Dr Oliver Mathematics
Trigonometry: Powers

In this note, we will investigate the powers of sine and cosine, and tangent of multiples angles.

1 Introduction

If you did Further Mathematics, then you were probably exposed to the following.

Let

$$z = \cos x + \mathrm{i} \sin x.$$

Now, for $n \in \mathbb{Z}$,

$$\begin{aligned} z^n &= (\cos x + \mathrm{i} \sin x)^n \\ &= \cos(nx) + \mathrm{i} \sin(nx) \end{aligned}$$

and

$$\begin{aligned} z^{-n} &= (\cos x + \mathrm{i} \sin x)^{-n} \\ &= \cos(-nx) + \mathrm{i} \sin(-nx) \\ &= \cos(nx) - \mathrm{i} \sin(nx). \end{aligned}$$

Next,

$$\begin{aligned} & 2\cos(nx) = [\cos(nx) + \mathrm{i} \sin(nx)] + [\cos(nx) - \mathrm{i} \sin(nx)] \\ \Rightarrow \quad & 2\cos(nx) = z^n + z^{-n} \\ \Rightarrow \quad & 2\cos(nx) = \left(z^n + \frac{1}{z^n} \right) \\ \Rightarrow \quad & \boxed{\cos(nx) = \tfrac{1}{2} \left(z^n + \frac{1}{z^n} \right)} \end{aligned}$$

and

$$\begin{aligned} & 2\,\mathrm{i} \sin(nx) = [\cos(nx) + \mathrm{i} \sin(nx)] - [\cos(nx) - \mathrm{i} \sin(nx)] \\ \Rightarrow \quad & 2\,\mathrm{i} \sin(nx) = z^n - z^{-n} \\ \Rightarrow \quad & 2\,\mathrm{i} \sin(nx) = \left(z^n - \frac{1}{z^n} \right) \\ \Rightarrow \quad & \boxed{\sin(nx) = \tfrac{1}{2\,\mathrm{i}} \left(z^n + \frac{1}{z^n} \right)}. \end{aligned}$$

And it is this that we use to discover powers of sine and cosine.

2 Powers of 2

2.1 $\cos^2 A$

$$\begin{aligned}
\cos^2 A &= (\cos A)^2 \\
&= \left[\tfrac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 \\
&= \tfrac{1}{4}\left[\binom{2}{0}(z)^2 + \binom{2}{1}(z)\left(\frac{1}{z}\right) + \binom{2}{2}\left(\frac{1}{z}\right)^2\right] \\
&= \tfrac{1}{4}\left[z^2 + 2 + \frac{1}{z^2}\right] \\
&= \tfrac{1}{4}\left[z^2 + \frac{1}{z^2}\right] + \tfrac{1}{2} \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}\left(z^2 + \frac{1}{z^2}\right)\right] + \tfrac{1}{2} \\
&= \underline{\underline{\tfrac{1}{2}\cos 2A + \tfrac{1}{2}}}.
\end{aligned}$$

2.2 $\sin^2 A$

$$\begin{aligned}
\sin^2 A &= (\sin A)^2 \\
&= \left[\tfrac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)\right]^2 \\
&= -\tfrac{1}{4}\left[\binom{2}{0}(z)^2 + \binom{2}{1}(z)\left(-\frac{1}{z}\right) + \binom{2}{2}\left(-\frac{1}{z}\right)^2\right] \\
&= -\tfrac{1}{4}\left[z^2 - 2 + \frac{1}{z^2}\right] \\
&= -\tfrac{1}{4}\left[z^2 + \frac{1}{z^2}\right] + \tfrac{1}{2} \\
&= -\tfrac{1}{2}\left[\tfrac{1}{2}\left(z^2 + \frac{1}{z^2}\right)\right] + \tfrac{1}{2} \\
&= \underline{\underline{-\tfrac{1}{2}\cos 2A + \tfrac{1}{2}}}.
\end{aligned}$$

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3 Powers of 3

3.1 $\cos^3 A$

$$\begin{aligned}
\cos^3 A &= (\cos A)^3 \\
&= \left[\tfrac{1}{2}\left(z + \frac{1}{z}\right)\right]^3 \\
&= \tfrac{1}{8}\left[\binom{3}{0}(z)^3 + \binom{3}{1}(z)^2\left(\frac{1}{z}\right) + \binom{3}{2}(z)\left(\frac{1}{z}\right)^2 + \binom{3}{3}\left(\frac{1}{z}\right)^3\right] \\
&= \tfrac{1}{8}\left[z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\right] \\
&= \tfrac{1}{4}\left[\tfrac{1}{2}\left(z^3 + \frac{1}{z^3}\right)\right] + \tfrac{3}{2}\left[\tfrac{1}{2}\left(z + \frac{1}{z}\right)\right] \\
&= \underline{\underline{\tfrac{1}{4}\cos 3A + \tfrac{3}{2}\cos A}}.
\end{aligned}$$

