Dr Oliver Mathematics
Further Mathematics
Conic Sections: Hyperbolas
Past Examination Questions

This booklet consists of 13 questions across a variety of examination topics.
The total number of marks available is 128.

1. Figure 1 shows the curve C which is part of the hyperbola with parametric equations

$$x = a\cosh t, y = 2a\sinh t,$$

where a is a positive constant and $x \geqslant a$.

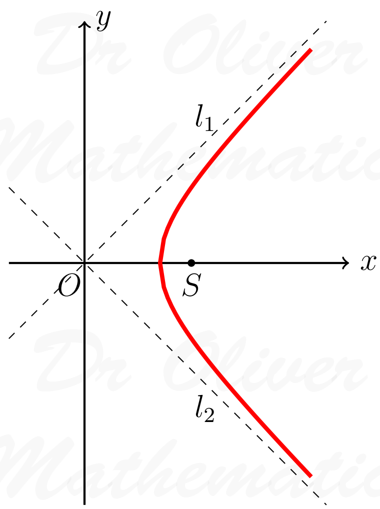


Figure 1: $x = a\cosh t, y = 2a\sinh t$

The lines l_1 and l_2 are asymptotes to C.

(a) Show that an equation for the tangent to C at a point $P(a\cosh p, 2a\sinh t)$ is (4)

$$2x\cosh p - y\sinh p = 2a.$$

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a\cosh t}{a\sinh t} = \frac{2\cosh t}{\sinh t}$$

and, at $P(a\cosh p, 2a\sinh t)$, we have

$$\begin{aligned} y - 2a\sinh p &= \frac{2\cosh p}{\sinh p}(x - a\cosh p) \\ \Rightarrow \quad \sinh p(y - 2a\sinh p) &= 2\cosh p(x - a\cosh p) \\ \Rightarrow \quad y\sinh p - 2a\sinh^2 p &= 2x\cosh p - 2a\cosh^2 p \\ \Rightarrow \quad 2x\cosh p - y\sinh p &= 2a\cosh^2 p - 2a\sinh^2 p \\ \Rightarrow \quad \underline{\underline{2x\cosh p - y\sinh p = 2a}}. \end{aligned}$$

The tangent to the curve C at P meets the asymptote l_1 at Q. Given that QS is parallel to the y-axis, where S is the focus,

(b) show that $p = \frac{1}{2}\ln 5$. (8)

Solution

l_1 is $y = 2x$ and l_2 is $y = -2x$ (why?). Now, for l_1:

$$\begin{aligned} 2x\cosh p - 2x\sinh p = 2a \Rightarrow x(\cosh p - \sinh p) &= a \\ \Rightarrow x &= \frac{a}{\cosh p - \sinh p}. \end{aligned}$$

Next, '$b^2 = a^2(e^2 - 1)$':

$$(2a)^2 = a^2(e^2 - 1) \Rightarrow e^2 - 1 = 4 \Rightarrow e = \sqrt{5};$$

we have $S(a\sqrt{5}, 0)$. Finally,

$$\begin{aligned} \frac{a}{\cosh p - \sinh p} = a\sqrt{5} &\Rightarrow \cosh p - \sinh p = \frac{\sqrt{5}}{5} \\ &\Rightarrow \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = \frac{\sqrt{5}}{5} \\ &\Rightarrow e^{-x} = \frac{\sqrt{5}}{5} \\ &\Rightarrow e^x = \sqrt{5} \\ &\Rightarrow x = \ln\sqrt{5} \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}\ln 5}}. \end{aligned}$$

2. The hyperbola C has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

(a) Show that an equation for the normal to C at a point $P(a \sec t, b \tan t)$ is (5)

$$ax \sin t + by = (a^2 + b^2) \tan t.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{a \sec^2 t}{b \sec t \tan t} = \frac{a}{b \sin t}$$

so the gradient of the normal is

$$-\frac{b \sin t}{a}.$$

Now,

$$\begin{aligned}
y - b \tan \theta &= -\frac{b \sin \theta}{a}(x - a \sec t) \\
\Rightarrow by - ab \tan t &= -b \sin \theta (x - a \sec t) \\
\Rightarrow 3y - 9 \tan t &= -bx \sin \theta + ab \tan t \\
\Rightarrow \underline{\underline{ax \sin t + by}} &\underline{\underline{= (a^2 + b^2) \tan t}}.
\end{aligned}$$

