Dr Oliver Mathematics
Advance Level Further Mathematics
Mechanics 2: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. A truck of mass 750 kg is moving with constant speed v ms^{-1} down a straight road inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{49}$. The resistance to motion of the truck is modelled as a constant force of magnitude 1200 N. The engine of the truck is working at a constant rate of 9 kW.

   (a) Find the value of v. (4)

   Solution

   $$\begin{aligned} \text{Parallel:} &\quad 750g\sin\theta + F = 1\,200 \\ \text{Perpendicular:} &\quad R = 750g\sin\theta \\ P = Fv: &\quad 9\,000 = Fv. \end{aligned}$$

   Now,

   $$\begin{aligned} F = \frac{9\,000}{v} &\Rightarrow 750g \times \tfrac{3}{49} + \frac{9\,000}{v} = 1\,200 \\ &\Rightarrow 450 + \frac{9\,000}{v} = 1\,200 \\ &\Rightarrow \frac{9\,000}{v} = 750 \\ &\Rightarrow v = \frac{9\,000}{750} \\ &\Rightarrow \underline{\underline{v = 12}}. \end{aligned}$$

   On another occasion the truck is moving up the same straight road. The resistance to motion of the truck from non-gravitational forces is modelled as a constant force of magnitude 1200 N. The engine of the truck is working at a constant rate of 9 kW.

   (b) Find the acceleration of the truck at the instant when it is moving with speed 4.5 ms^{-1} . (4)

Solution

$$\begin{aligned} \text{Parallel:} \quad & F - 750g\sin\theta - 1\,200 = 750a \\ \text{Perpendicular:} \quad & R = 750g\sin\theta \\ F = ma: \quad & F = 750a \\ P = Fv: \quad & 9\,000 = 4.5F. \end{aligned}$$

Now,

$$\begin{aligned} & F - 750g\sin\theta - 1\,200 = 750a \\ \Rightarrow \quad & \frac{9\,000}{4.5} - 750g \times \tfrac{3}{49} - 1\,200 = 750a \\ \Rightarrow \quad & 750a = 350 \\ \Rightarrow \quad & a = 0.4\dot{6} \\ \Rightarrow \quad & \underline{\underline{a = 0.47 \text{ ms}^{-2} \text{ (2 sf)}}}. \end{aligned}$$

2. The points A, B, and C lie on a smooth horizontal plane. A small ball of mass 0.2 kg is moving along the line AB with speed 4 ms^{-1}. When the ball is at B, the ball is given an impulse. Immediately after the impulse is given, the ball moves along the line BC with speed 7 ms^{-1}. The line BC makes an angle of 35° with the line AB, as shown in Figure 1.

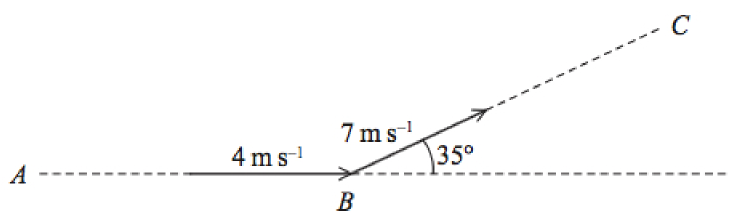


Figure 1: a small ball of mass 0.2 kg is moving along the line AB

(a) Find the magnitude of the impulse given to the ball. (4)

Solution

$$\begin{aligned} \mathbf{I} &= 0.2[(7\cos 35°\mathbf{i} + 7\sin 35°\mathbf{j}) - 4\mathbf{i}] \\ &= 0.2[(7\cos 35° - 4)\mathbf{i} + 7\sin 35°\mathbf{j}] \end{aligned}$$

and the magnitude is

$$0.2\sqrt{(7\cos 35^\circ - 4)^2 + (7\sin 35^\circ)^2} = 0.874\,699\,617\,5 \text{ (FCD)}$$
$$= \underline{\underline{0.87 \text{ Ns (2 sf)}}}.$$

(b) Find the size of the angle between the direction of the impulse and the original direction of motion of the ball. (3)

Solution
The angle is

$$\tan^{-1}\left(\frac{7\sin 35^\circ}{7\cos 35^\circ - 4}\right) = 66.640\,836\,92 \text{ (FCD)}$$
$$= \underline{\underline{67^\circ \text{ (2 sf)}}}.$$

3. (*The centre of mass of a semicircular lamina of radius r is $\frac{4r}{3\pi}$ from the centre.*)

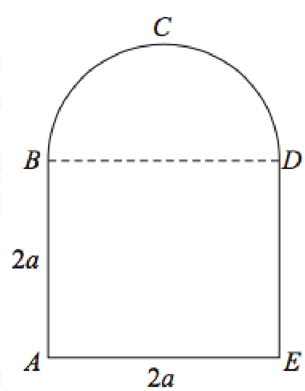


Figure 2: a uniform lamina $ABCDE$

Figure 2 shows the uniform lamina $ABCDE$, such that $ABDE$ is a square with sides of length $2a$ and BCD is a semicircle with diameter BD.

