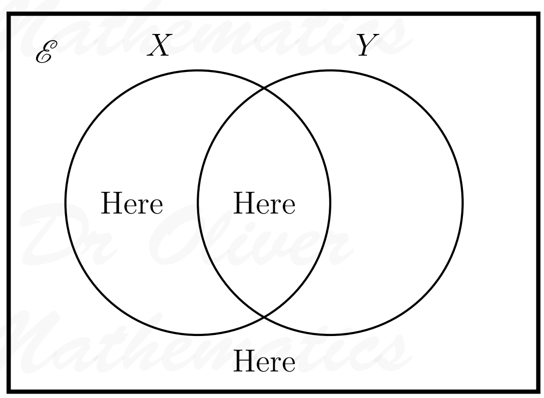
(ii) Every member of the rock-climbing section is also a member of the walking section. (1)

Solution
$\underline{\underline{R \subset W}}$.

(b) (i) On the diagram shade the region which represents the set $X \cup Y'$. (1)



Solution



(ii) Using set notation express the set $X \cup Y'$ in an alternative way. (1)

Solution
E.g., $\underline{\underline{(X' \cap Y)'}}$.

3. (a) Given that (3)

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix},$$

find the inverse of the matrix $\mathbf{A} + \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Solution

Well,

$$\begin{aligned} \mathbf{A} + \mathbf{I} &= \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ -2 & 6 \end{pmatrix}. \end{aligned}$$

Now,

$$\det(\mathbf{A} + \mathbf{I}) = 18 + 2 = 20$$

and the inverse is

$$\underline{\underline{\tfrac{1}{20}\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}}}.$$

(b) Hence, or otherwise, find the matrix $\mathbf{X}$ such that (2)

$$\mathbf{AX} + \mathbf{X} = \mathbf{B},$$

where

$$\mathbf{B} = \begin{pmatrix} 14 \\ 4 \end{pmatrix}.$$

Solution

Now,

$$\begin{aligned} \mathbf{AX} + \mathbf{X} = \mathbf{B} &\Rightarrow (\mathbf{A} + \mathbf{I})\mathbf{X} = \mathbf{B} \\ &\Rightarrow \mathbf{X} = (\mathbf{A} + \mathbf{I})^{-1}\mathbf{B} \\ &\Rightarrow \mathbf{X} = \tfrac{1}{20}\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \end{pmatrix} \\ &\Rightarrow \mathbf{X} = \tfrac{1}{20}\begin{pmatrix} 80 \\ 40 \end{pmatrix} \\ &\Rightarrow \underline{\underline{\mathbf{X} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}}}. \end{aligned}$$

4. (a) Prove that (3)

$$\frac{1}{1-\sin x} - \frac{1}{1+\sin x} \equiv 2\tan x \sec x.$$

Solution

Well,

$\times$	1	$-\sin x$
1	1	$-\sin x$
$+\sin x$	$+\sin x$	$-\sin^2 x$

and

$$\begin{aligned} & \frac{1}{1-\sin x} - \frac{1}{1+\sin x} \\ \equiv\ & \frac{1+\sin x}{(1-\sin x)(1+\sin x)} - \frac{1-\sin x}{(1-\sin x)(1+\sin x)} \\ \equiv\ & \frac{2\sin x}{1-\sin^2 x} \\ \equiv\ & \frac{2\sin x}{\cos^2 x} \\ \equiv\ & 2 \times \frac{\sin x}{\cos x} \times \frac{1}{\cos x} \\ \equiv\ & \underline{\underline{2\tan x \sec x}}, \end{aligned}$$

as required.

An acute angle x is such that

$$\sin x = p.$$

(b) Given that (3)

$$\sin 2x = 2\sin x \cos x,$$

find an expression, in terms of p, for $\operatorname{cosec} 2x$.

Solution

As it is an acute angle,

$$\begin{aligned}\cos x &= \sqrt{\cos^2 x}\\ &= \sqrt{1-\sin^2 x}\\ &= \sqrt{1-p^2}\end{aligned}$$

and

$$\begin{aligned}\operatorname{cosec} 2x &= \frac{1}{\sin 2x}\\ &= \frac{1}{2\sin x\cos x}\\ &= \underline{\underline{\frac{1}{2p\sqrt{1-p^2}}}}.\end{aligned}$$

5. (a) Given that (3)

$$y = x\sqrt{2x+15},$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k(x+5)}{\sqrt{2x+15}},$$

where k is a constant to be found.

