# Dr Oliver Mathematics
# Probability, Tree diagrams, Venn diagrams, and Two-way tables

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## 1 School A

**Example 1.1.** School A plays 60% of its basketball matches at home. The school wins 70% of its home matches. The school wins only 35% of its away matches.

(a) Represent this information using a probability tree diagram.

(b) Represent the probabilities found in part (a) using a Venn diagram.

(c) Given that School A played 150 games across the last four seasons represent their outcomes in a two-way table.

(d) Two (different) matches are chosen at random. What is the probability that they are both defeats?

**Solution** (a) Let $H$ represent Home, $A$ represent Away, $W$ represent Win, and $L$ represent Lose.

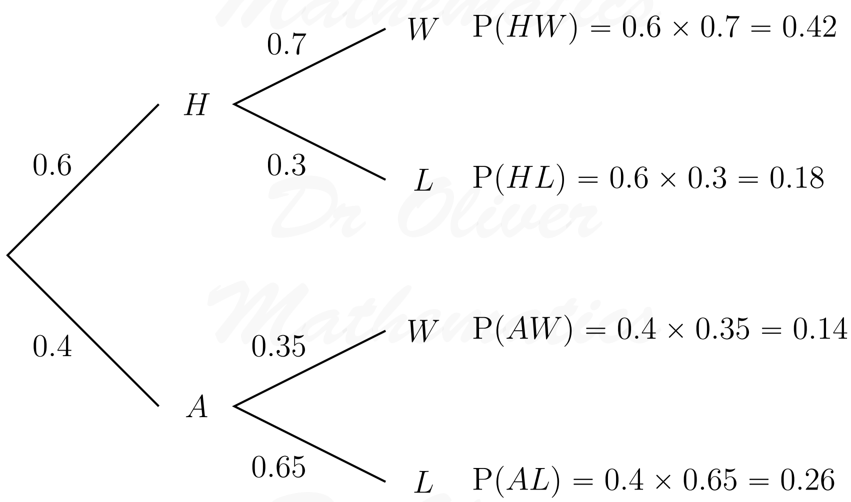


Figure 1: Probability tree diagram for school A

(b) For this part of the question it is simply a case of deciding which two sets are appropriate and then placing the four probabilities into the correct places.

The tree diagram has two parts. The first part of the tree diagram shows the location of the game and we could choose either $H$ or $A$ as our first set: we will choose $H$ and then any value that lies outside of $H$ tells us something about the probability of the outcome in an away game. The second part of the tree diagram shows the outcome of the game and we could choose either $W$ or $L$ as our second set: we will choose $W$ and then any value that lies outside of $H$ tells us something about the probability of the location of a defeat.

Next to the probability tree diagram (see Figure 1) we calculated four probabilities: we will work through these from top to bottom and assign them to their correct place in the Venn diagram.

(i) 0.42 is the probability of a home win and so this lies in both $H$ and $W$ — and hence this goes into the overlap;

(ii) 0.18 is the probability of a home defeat and so this lies inside $H$ and outside $W$;

(iii) 0.14 is the probability of an away win and so this lies outside $H$ and inside $W$;

(iv) 0.26 is the probability of an away defeat and so this lies outside both $H$ and $W$.

The result is shown in Figure 2.

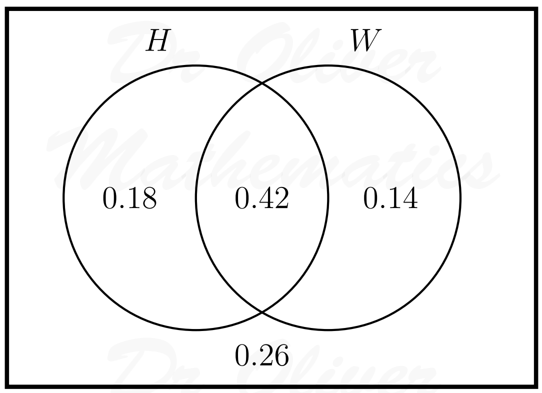


Figure 2: Venn diagram for school A

(c) We need to convert the probabilities into expected outcomes.