(a) Show that the distance of the centre of mass of the lamina from BD is (5)

$$\frac{20a}{3(8+\pi)}.$$

Solution

Let d be the centre of mass.

Section	Mass	Centre of mass
Square	$4a^2$	a
Hemisphere	$\frac{1}{2}\pi a^2$	$-\frac{4a}{3\pi}$
Whole shape	$4a^2 + \frac{1}{2}\pi a^2$	d

Now,

$$
\begin{aligned}
& (4a^2 \times a) + \left[\tfrac{1}{2}\pi a^2 \times (-\tfrac{4a}{3\pi})\right] = (4a^2 + \tfrac{1}{2}\pi a^2) \times d \\
\Rightarrow \quad & 4a^3 - \frac{2a^3}{3} = \frac{(8+\pi)a^2}{2} \times d \\
\Rightarrow \quad & \frac{10a^3}{3} = \frac{(8+\pi)a^2}{2} \times d \\
\Rightarrow \quad & \underline{\underline{d = \frac{20a}{3(8+\pi)}}},
\end{aligned}
$$

as required.

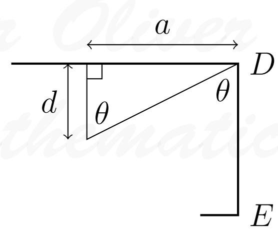
The lamina is freely suspended from D and hangs in equilibrium.

(b) Find, to the nearest degree, the angle that DE makes with the downward vertical. (3)

Solution

Let θ be the angle that DE makes with the downward vertical.

A picture could help ...



Now,

$$\tan\theta = \frac{a}{d} \Rightarrow \tan\theta = \frac{3(8+\pi)}{20}$$
$$\Rightarrow \theta = 59.105\,448\,72 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 59^\circ \text{ (nearest degree)}}}.$$

4. A uniform rod AB, of mass m and length $2a$, rests with its end A on rough horizontal ground. The rod is held in limiting equilibrium at an angle θ to the horizontal by a light string attached to the rod at B, as shown in Figure 3. (10)

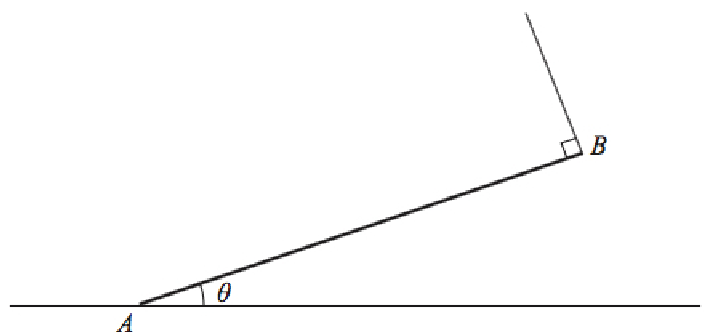


Figure 3: a uniform rod AB, of mass m and length $2a$

The string is perpendicular to the rod and lies in the same vertical plane as the rod.

The coefficient of friction between the ground and the rod is μ.

Show that

$$\mu = \frac{\cos\theta\sin\theta}{2-\cos^2\theta}.$$

Solution

We take moments about B (that way, we don't need to bother with the tension).

$$\begin{aligned} M(B) \quad & (mg)(a\cos\theta) + (F)(2a\sin\theta) = (R)(2a\cos\theta) \\ \text{Parallel:} \quad & (F)(\cos\theta) + (R)(\sin\theta) = (mg)(\sin\theta) \\ F = \mu R: \quad & \end{aligned}$$