The normal to C at P cuts the x-axis at the point A and S is a focus of C. Given that the eccentricity of C is $\frac{3}{2}$, and that $OA = 3OS$, where O is the origin,

(b) determine the possible values of t, for $0 \leqslant t < 2\pi$. (8)

Solution

$$\begin{aligned}
y = 0 \Rightarrow ax \sin t &= (a^2 + b^2) \tan t \\
\Rightarrow x &= \frac{(a^2 + b^2) \tan t}{a \sin t} \\
\Rightarrow x &= \frac{a^2 + b^2}{a \cos t}.
\end{aligned}$$

Now,

$$b^2 = a^2(e^2 - 1) = a^2 \left[(\tfrac{3}{2})^2 - 1 \right] = \frac{5a^2}{4},$$

$OS = ae$, $OA = 3OS$, and $a^2 + b^2 = ax \cos t$:

$$a^2 + \frac{5a^2}{4} = 3a \times \frac{3}{2}a \times \cos t \Rightarrow \frac{9}{4} = \frac{9\cos t}{2}$$
$$\Rightarrow \cos t = \tfrac{1}{2}$$
$$\Rightarrow \underline{\underline{t = \tfrac{\pi}{3}, \tfrac{5\pi}{3}}}$$

or

$$-\left(a^2 + \frac{5a^2}{4}\right) = 3a^2 \times \frac{3}{2} \times \cos t \Rightarrow -\frac{9}{4} = \frac{9\cos t}{2}$$
$$\Rightarrow \cos t = -\tfrac{1}{2}$$
$$\Rightarrow \underline{\underline{t = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}}}.$$

3. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{4} = 1.$$

Find

(a) the value of the eccentricity of H, (2)

Solution

$a = 4$, $b = 2$, and

$$e = \sqrt{\frac{b^2}{a^2} + 1} = \underline{\underline{\frac{\sqrt{5}}{2}}}.$$

(b) the distance between the foci of H. (2)

Solution

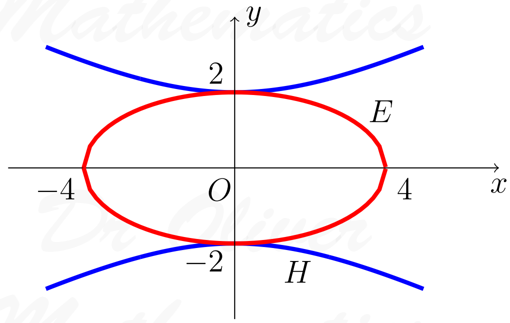
The foci are at $(2\sqrt{5}, 0)$ and $(-2\sqrt{5}, 0)$ which gives the distance between the foci of H $\underline{\underline{4\sqrt{5}}}$.

The ellipse E has equation

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

(c) Sketch H and E on the same diagram, showing the coordinates of the points where each curve crosses the axes. (3)

Solution



4. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

(a) Show that an equation for the normal to H at a point $P(4\sec t, 3\tan t)$ is (6)

$$4x\sin t + 3y = 25\tan t.$$

Solution

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow \frac{x}{8} - \frac{2y}{9}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9x}{16y},$$

and, at the point $P(4\sec t, 3\tan t)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{36\sec t}{48\tan t} = \frac{3}{4\sin t}$$

and so the gradient of the normal is

$$m = -\frac{4\sin t}{3}.$$

Now,

$$y - 3\tan t = -\frac{4\sin t}{3}(x - 4\sec t) \Rightarrow 3y - 9\tan t = -4x\sin t + 16\tan t$$
$$\Rightarrow \underline{\underline{4x\sin t + 3y = 25\tan t}}.$$