3.2 $\sin^3 A$

$$\begin{aligned}
\sin^3 A &= (\sin A)^3 \\
&= \left[\tfrac{1}{2\,\mathrm{i}}\left(z - \frac{1}{z}\right)\right]^3 \\
&= -\tfrac{1}{8\,\mathrm{i}}\left[\binom{3}{0}(z)^3 + \binom{3}{1}(z)^2\left(-\frac{1}{z}\right) + \binom{3}{2}(z)\left(-\frac{1}{z}\right)^2 + \binom{3}{3}\left(-\frac{1}{z}\right)^3\right] \\
&= -\tfrac{1}{8\,\mathrm{i}}\left[z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}\right] \\
&= -\tfrac{1}{4}\left[\tfrac{1}{2\,\mathrm{i}}\left(z^3 - \frac{1}{z^3}\right)\right] + \tfrac{3}{4}\left[\tfrac{1}{2\,\mathrm{i}}\left(z - \frac{1}{z}\right)\right] \\
&= \underline{\underline{-\tfrac{1}{4}\sin 3A + \tfrac{3}{4}\sin A}}.
\end{aligned}$$

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4 Powers of 4

4.1 $\cos^4 A$

$$\begin{aligned}
\cos^4 A &= (\cos A)^4 \\
&= \left[\tfrac{1}{2}\left(z + \frac{1}{z}\right)\right]^4 \\
&= \tfrac{1}{16}\left[z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}\right] \\
&= \tfrac{1}{8}\left[\tfrac{1}{2}\left(z^4 + \frac{1}{z^4}\right) + \tfrac{4}{2}\left(z^2 + \frac{1}{z^2}\right) + 6\right] \\
&= \tfrac{1}{8}\left[\cos 4A + 4\cos 2A + 3\right] \\
&= \underline{\underline{\tfrac{1}{8}\cos 4A + \tfrac{1}{2}\cos 2A + \tfrac{3}{8}}}.
\end{aligned}$$

4.2 $\sin^4 A$

$$\begin{aligned}
\sin^4 A &= (\sin A)^4 \\
&= \left[\tfrac{1}{2\,\mathrm{i}}\left(z + \frac{1}{z}\right)\right]^4 \\
&= \tfrac{1}{16}\left[z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right] \\
&= \tfrac{1}{8}\left[\tfrac{1}{2}\left(z^4 + \frac{1}{z^4}\right) - \tfrac{4}{2}\left(z^2 + \frac{1}{z^2}\right) + 12\right] \\
&= \tfrac{1}{8}\left[\cos 4A - 4\cos 2A + 3\right] \\
&= \underline{\underline{\tfrac{1}{8}\cos 4A - \tfrac{1}{2}\cos 2A + \tfrac{3}{8}}}.
\end{aligned}$$

5 We Can Multiply . . .

What is

$$\sin^5 A \cos^4 A?$$

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Well,

$$
\begin{aligned}
\sin^5 A \cos^4 A &= (\sin A)^5 (\cos A)^4 \\
&= \left[\tfrac{1}{2\,\mathrm{i}}\left(z - \frac{1}{z}\right)\right]^5 \left[\tfrac{1}{2}\left(z + \frac{1}{z}\right)\right]^4 \\
&= \tfrac{1}{2^9\,\mathrm{i}}\left(z - \frac{1}{z}\right)^5 \left(z + \frac{1}{z}\right)^4 \\
&= \tfrac{1}{2^9\,\mathrm{i}}\left[\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^4\right]\left(z - \frac{1}{z}\right) \\
&= \tfrac{1}{2^9\,\mathrm{i}}\left[\left(z - \frac{1}{z}\right)\left(z + \frac{1}{z}\right)\right]^4\left(z - \frac{1}{z}\right) \\
&= \tfrac{1}{2^9\,\mathrm{i}}\left(z^2 - \frac{1}{z^2}\right)^4\left(z - \frac{1}{z}\right) \\
&= \tfrac{1}{2^9\,\mathrm{i}}\left(z^8 - 4z^4 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z - \frac{1}{z}\right)
\end{aligned}
$$

$\times$	z^8	$-4z^4$	$+6$	$-\frac{4}{z^4}$	$+\frac{1}{z^8}$
z	z^9	$-4z^5$	$+6z$	$-\frac{4}{z^3}$	$+\frac{1}{z^7}$
$-\frac{1}{z}$	$-z^7$	$+4z^3$	$-\frac{6}{z}$	$+\frac{4}{z^5}$	$-\frac{1}{z^9}$

$$
\begin{aligned}
&= \tfrac{1}{2^9\,\mathrm{i}}\left(z^9 - z^7 - 4z^5 + 4z^3 + 6z - \frac{6}{z} - \frac{4}{z^3} + \frac{4}{z^5} + \frac{1}{z^7} - \frac{1}{z^9}\right) \\
&= \tfrac{1}{2^8}\left[\tfrac{1}{2\,\mathrm{i}}\left(z^9 - \frac{1}{z^9}\right) - \tfrac{1}{2\,\mathrm{i}}\left(z^7 - \frac{1}{z^7}\right) - \tfrac{4}{2\,\mathrm{i}}\left(z^5 - \frac{1}{z^5}\right)\right. \\
&\qquad\qquad \left. + \tfrac{4}{2\,\mathrm{i}}\left(z^3 - \frac{1}{z^3}\right) + \tfrac{6}{2\,\mathrm{i}}\left(z - \frac{1}{z}\right)\right] \\
&= \underline{\underline{\tfrac{1}{256}(\sin 9A - \sin 7A - 4\sin 5A + 4\sin 3A + 6\sin A)}}.
\end{aligned}
$$