$$\begin{aligned} \text{Home wins:} \quad & 150 \times 0.42 = 63, \\ \text{Home losses:} \quad & 150 \times 0.18 = 27, \\ \text{Away wins:} \quad & 150 \times 0.14 = 21, \\ \text{Away losses:} \quad & 150 \times 0.26 = 39. \end{aligned}$$

We then simply transfer these values into Table 1.

| | Win | Loss | Total |
|---|---|---|---|
| Home | 63 | 27 | 90 |
| Away | 21 | 39 | 60 |
| Total | 84 | 66 | 150 |

Table 1: Two-way table for school A

(d) For the first game there are 66 defeats to choose from 150. For the second game there are now 65 defeats to choose from 149. So the probability of two defeats is

$$\frac{66}{150} \times \frac{65}{149} = \underline{\underline{\frac{143}{745} = 0.192 \text{ (3 dp)}}}.$$

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## 2 School B

**Example 2.1.** The Venn diagram shown in Figure 3 shows the probability of School B playing home games ($H$) and winning games ($W$) in their last basketball season.

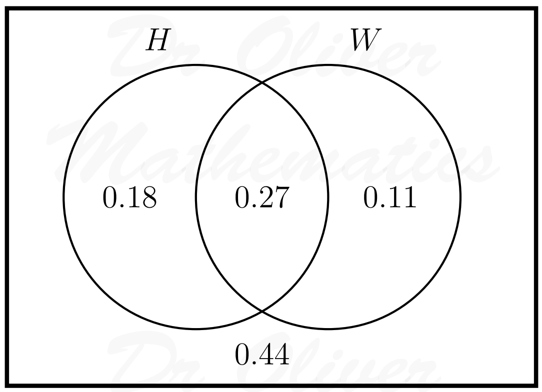


Figure 3: Venn diagram for school B

(a) Construct a probability tree diagram for these probabilities.

(b) One of school B's defeats is chosen at random. What is the probability that they lost at home? Given your answer as a fraction in lowest terms.

**Solution** (a) We will work across the sets from left to right and then place the number outside the two sets. Let $A$ represent Away and $L$ represent Lose.

(i) 0.18 is in $H$ but is outside of $W$ and so this is $\mathrm{P}(HL)$;

(ii) 0.27 is in both $H$ and $W$ and so this is $\mathrm{P}(HW)$;

(iii) 0.11 is outside of $H$ but is inside $W$ and so this is $\mathrm{P}(AW)$;

(iv) 0.44 is outside of both $H$ and $W$ and so this is $\mathrm{P}(AL)$.

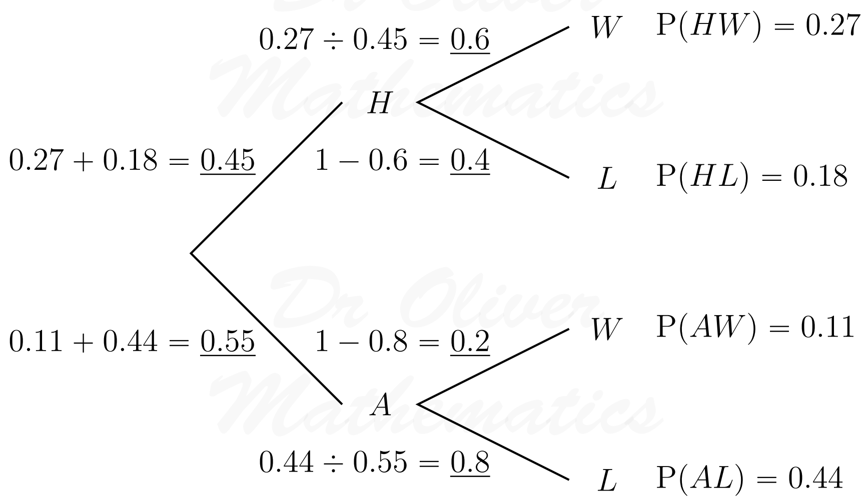


Figure 4: Probability tree diagram for school B

(b) The probability of a defeat is $0.18 + 0.44 = 0.62$. Hence the required probability is