The point S, which lies on the positive x-axis, is a focus of H. Given that PS is parallel to the y-axis and the the y-coordinate of P is positive,

(b) find the values of the coordinates of P. (5)

> **Solution**
>
> $a = 4$, $b = 3$, and
>
> $$e = \sqrt{\frac{b^2}{a^2} + 1} = \frac{5}{4}$$
>
> and so $S(5, 0)$ is a focus of H. Now,
>
> $$\frac{25}{16} - \frac{y^2}{9} = 1 \Rightarrow y = \pm\frac{9}{4};$$
>
> so $\underline{\underline{P(5, \frac{9}{4})}}$.

Given that the normal to H at this point P intersects the x-axis at the point R,

(c) find the area of triangle PRS. (3)

> **Solution**
>
> $$y = 0 \Rightarrow 4x \sin t = 25 \tan t \Rightarrow x = \frac{25 \sec t}{4} = \frac{125}{16}$$
>
> since $\sec t = \frac{5}{4}$. Now,
>
> $$\begin{aligned} \text{area} &= \frac{1}{2} \times \left(\frac{125}{16} - 5\right) \times \frac{9}{4} \\ &= \underline{\underline{\frac{405}{128}}}. \end{aligned}$$

5. The hyperbola H has equation

$$x^2 - 4y^2 = 4a^2, a > 0.$$

(a) Find the eccentricity of H. (3)

> **Solution**
>
> $$x^2 - 4y^2 = 4a^2 \Rightarrow \frac{x^2}{4a^2} - \frac{y^2}{a^2} = 1;$$
>
> now, '$b^2 = a^2(e^2 - 1)$' and
>
> $$e = \sqrt{\frac{b^2}{a^2} + 1} = \underline{\underline{\frac{\sqrt{5}}{2}}}.$$

Given that $x = 10$ is an equation of a directrix of H,

(b) find the value of a. (2)

Solution

'$x = \frac{a}{e}$': $\frac{2a}{e} = 10 \Rightarrow \underline{\underline{a = \frac{5\sqrt{5}}{2}}}$.

6. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are a positive constants. The line L has equation

$$y = mx + c,$$

where m and c are constants.

(a) Given that L and H meet, show that the x-coordinates of the points of intersection are the roots of the equation (2)

$$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0.$$

Solution

$$\begin{aligned} y = mx + c &\Rightarrow \frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \\ &\Rightarrow b^2x^2 - a^2(mx+c)^2 = a^2b^2 \\ &\Rightarrow a^2(m^2x^2 + 2mcx + c^2) - b^2x^2 + a^2b^2 = 0 \\ &\Rightarrow \underline{\underline{(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0}}. \end{aligned}$$

Hence, given that L is a tangent to H,

(b) show that (2)

$$a^2m^2 = b^2 + c^2.$$

Solution

Since the line is a tangent, the discriminant must be zero:

$$\begin{aligned}
& (2a^2mc)^2 - 4 \times (a^2m^2 - b^2) \times a^2(c^2 + b^2) = 0 \\
\Rightarrow\ & 4a^4m^2c^2 - 4a^2(a^2m^2 - b^2)(c^2 + b^2) = 0 \\
\Rightarrow\ & a^2m^2c^2 - (a^2c^2m^2 - b^2c^2 + a^2b^2m^2 - b^4) = 0 \\
\Rightarrow\ & b^2c^2 - a^2b^2m^2 + b^4 = 0 \\
\Rightarrow\ & c^2 - a^2m^2 + b^2 = 0 \\
\Rightarrow\ & \underline{\underline{a^2m^2 = b^2 + c^2}}.
\end{aligned}$$

The hyperbola H' has equation

$$\frac{x^2}{25} - \frac{y^2}{16} = 1.$$

(c) Find the equations of the tangents to H' which pass through the point $(1, 4)$. (7)