6 . . . and Integrate

1. (a) Prove that

$$\left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^2 \equiv \left(z^5 + \frac{1}{z^5}\right) + \left(z^3 + \frac{1}{z^3}\right) - 2\left(z + \frac{1}{z}\right).$$

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Solution

Well,

$$\begin{aligned}\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^2 &\equiv \left[\left(z+\frac{1}{z}\right)^2\left(z-\frac{1}{z}\right)^2\right]\left(z+\frac{1}{z}\right)\\ &\equiv \left[\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)\right]^2\left(z+\frac{1}{z}\right)\\ &\equiv \left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)\\ &\equiv \left(z^4-2+\frac{1}{z^4}\right)^2\left(z+\frac{1}{z}\right)\end{aligned}$$

$\times$	z^4	-2	$+\frac{1}{z^4}$
z	z^5	$-2z$	$+\frac{1}{z^3}$
$+\frac{1}{z}$	$+z^3$	$-\frac{2}{z}$	$+\frac{1}{z^5}$

$$\begin{aligned}&\equiv z^5+z^3-2z-\frac{2}{z}+\frac{1}{z^3}+\frac{1}{z^5}\\ &\equiv \underline{\underline{\left(z^5+\frac{1}{z^5}\right)+\left(z^3+\frac{1}{z^3}\right)-2\left(z+\frac{1}{z}\right)}},\end{aligned}$$

as required.

(b) Hence find

$$\cos^3\theta\sin^2\theta.$$

Solution

Now,

$$\begin{aligned}
\cos^3\theta\sin^2\theta &\equiv [\cos\theta]^3\,[\sin\theta]^2 \\
&\equiv \left[\tfrac{1}{2}\left(z+\frac{1}{z}\right)\right]^3\left[\tfrac{1}{2\,\mathrm{i}}\left(z-\frac{1}{z}\right)\right]^2 \\
&\equiv -\tfrac{1}{32}\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^2 \\
&\equiv -\tfrac{1}{32}\left[\left(z^5+\frac{1}{z^5}\right)+\left(z^3+\frac{1}{z^3}\right)-2\left(z+\frac{1}{z}\right)\right] \\
&\equiv -\tfrac{1}{16}\left[\tfrac{1}{2}\left(z^5+\frac{1}{z^5}\right)+\tfrac{1}{2}\left(z^3+\frac{1}{z^3}\right)-\tfrac{2}{2}\left(z+\frac{1}{z}\right)\right] \\
&\equiv -\tfrac{1}{16}\left(\cos 5\theta+\cos 3\theta-2\cos\theta\right) \\
&\equiv \underline{\underline{\tfrac{1}{16}\left(2\cos\theta-\cos 3\theta-\cos 5\theta\right).}}
\end{aligned}$$

(c) Hence find

$$\int \cos^3\theta\sin^2\theta\,\mathrm{d}\theta.$$

Solution

$$\begin{aligned}
\int \cos^3\theta\sin^2\theta\,\mathrm{d}\theta &= \tfrac{1}{16}\int\left(2\cos\theta-\cos 3\theta-\cos 5\theta\right)\,\mathrm{d}\theta \\
&= \tfrac{1}{16}\left(2\sin\theta-\tfrac{1}{3}\sin 3\theta-\tfrac{1}{5}\sin 5\theta\right)+c \\
&= \underline{\underline{\tfrac{1}{8}\sin\theta-\tfrac{1}{48}\sin 3\theta-\tfrac{1}{80}\sin 5\theta+c.}}
\end{aligned}$$

(d) Hence find

$$\int_0^{\frac{1}{2}\pi}\cos^3\theta\sin^2\theta\,\mathrm{d}\theta.$$

Solution

$$
\begin{aligned}
\int_0^{\frac{1}{2}\pi} \cos^3\theta \sin^2\theta \, \mathrm{d}\theta &= \left[\tfrac{1}{8}\sin\theta - \tfrac{1}{48}\sin 3\theta - \tfrac{1}{80}\sin 5\theta\right]_{x=0}^{\frac{1}{2}\pi} \\
&= \left(\tfrac{1}{8} + \tfrac{1}{48} - \tfrac{1}{80}\right) - (0 - 0 - 0) \\
&= \underline{\underline{\tfrac{2}{15}}}.
\end{aligned}
$$