$$\frac{0.18}{0.62} = \underline{\underline{\frac{9}{31}}}.$$

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# 3 School C

**Example 3.1.** The Venn diagram in Figure 5 shows the comparable information for school C.

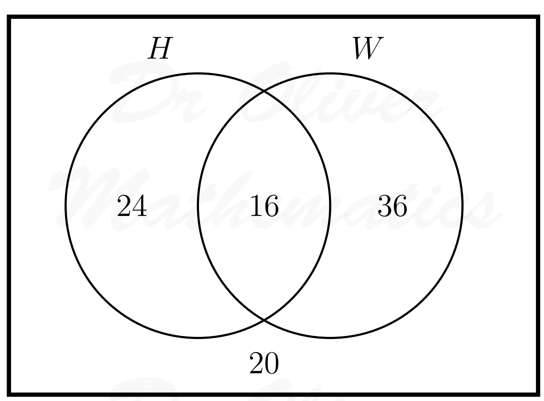


Figure 5: Venn diagram for school C

(a) An away match is chosen at random. What is the probability that it was a win? Give your answer as a fraction in its simplest form.

(b) Construct a probability tree diagram for school C's outcomes.

(c) Construct a two-way table for school C's outcomes.

**Solution** (a) There were $36 + 20 = 56$ away matches and, of these, 36 were wins. Hence the probability of an away match being a win is

$$\frac{36}{56} = \underline{\underline{\frac{9}{14}}}.$$

(b) In total, there were $24 + 16 + 36 + 20 = 96$ matches played and we can convert the Venn diagram into one that shows probabilities, as shown in Figure 6.

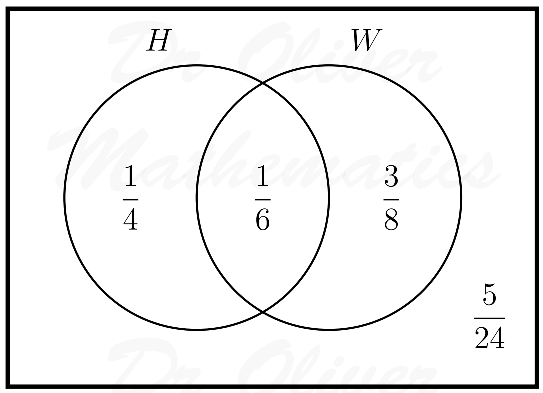


Figure 6: Venn diagram for school C

(i) $\frac{1}{4}$ is in $H$ but is outside of $W$ and so this is $\text{P}(HL)$;

(ii) $\frac{1}{6}$ is in both $H$ and $W$ and so this is $\text{P}(HW)$;

(iii) $\frac{3}{8}$ is outside of $H$ but is inside $W$ and so this is $\text{P}(AW)$;

(iv) $\frac{5}{24}$ is outside of both $H$ and $W$ and so this is $\text{P}(AL)$.

These are then transferred to Figure 7.

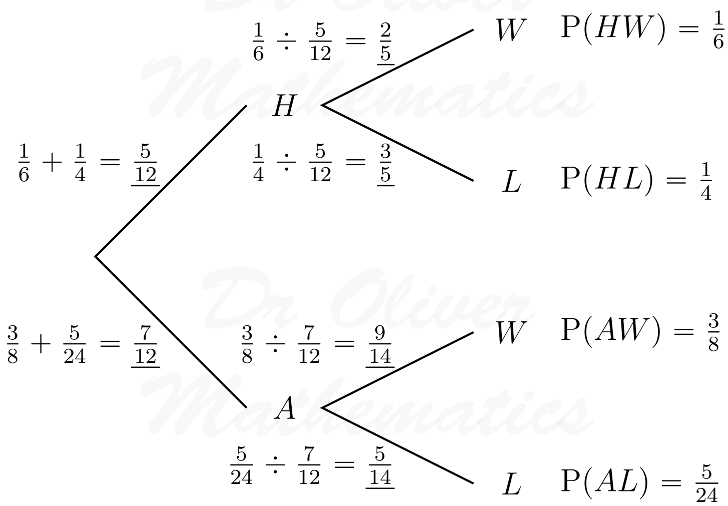


Figure 7: Probability tree diagram for school C

(c) We simply transfer the information from the Venn diagram, as shown in Table 2.