Solution

Product rule:

$$\begin{aligned}u = x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1\\ v = (2x+15)^{\frac{1}{2}} &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = (2x+15)^{-\frac{1}{2}}\end{aligned}$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x)[(2x+15)^{-\frac{1}{2}}] + (1)[(2x+15)^{\frac{1}{2}}] \\ &= \frac{x}{(2x+15)^{\frac{1}{2}}} + (2x+15)^{\frac{1}{2}} \\ &= \frac{x+(2x+15)}{(2x+15)^{\frac{1}{2}}} \\ &= \frac{3x+15}{\sqrt{2x+15}} \\ &= \underline{\underline{\frac{3(x+5)}{\sqrt{2x+15}}}};\end{aligned}$$

hence, $\underline{\underline{k=3}}$.

(b) Hence find (3)

$$\int \frac{x+5}{\sqrt{2x+15}}\,\mathrm{d}x,$$

and evaluate

$$\int_{-3}^{5} \frac{x+5}{\sqrt{2x+15}}\,\mathrm{d}x.$$

Solution

Now,

$$\begin{aligned}\int \frac{x+5}{\sqrt{2x+15}}\,\mathrm{d}x &= \tfrac{1}{3}\int \frac{3(x+5)}{\sqrt{2x+15}}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3}x\sqrt{2x+15}+c}}\end{aligned}$$

and

$$\begin{aligned}\int_{-3}^{5} \frac{x+5}{\sqrt{2x+15}}\,\mathrm{d}x &= \tfrac{1}{3}\int_{-3}^{5} \frac{3(x+5)}{\sqrt{2x+15}}\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x\sqrt{2x+15}\right]_{x=-3}^{5} \\ &= \tfrac{25}{3} - (-3) \\ &= \underline{\underline{11\tfrac{1}{3}}}.\end{aligned}$$

6. The line (7)

$$y = 3x - 9$$

intersects the curve

$$49x^2 - y^2 + 42x + 8y = 247$$

at the points A and B.

Find the length of the line AB.

Solution

Well,

$\times$	$3x$	-9
$3x$	$9x^2$	$-27x$
-9	$-27x$	$+81$

and

$$\begin{aligned} 49x^2 - y^2 + 42x + 8y = 247 &\Rightarrow 49x^2 - (3x-9)^2 + 42x + 8(3x-9) = 247 \\ &\Rightarrow 49x^2 - (9x^2 - 54x + 81) + 42x + 8(3x-9) = 247 \\ &\Rightarrow 40x^2 + 120x - 400 = 0 \\ &\Rightarrow 40(x^2 + 3x - 10) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & -10 \end{array}\right\} +5, -2$$

$$\begin{aligned} &\Rightarrow 40(x+5)(x-2) = 0 \\ &\Rightarrow x = -5 \text{ or } x = 2 \\ &\Rightarrow y = -24 \text{ or } y = -3; \end{aligned}$$

so, $A(-5, -24)$ and $B(2, -3)$.

Finally,

$$\begin{aligned} AB &= \sqrt{[2-(-5)]^2 + [-3-(-24)]^2} \\ &= \sqrt{7^2 + 21^2} \\ &= \sqrt{490} \\ &= \underline{\underline{7\sqrt{10}}}. \end{aligned}$$

7. A particle moves in a straight line so that, t s after passing through a fixed point O, its velocity, v ms^{-1}, is given by

$$v = \frac{60}{(3t+4)^2}.$$

(a) Find the velocity of the particle as it passes through O. (1)

Solution

$$t = 0 \Rightarrow \underline{\underline{v = 3.75 \text{ ms}^{-1}}}.$$

(b) Find the acceleration of the particle when $t = 2$. (3)

Solution

Now,

$$\begin{aligned} v = \frac{60}{(3t+4)^2} &\Rightarrow v = 60(3t+4)^{-2} \\ &\Rightarrow a = -360(3t+4)^{-3} \end{aligned}$$

and

$$t = 2 \Rightarrow \underline{\underline{a = -0.36 \text{ ms}^{-2}}}.$$

(c) Find an expression for the displacement of the particle from O, t s after it has passed through O. (4)

Solution

Well,

$$v = 60(3t+4)^{-2} \Rightarrow s = -20(3t+4)^{-1} + c.$$

Now,

$$\begin{aligned} t = 0, s = 0 &\Rightarrow -5 + c = 0 \\ &\Rightarrow c = 5 \end{aligned}$$

and, hence,

$$\underline{\underline{s = -20(3t+4)^{-1} + 5}}.$$

8. (a) (i) Solve (2)

$$3^x = 200,$$

giving your answer to 2 decimal places.