Solution

So, $4 = m + c$ and we substitute in to equation $a^2m^2 = b^2 + c^2$:

$$\begin{aligned}
c = 4 - m &\Rightarrow 25m^2 = 16 + (4 - m)^2 \\
&\Rightarrow 25m^2 = 16 + (16 - 8m + m^2) \\
&\Rightarrow 24m^2 + 8m - 32 = 0 \\
&\Rightarrow 3m^2 + m - 4 = 0 \\
&\Rightarrow (3m + 4)(m - 1) = 0 \\
&\Rightarrow m = -\tfrac{4}{3} \text{ or } m = 1;
\end{aligned}$$

hence, the equations are

$$\underline{\underline{y = -\tfrac{4}{3}x + \tfrac{16}{3}}} \text{ or } \underline{\underline{y = x + 3}}.$$

7. A hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{4} = 1.$$

The line l_1 is the tangent to H at the point $P(4\sec t, 2\tan t)$.

(a) Use calculus to show that an equation of l_1 is (5)

$$2y \sin t = x - 4\cos t.$$

Solution

$$\frac{x^2}{16} - \frac{y^2}{4} = 1 \Rightarrow \frac{x}{8} - \frac{y}{2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{4y},$$

and, at the point $P(4\sec t, 2\tan t)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sec t}{8\tan t} = \frac{1}{2\sin t}.$$

Now,

$$\begin{aligned} y - 2\tan t = \frac{1}{2\sin t}(x - 4\sec t) &\Rightarrow 2\sin t(y - 2\tan t) = x - 4\sec t \\ &\Rightarrow 2y\sin t - 4\sin t\tan t = x - 4\sec t \\ &\Rightarrow 2y\sin t = x - 4(\sec t - \sin t\tan t) \\ &\Rightarrow 2y\sin t = x - 4\left(\frac{1}{\cos t} - \frac{\sin^2 t}{\cos t}\right) \\ &\Rightarrow 2y\sin t = x - 4\left(\frac{1 - \sin^2 t}{\cos t}\right) \\ &\Rightarrow \underline{\underline{2y\sin t = x - 4\cos t}}. \end{aligned}$$

The line l_2 passes through the origin and is perpendicular to l_1. The lines l_1 and l_2 intersect at the point Q.

(b) Show that, at t varies, an equation of the locus of Q is (8)

$$(x^2 + y^2)^2 = 16x^2 - 4y^2.$$

Solution

The gradient of l_2 is $-2\sin t$ and the equation is

$$y = -2x\sin t.$$

Substitute:

$$\begin{aligned}
&2\sin t(-2x\sin t) = x - 4\cos t \\
\Rightarrow\quad & -4x\sin^2 t = x - 4\cos t \\
\Rightarrow\quad & x(1+4\sin^2 t) = 4\cos t \\
\Rightarrow\quad & x = \frac{4\cos t}{1+4\sin^2 t} \\
\Rightarrow\quad & y = -\frac{8\sin t\cos t}{1+4\sin^2 t}.
\end{aligned}$$

Now,

$$\begin{aligned}
(x^2+y^2)^2 &= \left[\left(\frac{4\cos t}{1+4\sin^2 t}\right)^2 + \left(-\frac{8\sin t\cos t}{1+4\sin^2 t}\right)^2\right]^2 \\
&= \frac{1}{(1+4\sin^2 t)^4}\left[16\cos^2 t + 64\sin^2 t\cos^2 t\right]^2 \\
&= \frac{256\cos^4 t}{(1+4\sin^2 t)^4}\left[1+4\sin^2 t\right]^2 \\
&= \frac{256\cos^4 t}{(1+4\sin^2 t)^2}
\end{aligned}$$

whilst

$$\begin{aligned}
16x^2 - 4y^2 &= 16\left(\frac{4\cos t}{1+4\sin^2 t}\right)^2 - 4\left(-\frac{8\sin t\cos t}{1+4\sin^2 t}\right)^2 \\
&= \frac{256\cos^2 t}{(1+4\sin^2 t)^2} - \frac{256\sin^2 t\cos^2 t}{(1+4\sin^2 t)^2} \\
&= \frac{256\cos^2 t(1-\sin^2 t)}{(1+4\sin^2 t)^2} \\
&= \frac{256\cos^4 t}{(1+4\sin^2 t)^2};
\end{aligned}$$