| | Win | Loss | Total |
|---|---|---|---|
| Home | 16 | 24 | 40 |
| Away | 36 | 20 | 56 |
| Total | 52 | 44 | 96 |

Table 2: Two-way table for school C

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# 4 School D

**Example 4.1.** Table 3 shows a partially completed two-way table for school D's outcomes.

| | Win | Loss | Total |
|---|---|---|---|
| Home | 18 | | 60 |
| Away | | 36 | |
| Total | | | 144 |

Table 3: Partially completed two-way table for school D

(a) Complete the two-way table.

(b) A defeat is chosen at random. What is the probability that it occurred at home? Given your answer as a fraction in its lowest terms.

(c) Construct a Venn diagram to show school D's results.

(d) Construct a probability tree diagram to show school D's results.

**Solution** (a) For example, we can begin by completing the top row. The completed two-way is shown in Table 4.

| | Win | Loss | Total |
|---|---|---|---|
| Home | 18 | **42** | 60 |
| Away | **48** | 36 | **84** |
| Total | **66** | **78** | 144 |

Table 4: Completed two-way table for school D

(b) There were 78 defeats and, of these, 42 occurred at home. Hence the probability of a defeat happening at home is

$$\frac{42}{78} = \underline{\underline{\frac{7}{13}}}.$$

(c) We can represent the information using totals (as in Figure 8) or, if we divide by the total number of games, using probabilities (as in Figure 9).

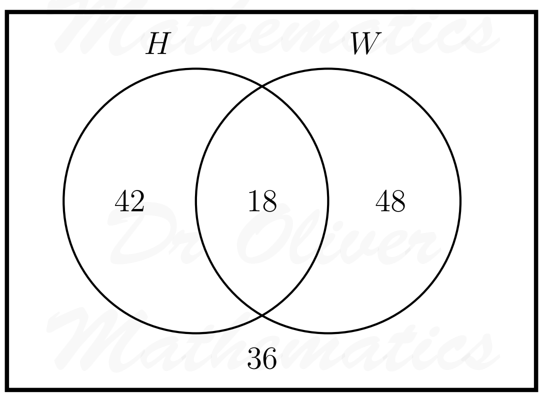


Figure 8: Venn diagram for school D (totals)

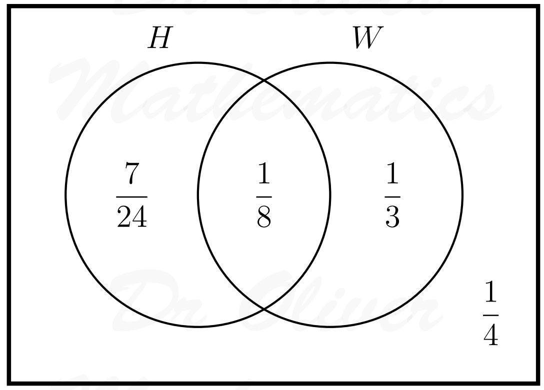


Figure 9: Venn diagram for school D (probabilities)

(d) (i) $\frac{7}{24}$ is in $H$ but is outside of $W$ and so this is $\mathrm{P}(HL)$;

(ii) $\frac{1}{8}$ is in both $H$ and $W$ and so this is $\mathrm{P}(HW)$;

(iii) $\frac{1}{3}$ is outside of $H$ but is inside $W$ and so this is $\mathrm{P}(AW)$;

(iv) $\frac{1}{4}$ is outside of both $H$ and $W$ and so this is $\mathrm{P}(AL)$.

We can transfer the probabilities from Figure 9 to a tree diagram shown in Figure 10.