Solution

$$\begin{aligned} 3^x = 200 &\Rightarrow x = \log_3 200 \\ &\Rightarrow x = 4.822\,736\,302 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 4.82 \text{ (2 dp)}}}. \end{aligned}$$

(ii) Solve (4)

$$\log_5(5y+40) - \log_5(y+2) = 2.$$

Solution

Now,

$$\begin{aligned} \log_5(5y+40) - \log_5(y+2) = 2 &\Rightarrow \log_5\left(\frac{5y+40}{y+2}\right) = 2 \\ &\Rightarrow \frac{5y+40}{y+2} = 5^2 \\ &\Rightarrow \frac{5y+40}{y+2} = 25 \\ &\Rightarrow 5y+40 = 25(y+2) \\ &\Rightarrow 5y+40 = 25y+50 \\ &\Rightarrow 20y = -10 \\ &\Rightarrow \underline{\underline{y = -\tfrac{1}{2}}}. \end{aligned}$$

(b) Given that (3)

$$\frac{(24z^3)^2}{27 \times 12z} = 2^a 3^b z^c,$$

evaluate a, b, and c.

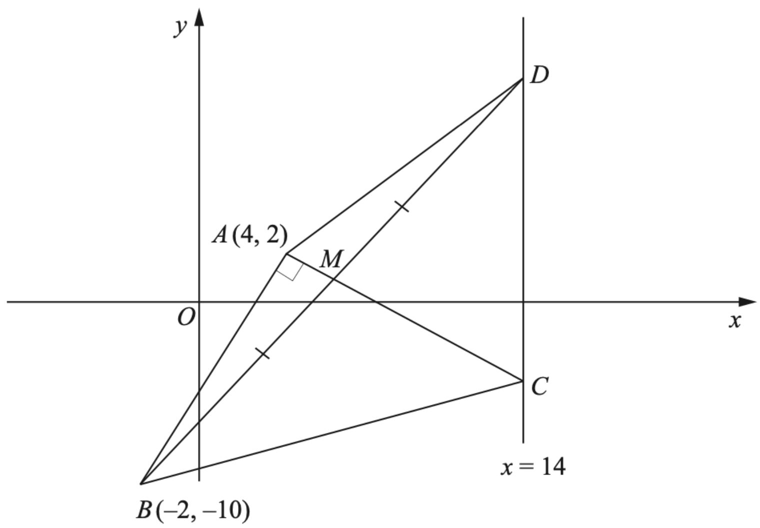
Solution

Well,

$$\begin{aligned} \frac{(24z^3)^2}{27 \times 12z} &= \frac{(2^3 \times 3 \times z^3)^2}{3^3 \times 2^2 \times 3 \times z} \\ &= \frac{2^6 \times 3^2 \times z^6}{2^2 \times 3^4 \times z} \\ &= \underline{\underline{2^4 \times 3^{-2} \times z^5}}; \end{aligned}$$

hence, $\underline{\underline{a=4}}$, $\underline{\underline{b=-2}}$, and $\underline{\underline{c=5}}$.

9. **Solutions to this question by accurate drawing will not be accepted.** (9)
The diagram shows the quadrilateral $ABCD$ in which A is the point $(4,2)$ and B is the point $(-2,-10)$.



- The points C and D lie on the line $x = 14$.
- The diagonal AC is perpendicular to AB and passes through the mid-point, M, of the diagonal BD.

Find the area of the quadrilateral $ABCD$.

Solution

Well,

$$\begin{aligned} m_{AB} &= \frac{2-(-10)}{4-(-2)} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

and so

$$m_{AC} = -\tfrac{1}{2}.$$

Now, the equation of AC is

$$y - 2 = -\tfrac{1}{2}(x - 4)$$

and

$$\begin{aligned} x = 14 \Rightarrow y - 2 &= -\tfrac{1}{2}(14 - 4) \\ \Rightarrow y - 2 &= -5 \\ \Rightarrow y &= -3; \end{aligned}$$

so, $C(14, -3)$.