Now,

$$\begin{aligned}
& (mg)(a\cos\theta) + (F)(2a\sin\theta) = (R)(2a\cos\theta) \\
\Rightarrow\quad & mg\cos\theta + 2\mu R\sin\theta = 2R\cos\theta \\
\Rightarrow\quad & 2R\cos\theta - 2\mu R\sin\theta = mg\cos\theta \\
\Rightarrow\quad & 2R(\cos\theta - \mu\sin\theta) = mg\cos\theta \\
\Rightarrow\quad & R = \frac{mg\cos\theta}{2(\cos\theta - \mu\sin\theta)} \quad (1)
\end{aligned}$$

and

$$\begin{aligned}
& (F)(\cos\theta) + (R)(\sin\theta) = (mg)(\sin\theta) \\
\Rightarrow\quad & \mu R\cos\theta + R\sin\theta = mg\sin\theta \\
\Rightarrow\quad & R(\mu\cos\theta + \sin\theta) = mg\sin\theta \\
\Rightarrow\quad & R = \frac{mg\sin\theta}{\mu\cos\theta + \sin\theta} \quad (2).
\end{aligned}$$

Now, set (1) equal to (2):

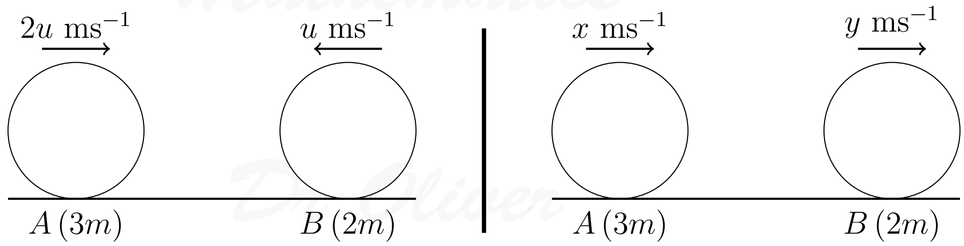
$$\begin{aligned}
& \frac{mg\cos\theta}{2(\cos\theta - \mu\sin\theta)} = \frac{mg\sin\theta}{\mu\cos\theta + \sin\theta} \\
\Rightarrow\quad & \cos\theta(\mu\cos\theta + \sin\theta) = \sin\theta\,[2(\cos\theta - \mu\sin\theta)] \\
\Rightarrow\quad & \mu\cos^2\theta + \sin\theta\cos\theta = 2\sin\theta\cos\theta - 2\mu\sin^2\theta \\
\Rightarrow\quad & \mu\cos^2\theta + 2\mu\sin^2\theta = \sin\theta\cos\theta \\
\Rightarrow\quad & \mu(\cos^2\theta + 2\sin^2\theta) = \sin\theta\cos\theta \\
\Rightarrow\quad & \mu = \frac{\sin\theta\cos\theta}{\cos^2\theta + 2(1 - \cos^2\theta)} \\
\Rightarrow\quad & \underline{\underline{\mu = \frac{\sin\theta\cos\theta}{2 - \cos^2\theta}}},
\end{aligned}$$

as required.

5. A particle A of mass $3m$ is moving in a straight line with speed $2u$ on a smooth horizontal floor. Particle A collides directly with another particle B of mass $2m$ which is moving along the same straight line with speed u but in the opposite direction to A. The coefficient of restitution between A and B is $\frac{1}{3}$.

   (a) (i) Show that the speed of B immediately after the collision is $\frac{7}{5}u$. (7)

Solution



$$\begin{aligned} \text{Cons. of momentum} : &\quad (3m \times 2u) - (2m \times u) = (3m \times x) + (2m \times y) \\ \text{Newton's Law of Rest.} : &\quad \frac{y - x}{2u - (-u)} = \tfrac{1}{3}. \end{aligned}$$

Now,

$$\begin{aligned} 4u = 3x + 2y \Rightarrow 3x &= 4u - 2y \\ \Rightarrow x &= \tfrac{1}{3}(4u - 2y) \quad (1) \end{aligned}$$

and

$$\begin{aligned} \frac{y - x}{2u - (-u)} = \tfrac{1}{3} &\Rightarrow y - x = u \\ &\Rightarrow x = y - u \quad (2). \end{aligned}$$

Finally, set (1) equal to (2):

$$\begin{aligned} \tfrac{1}{3}(4u - 2y) = y - u &\Rightarrow 4u - 2y = 3(y - u) \\ &\Rightarrow 4u - 2y = 3y - 3u \\ &\Rightarrow 5y = 7u \\ &\Rightarrow \underline{\underline{y = \tfrac{7}{5}u}}, \end{aligned}$$

as required.