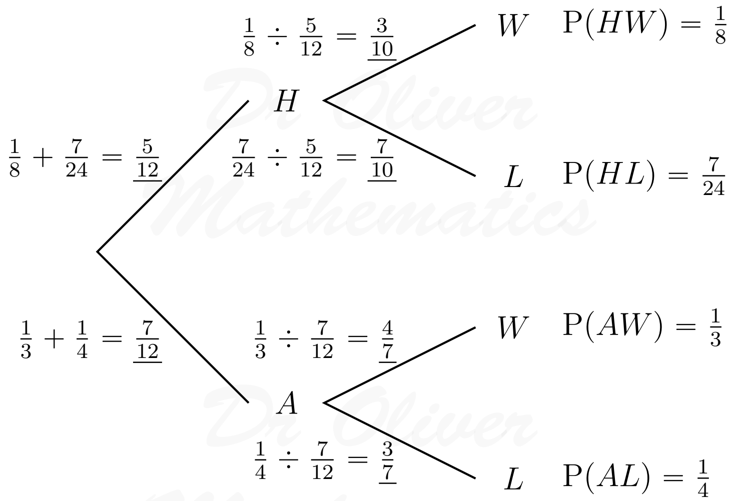


Figure 10: Probability tree diagram for school D

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## 5 School E

**Example 5.1.** School E played 40 matches last season. They played 15 matches at home, they won 19 matches, and they suffered 13 defeats on their travels.

(a) Represent this information on a Venn diagram.

(b) Three games are chosen at random. Find the probability that they consist of two wins and a defeat.

**Solution** (a) We need to do this algebraically. Let $x$ be the number of home wins. Then $15 - x$ is the number of home defeats and $19 - x$ is the number of away wins. We can represent these as shown in Figure 11.

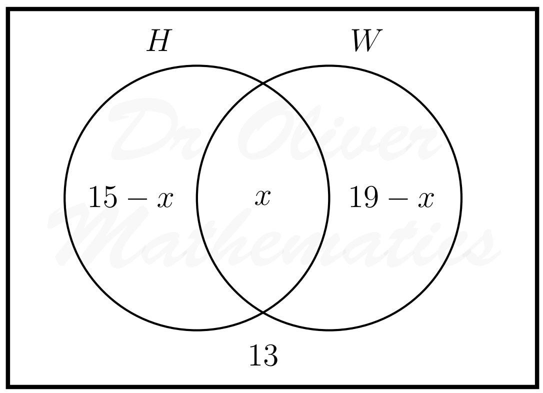


Figure 11: Partial Venn diagram for school E

So

$$(15 - x) + x + (19 - x) + 13 = 40 \Rightarrow 47 - x = 40 \Rightarrow x = 7$$

and hence the final Venn diagram is given in Figure 12.

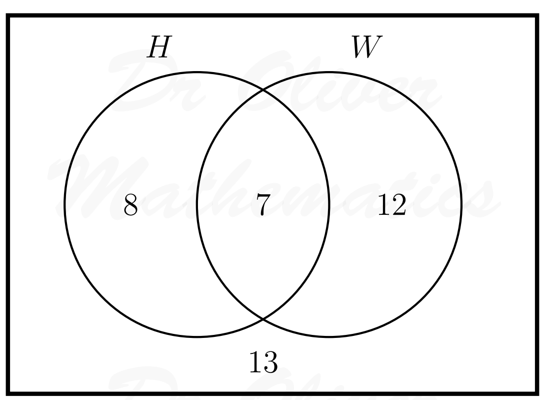


Figure 12: Final Venn diagram for school E

(b) If we were to draw a Venn diagram (we won't) then we would see that there are three routes that give the desired outcome: $WWL$, $WLW$, and $LWW$. Each of these has the same probability so we can find just one of these values (we will take the first one) and then multiply by 3:

$$\text{P}(2\text{ wins, }1\text{ defeat}) = 3 \times \frac{19}{40} \times \frac{18}{39} \times \frac{21}{38} = \underline{\underline{\frac{189}{520}}}.$$

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# 6 Tree diagrams

**Example 6.1.** A bag contains six yellow discs and four green discs. Three discs are chosen at random without replacement. What is the probability that two of the discs are yellow and the other is green?