$D(14, a)$ and $M(6, b)$ for some constants a and b. Now, since M lies on AC, we have

$$\begin{aligned} x = 8 \Rightarrow y - 2 &= -\tfrac{1}{2}(6 - 4) \\ \Rightarrow y - 2 &= -1 \\ \Rightarrow y &= 1; \end{aligned}$$

hence, $M(6, 1)$. Next,

$$\begin{aligned} \overrightarrow{OD} &= \overrightarrow{OB} + \overrightarrow{BD} \\ &= \overrightarrow{OA} + 2\overrightarrow{BM} \\ &= \begin{pmatrix} -2 \\ -10 \end{pmatrix} + 2\begin{pmatrix} 8 \\ 11 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ 22 \end{pmatrix}; \end{aligned}$$

so, $D(14, 12)$.

<u>Area of ACD</u> :

$$\begin{aligned} \text{Area of } ACD &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2} \times 10 \times 15 \\ &= 75. \end{aligned}$$

<u>Area of ABC</u> :
Well,

$$\begin{aligned} AB &= \sqrt{[4 - (-2)]^2 + [2 - (-10)]^2} \\ &= \sqrt{6^2 + 12^2} \\ &= 6\sqrt{5}, \end{aligned}$$

$$
\begin{aligned}
AC &= \sqrt{(4-14)^2 + [2-(-3)]^2} \\
&= \sqrt{10^2 + 5^2} \\
&= 5\sqrt{5},
\end{aligned}
$$

and

$$
\begin{aligned}
\text{area of } ABC &= \tfrac{1}{2} \times 6\sqrt{5} \times 5\sqrt{5} \\
&= 75.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\text{area of } ABCD &= \text{area of } ACD + \text{area of } ABC \\
&= 75 + 75 \\
&= \underline{\underline{150}}.
\end{aligned}
$$

10. (a) (i) Express (3)

$$18 + 16x - 2x^2$$

in the form

$$a + b(x + c)^2,$$

where a, b, and c are integers.

Solution
Well,

$$
\begin{aligned}
18 + 16x - 2x^2 &\equiv 18 - 2(x^2 - 8x) \\
&\equiv 18 - 2[(x^2 - 8x + 16) - 16] \\
&\equiv 18 - 2(x-4)^2 + 32 \\
&\equiv \underline{\underline{50 - 2(x-4)^2}};
\end{aligned}
$$

hence, $\underline{\underline{a = 50}}$, $\underline{\underline{b = -2}}$, and $\underline{\underline{c = -4}}$.

A function f is defined by

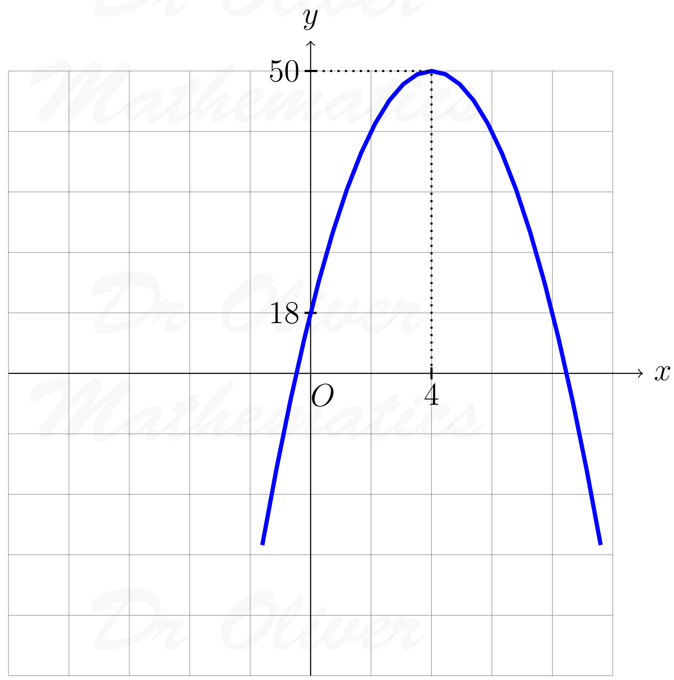
$$\mathrm{f} : x \mapsto 18 + 16x - 2x^2 \text{ for } x \in \mathbb{R}.$$

(ii) Write down the coordinates of the stationary point on the graph of $y = \mathrm{f}(x)$. (1)

Solution
$\underline{\underline{(4, 50)}}$.