(ii) Find the speed of A immediately after the collision.

Solution

$$x = \tfrac{7}{5}u - u = \underline{\underline{\tfrac{2}{5}u}}.$$

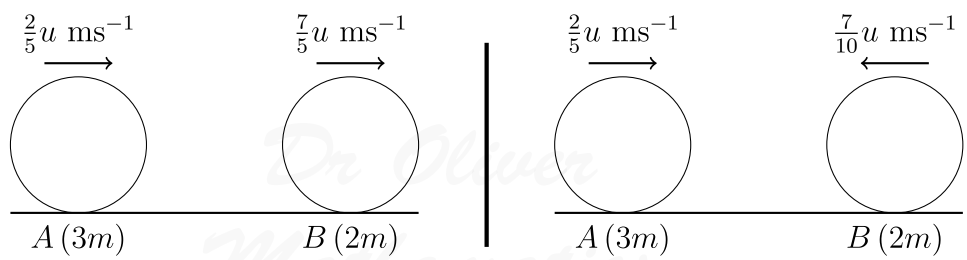
After the collision, B hits a smooth vertical wall which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the wall is $\frac{1}{2}$. The first collision between A and B occurred at a distance x from the wall. The particles collide again at a distance y from the wall.

(b) Find y in terms of x. (6)

Solution

The speed of B after collision with the wall is

$$\tfrac{1}{2} \times \tfrac{7}{5}u = \tfrac{7}{10}u.$$



The time for B to travel is

$$\begin{aligned}\frac{x}{\frac{7}{5}u} + \frac{y}{\frac{7}{10}u} &= \frac{5x}{7u} + \frac{10y}{7u} \\ &= \frac{5x + 10y}{7u}.\end{aligned}$$

Now, the distance moved by A is

$$\begin{aligned}\frac{2}{5}u \times \frac{5x + 10y}{7u} &= \frac{2(5x + 10y)}{35} \\ &= \frac{10x + 20y}{35} \\ &= \frac{2x + 4y}{7}.\end{aligned}$$

Finally,

$$\begin{aligned}\frac{2x + 4y}{7} + y = x &\Rightarrow 2x + 4y + 7y = 7x \\ &\Rightarrow 11y = 5x \\ &\Rightarrow \underline{\underline{y = \tfrac{5}{11}x}}.\end{aligned}$$

6. A particle P of mass 0.5 kg moves under the action of a single force $\mathbf{F}$ newtons. At time t seconds, $t \geqslant 0$, P has velocity $\mathbf{v}$ ms^{-1}, where

$$\mathbf{v} = (4t - 3t^2)\mathbf{i} + (t^2 - 8t - 40)\mathbf{j}.$$

(a) Find (9)

(i) the magnitude of $\mathbf{F}$ when $t = 3$,

Solution

$$\mathbf{v} = (4t - 3t^2)\mathbf{i} + (t^2 - 8t - 40)\mathbf{j} \Rightarrow \mathbf{a} = (4 - 6t)\mathbf{i} + (2t - 8)\mathbf{j},$$

and, when $t = 3$,

$$\mathbf{a} = -14\mathbf{i} - 2\mathbf{j}.$$

Finally,

$$\begin{aligned} F &= 0.5\sqrt{(-14)^2 + (-2)^2} \\ &= 5\sqrt{2} \\ &= \underline{\underline{7.1 \text{ ms}^{-2} \text{ (2 sf)}}}. \end{aligned}$$

(ii) the acceleration of P at the instant when it is moving in the direction of the vector $-\mathbf{i} - \mathbf{j}$.

Solution

$$\begin{aligned} \frac{4t - 3t^2}{t^2 - 8t - 40} = \frac{-1}{-1} &\Rightarrow 4t - 3t^2 = t^2 - 8t - 40 \\ &\Rightarrow 4t^2 - 12t - 40 = 0 \\ &\Rightarrow t^2 - 3t - 10 = 0 \\ &\Rightarrow (t + 2)(t - 5) = 0 \\ &\Rightarrow t = -2 \text{ or } t = 5. \end{aligned}$$

Finally, the acceleration is

$$\underline{\underline{\mathbf{a} = -26\mathbf{i} + 2\mathbf{j}}}.$$

When $t = 1$, P is at the point A. When $t = 2$, P is at the point B.

