Dr Oliver Mathematics
Direct Proof

In this note, we explore the idea of direct proof.

Example 1
The product of any two odd integers is odd.

Solution
Suppose m and n are any particular but arbitrarily chosen odd integers. By the definition of odd numbers, we have

$$n = 2r + 1$$

for some integer r and

$$m = 2s + 1$$

for some integer s. Then, by substitution, we have

$$\begin{aligned} mn &= (2r + 1)(2s + 1) \\ &= 4rs + 2r + 2s + 1 \\ &= 2(2rs + r + s) + 1. \end{aligned}$$

Now, $(2rs + r + s)$ is an integer (because products and sums of integers are integers and 2, r, and s are all integers) and therefore, by definition of odd number, mn is odd. $\blacksquare$

Example 2
For all integers n,

$$4(n^2 + n + 1) - 3n^2$$

is a perfect square.

Solution
Let n is any particular but arbitrarily chosen integer. Then

$$\begin{aligned} 4(n^2 + n + 1) - 3n^2 &= (4n^2 + 4n + 4) - 3n^2 \\ &= n^2 + 4n + 4 \\ &= (n + 2)^2. \end{aligned}$$

Now, $4(n^2 + n + 1) - 3n^2$ is a perfect square because $(n + 2)$ is an integer (being a sum of n and 2). $\blacksquare$

Example 3
For all integers a, b and c, if a divides b and b divides c, then a divides c.

Solution
Since a divides b,

$$b = ar \text{ for some integer } r$$

and, since b divides c,

$$c = bs \text{ for some integer } s.$$

Now,

$$\begin{aligned} c &= bs \\ &= (ar)s \\ &= a(rs) \end{aligned}$$

and a divides c. ■

Here are some examples for you to try.

1. For all integer a, b and c, if $a|b$ and $a|c$, then $a|(b+c)$.
2. Suppose a, b, c, and $d \in \mathbb{Z}$. If $a|b$ and $c|d$, then $ac|bd$.
3. If two integers have opposite parity, then their sum is odd.
4. Let x and y be positive numbers. If $x \leqslant y$, then $\sqrt{x} \leqslant \sqrt{y}$.
5. If a is an odd integer, then $a^2 + 3a + 5$ is odd.
6. Suppose a is an integer. If $7|4a$, then $7|a$.
7. If a, b, and $c \in \mathbb{N}$, and $c \leqslant b \leqslant a$, then
$$\binom{a}{b}\binom{b}{c} = \binom{a}{b-c}\binom{a-b+c}{c}.$$
8. Prove that the sum of three consecutive numbers is a multiple of 3.
9. Prove that the sum of two consecutive odd numbers is a multiple of 4.
10. Denise has a multiple of 8. She adds 3 to this number and then squares the number. Prove that the resulting number is odd.
11. (a) Find an algebraic expression for the difference between the squares of any two consecutive numbers.

    (b) Hence, prove that the difference between the squares of any two consecutive numbers leaves a remainder of 1 when divided by 2.