(iii) Sketch the graph of $y = \mathrm{f}(x)$. (2)

Solution



(b) A function g is defined by

$$\mathrm{g} : x \mapsto (x+3)^2 - 7 \text{ for } x > -3.$$

(i) Find an expression for $\mathrm{g}^{-1}(x)$. (2)

Solution
Well,

$$\begin{aligned} y = (x+3)^2 - 7 \Rightarrow y + 7 &= (x+3)^2 \\ \Rightarrow \sqrt{y+7} &= x + 3 \\ \Rightarrow \sqrt{y+7} - 3 &= x; \end{aligned}$$

hence,

$$\underline{\underline{y = \sqrt{x+7} - 3}}.$$

(ii) Solve the equation (3)

$$g^{-1}(x) = g(0).$$

Solution

Now,

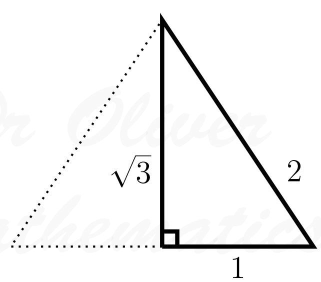
$$\begin{aligned} g^{-1}(x) = g(0) &\Rightarrow \sqrt{x+7} - 3 = 2 \\ &\Rightarrow \sqrt{x+7} = 5 \\ &\Rightarrow x + 7 = 25 \\ &\Rightarrow \underline{\underline{x = 18}}. \end{aligned}$$

EITHER

11. (a) Using an equilateral triangle of side 2 units, find the exact value of $\sin 60°$ and of $\cos 60°$. (3)

Solution

We cut the triangle in half:



Now,

$$\begin{aligned} \text{opp}^2 + \text{adj}^2 = \text{hyp}^2 &\Rightarrow 1^2 + \text{adj}^2 = 2^2 \\ &\Rightarrow \text{adj}^2 = 3 \\ &\Rightarrow \text{adj} = \sqrt{3}. \end{aligned}$$

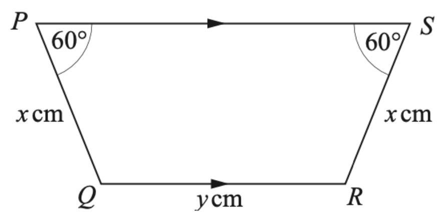
Next,

$$\sin = \frac{\text{opp}}{\text{hyp}} \Rightarrow \underline{\underline{\sin 60° = \tfrac{\sqrt{3}}{2}}}$$

and

$$\cos = \frac{\text{adj}}{\text{hyp}} \Rightarrow \underline{\underline{\cos 60° = \tfrac{1}{2}}}.$$

(b) Consider the diagram below.



- $PQRS$ is a trapezium in which $PQ = RS = x$ cm and $QR = y$ cm.
- Angle QPS = angle $RSP = 60°$ and QR is parallel to PS.

(i) Given that the perimeter of the trapezium is 60 cm, express y in terms of x. (2)

Solution
Well,

$$\begin{aligned} & x + (x\cos 60^\circ) + y + (x\cos 60^\circ) + x + y = 60 \\ \Rightarrow\ & 2y = 60 - 2x - 2x\cos 60^\circ \\ \Rightarrow\ & 2y = 60 - 2x - x \\ \Rightarrow\ & 2y = 60 - 3x \\ \Rightarrow\ & \underline{\underline{y = \tfrac{1}{2}(60 - 3x)}}. \end{aligned}$$

(ii) Given that the area of the trapezium is A cm^2, show that (3)

$$A = \frac{\sqrt{3}(30x - x^2)}{2}.$$

Solution
Now,

$$\begin{aligned} A &= (x\cos 60^\circ \times x\sin 60^\circ) + (y \times x\sin 60^\circ) \\ &= \tfrac{\sqrt{3}}{4}x^2 + \tfrac{\sqrt{3}}{2}x[\tfrac{1}{2}(60 - 3x)] \\ &= \tfrac{\sqrt{3}}{4}x^2 + 15\sqrt{3}x - \tfrac{3\sqrt{3}}{4}x^2 \\ &= 15\sqrt{3}x - \tfrac{\sqrt{3}}{2}x^2 \\ &= \underline{\underline{\frac{\sqrt{3}(30x - x^2)}{2}}}, \end{aligned}$$

as required.