(b) Find, in terms of $\mathbf{i}$ and $\mathbf{j}$, the vector $\overrightarrow{AB}$. (5)

Solution

$$\begin{aligned} & \mathbf{v} = (4t - 3t^2)\mathbf{i} + (t^2 - 8t - 40)\mathbf{j} \\ \Rightarrow\ & \mathbf{r} = (2t^2 - t^3 + c)\mathbf{i} + (\tfrac{1}{3}t^3 - 4t^2 - 40t + d)\mathbf{j} \end{aligned}$$

for some constants c and d. Now,

$$\begin{aligned} t = 1 &\Rightarrow \mathbf{r} = (1 + c)\mathbf{i} + (-\tfrac{131}{3} + d)\mathbf{j} \\ t = 2 &\Rightarrow \mathbf{r} = c\mathbf{i} + (-\tfrac{280}{3} + d)\mathbf{j}. \end{aligned}$$

Finally,

$$\begin{aligned} \overrightarrow{AB} &= [c - (1 + c)]\mathbf{i} + [(-\tfrac{280}{3} + d) - (-\tfrac{131}{3} + d)]\mathbf{j} \\ &= \underline{\underline{-\mathbf{i} - \tfrac{149}{3}\mathbf{j}}}. \end{aligned}$$

7. A particle, of mass 0.3 kg, is projected from a point O on horizontal ground with speed u. The particle is projected at an angle α above the horizontal, where $\tan\alpha = 2$, and moves freely under gravity. When the particle has moved a horizontal distance x from O, its height above the ground is y.

   (a) Show that (5)

$$y = 2x - \frac{5g}{2u^2}x^2.$$

Solution

$$\tan\alpha = 2 \Rightarrow \sin\alpha = \tfrac{2\sqrt{5}}{5},\ \cos\alpha = \tfrac{\sqrt{5}}{5}$$

and

$$s = xt = (u\cos\alpha)t \Rightarrow t = \frac{x}{u\cos\alpha}.$$

Now, $s = y\,(\uparrow)$, 'u'$= u\sin\alpha$, $v = ?$, $a = -g$, and $t = ?$:

$$\begin{aligned} s = ut + \tfrac{1}{2}at^2 &\Rightarrow y = (u\sin\alpha)t - \tfrac{1}{2}gt^2 \\ &\Rightarrow y = (u\sin\alpha)\left(\frac{x}{u\cos\alpha}\right) - \tfrac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2 \\ &\Rightarrow y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \\ &\Rightarrow y = 2x - \frac{gx^2}{2u^2(\frac{\sqrt{5}}{5})^2} \\ &\Rightarrow \underline{\underline{y = 2x - \frac{5g}{2u^2}x^2}}, \end{aligned}$$

as required.

The particle hits the ground at the point A, where $OA = 36$ m.

(b) Find u, the speed of projection. (2)

Solution

$$\begin{aligned} x = 36\,y = 0 \Rightarrow 0 &= 2(36) - \frac{5g}{2u^2}(36)^2 = 0 \\ \Rightarrow 72 &= \frac{6\,480g}{2u^2} = 0 \\ \Rightarrow u^2 &= 441 \\ \Rightarrow \underline{\underline{u = 21}}. \end{aligned}$$

(c) Find the minimum kinetic energy of the particle as it moves between O and A. (3)

Solution

The minimum speed is

$$x = u\cos\alpha = \tfrac{21\sqrt{5}}{5} \text{ (why?)}$$

and the minimum kinetic energy is

$$\begin{aligned} \tfrac{1}{2}(0.3)(\tfrac{21\sqrt{5}}{5})^2 &= 13.23 \\ &= \underline{\underline{13 \text{ J (2 sf)}}}. \end{aligned}$$

The point B lies on the path of the particle. The direction of motion of the particle at B is perpendicular to the initial direction of motion of the particle.

(d) Find the horizontal distance between O and B. (5)

Solution

Now, the gradient of the trajectory at B is

$$-\frac{1}{\tan\alpha} = -\tfrac{1}{2}.$$

Next,

$$y = 2x - \frac{5g}{2u^2}x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{5g}{u^2}x$$

and

$$\begin{aligned} 2 - \frac{5g}{u^2}x = -\tfrac{1}{2} &\Rightarrow \frac{5g}{u^2}x = \tfrac{5}{2} \\ &\Rightarrow x = 22.5 \\ &= \underline{\underline{x = 23 \text{ m } (2 \text{ sf})}}. \end{aligned}$$