**Solution** If we were to draw a tree diagram (we won't) then it would be easy to see the number of routes through the diagram and the combination of yellows and greens that would give us the answer.

$$\begin{aligned}\text{P(two yellow and one green)} &= \text{P(YYG)} + \text{P(YGY)} + \text{P(GYY)}\\ &= \tfrac{6}{10} \times \tfrac{5}{9} \times \tfrac{4}{8} + \tfrac{6}{10} \times \tfrac{4}{9} \times \tfrac{5}{8} + \tfrac{4}{10} \times \tfrac{6}{9} \times \tfrac{5}{8}\\ &= \tfrac{120}{720} + \tfrac{120}{720} + \tfrac{120}{720}\\ &= \underline{\underline{\tfrac{1}{2}}}.\end{aligned}$$

■

Hopefully you spotted the each of the three possibilities gave the same outcome; if you did not, look back at the solution and pay attention this time. Each of the calculations involves a 10 (the number of discs to choose from the first time), a 9 (the number of discs to choose from for the second selection), and an 8 (the number of discs to choose from for the third selection) on the denominator. The numerators involve a 6 (the original number of yellow discs), a 5 (the number of yellow discs left when one has been chosen), and a 4 (the number of green discs) in some order.

This is not a coincidence.

When working through a tree diagram problem such as this a change in the ordering of the discs does not change the probability. So we can simplify the problem by trying to find the probability of any relevant route, the number of such routes, and multiplying these together.

**Flipping a coin once**

| | |
|---|---|
| 0 heads: | 1 way (T) |
| 1 head: | 1 way (H) |

In this case there are a total of two routes.

**Flipping a coin twice**

| | |
|---|---|
| 0 heads: | 1 way (TT) |
| 1 head: | 2 ways (HT, TH) |
| 2 heads: | 1 way (HH) |

In this case there are a total of four routes.

**Flipping a coin three times**

| | |
|---|---|
| 0 heads: | 1 way (TTT) |
| 1 head: | 3 ways (HTT, THT, TTH) |
| 2 heads: | 3 ways (HHT, HTH, THH) |
| 3 heads: | 1 way (HHH) |

In this case there are a total of eight routes.

**Flipping a coin four times**

| | |
|---|---|
| 0 heads: | 1 way (TTTT) |
| 1 head: | 4 ways (HTTT, THTT, TTHT, TTTH) |
| 2 heads: | 6 ways (HHTT, HTHT, HTTH, THHT, THTH, TTHH) |
| 3 heads: | 4 ways (HHHT, HHTH, HTHH, THHH) |
| 4 heads: | 1 way (HHHH) |

In this case there are a total of sixteen routes.

Can you make a prediction for the total number of routes if we were flipping a coin five times? Hopefully, you predicted 32 routes[1]. You should be able to see that these powers of two are related to the number of choices that are being made: if there are $n$ coin flips then there are a total of $2^n$ different routes through the tree diagram.

The good news, however, is that you do not need to laborious count up how many ways there are for a specific outcome (such as four heads from ten coin flips): Pascal's triangle[2] allows you to simply read off the values.

[1]If you predicted some other value then try listing all of the outcomes and verify that 32 is the correct answer. Then predict the total number of routes if we were flipping a coin six times.

[2]Not due to Blaise Pascal despite being named after him — but that's another story.

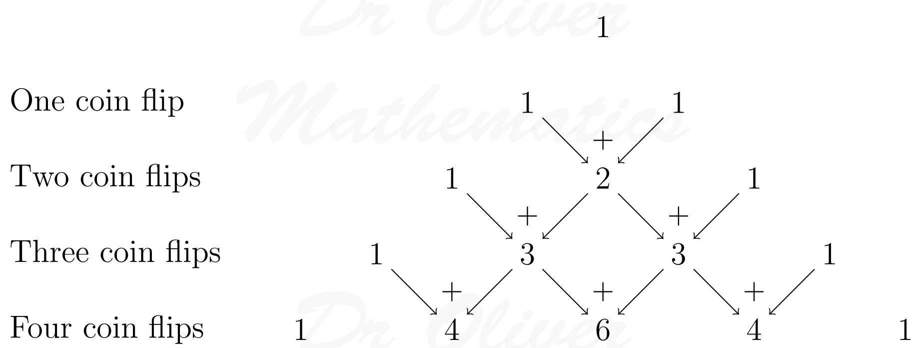


Figure 13: Pascal's triangle