(iii) Given that x can vary, find the value of x for which A has a stationary value and determine the nature of this stationary value. (4)

Solution
Well,

$$A = \frac{\sqrt{3}(30x - x^2)}{2} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\sqrt{3}(30 - 2x)}{2}$$

and

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} = 0 &\Rightarrow \frac{\sqrt{3}(30 - 2x)}{2} = 0 \\ &\Rightarrow 2x = 30 \\ &\Rightarrow x = 15; \end{aligned}$$

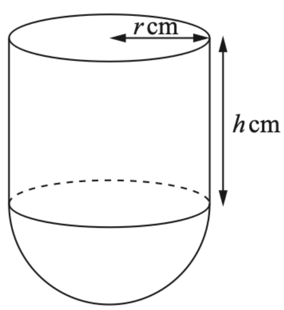
hence, the stationary value is $\underline{\underline{x = 15}}$. Now,

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{\sqrt{3}(-2)}{2} < 0$$

and this stationary value is $\underline{\underline{\text{maximum}}}$.

OR

12. The diagram shows a solid object in the form of a cylinder of height h cm and radius r cm on top of a hemisphere of radius r cm.



For a sphere of radius r:

Volume = $\frac{4}{3}\pi r^3$

Surface area = $4\pi r^2$

Given that the volume of the object is $2\,880\pi$ cm^3,

(a) express h in terms of r, (2)

Solution

The the volume of the object is the sum of the hemisphere and the cylinder:

$$\begin{aligned}\tfrac{2}{3}\pi r^3 + \pi r^2 h = 2\,880\pi &\Rightarrow \tfrac{2}{3}r^3 + r^2 h = 2\,880\\ &\Rightarrow r^2 h = 2\,880 - \tfrac{2}{3}r^3\\ &\Rightarrow \underline{\underline{h = \frac{2\,880}{r^2} - \tfrac{2}{3}r}}.\end{aligned}$$

(b) show that the external surface area, A cm^2, of the object is given by (3)

$$A = \tfrac{5}{3}\pi r^2 + \frac{5\,760\pi}{r}.$$

Solution

$$\begin{aligned}A &= \text{hemisphere} + \text{cylinder} + \text{base}\\ &= 2\pi r^2 + 2\pi r h + \pi r^2\\ &= 3\pi r^2 + 2\pi r\left(\frac{2\,880}{r^2} - \tfrac{2}{3}r\right)\\ &= 3\pi r^2 + \frac{5\,760\pi}{r} - \tfrac{4}{3}\pi r^2\\ &= \underline{\underline{\tfrac{5}{3}\pi r^2 + \frac{5\,760\pi}{r}}},\end{aligned}$$

as required.

Given that r can vary,

(c) find the value of r for which A has a stationary value, (4)

Solution

$$\begin{aligned}A = \tfrac{5}{3}\pi r^2 + \frac{5\,760\pi}{r} &\Rightarrow A = \tfrac{5}{3}\pi r^2 + 5\,760\pi r^{-1}\\ &\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = \tfrac{10}{3}\pi r - 5\,760\pi r^{-2}\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}r} = 0 &\Rightarrow \tfrac{10}{3}\pi r - 5\,760\pi r^{-2} = 0\\ &\Rightarrow \tfrac{10}{3}r = 5\,760 r^{-2}\\ &\Rightarrow r^3 = 1\,728\\ &\Rightarrow \underline{\underline{r = 12}}.\end{aligned}$$

(d) find this stationary value of A, leaving your answer in terms of π, (2)

Solution
Well,

$$\begin{aligned}r = 12 &\Rightarrow A = \tfrac{5}{3}\pi(12^2) + \frac{5\,760\pi}{12}\\ &\Rightarrow \underline{\underline{A = 720\pi \text{ cm}^2}}.\end{aligned}$$

(e) determine the nature of this stationary value. (1)

Solution
Now,

$$\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = \tfrac{10}{3}\pi + 11\,520\pi r^{-3}$$

and

$$r = 12 \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}r^2} = 10\pi > 0$$

and so the nature of this stationary value is <u>minimum</u>.