hence,

$$\underline{\underline{(x^2+y^2)^2 = 16x^2 - 4y^2}}.$$

8. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

(a) Use calculus to show that an equation of the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$ (4)

may be written in the form

$$xb\cosh\theta - ya\sinh\theta = ab.$$

Solution

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b^2x}{a^2y},$$

and, at the point $(a\cosh\theta, b\sinh\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ab^2\cosh\theta}{a^2b\sinh\theta} = \frac{b\cosh\theta}{a\sinh\theta}.$$

Now,

$$\begin{aligned} & y - b\sinh\theta = \frac{b\cosh\theta}{a\sinh\theta}(x - a\cosh\theta) \\ \Rightarrow\quad & a\sinh\theta(y - b\sinh\theta) = b\cosh\theta(x - a\cosh\theta) \\ \Rightarrow\quad & yb\sinh\theta - ab\sinh^2\theta = bx\cosh\theta - ab\cosh^2\theta \\ \Rightarrow\quad & xb\cosh\theta - ya\sinh\theta = ab(\cosh^2\theta - \sinh^2\theta) \\ \Rightarrow\quad & \underline{\underline{xb\cosh\theta - ya\sinh\theta = ab}}. \end{aligned}$$

The line l_1 is the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$, $\theta \neq 0$. Given that l_1 meets the x-axis at the point P,

(b) find, in terms of a and θ, the coordinates of P. (2)

Solution

$$y = 0 \Rightarrow xb\cosh\theta = ab \Rightarrow x = a\,\mathrm{sech}\,\theta;$$

hence, $\underline{\underline{P(a\,\mathrm{sech}\,\theta, 0)}}$.

The line l_2 is the tangent to H at the point $(a, 0)$. Given that l_1 and l_2 meet at the point Q,

(c) find, in terms of a, b, and θ, the coordinates of Q. (2)

Solution

l_2 has equation $x = a$ (why?):

$$ab\cosh\theta - ya\sinh\theta = ab \Rightarrow y\sinh\theta = b\cosh\theta - b$$
$$\Rightarrow y = \frac{b(\cosh\theta - 1)}{\sinh\theta};$$

hence,

$$\underline{\underline{Q\left(a, \frac{b(\cosh\theta - 1)}{\sinh\theta}\right)}}.$$

(d) Show that, at θ varies, the locus of the mid-point of PQ has equation (6)

$$x(4y^2 + b^2) = ab^2.$$

Solution

$$x = \frac{a + a\operatorname{sech}\theta}{2} = \frac{a(1 + \operatorname{sech}\theta)}{2}$$

and

$$y = \frac{0 + \frac{b(\cosh\theta - 1)}{\sinh\theta}}{2} = \frac{b(\cosh\theta - 1)}{2\sinh\theta}.$$

Now,

$$
\begin{aligned}
x(4y^2+b^2) &= \frac{a(1+\operatorname{sech}\theta)}{2}\left(\frac{4b^2(\cosh\theta-1)^2}{4\sinh^2\theta}+b^2\right) \\
&= \frac{4ab^2(1+\operatorname{sech}\theta)}{8\sinh^2\theta}\left[(\cosh\theta-1)^2+\sinh^2\theta\right] \\
&= \frac{4ab^2(1+\operatorname{sech}\theta)}{8\sinh^2\theta}\left[(\cosh^2\theta-2\cosh\theta+1)+(\cosh^2\theta-1)\right] \\
&= \frac{4ab^2(1+\operatorname{sech}\theta)}{8\sinh^2\theta}(2\cosh^2\theta-2\cosh\theta) \\
&= \frac{ab^2(1+\operatorname{sech}\theta)}{\sinh^2\theta}(\cosh^2\theta-\cosh\theta) \\
&= \frac{ab^2\cosh\theta(1+\operatorname{sech}\theta)}{\sinh^2\theta}(\cosh\theta-1) \\
&= \frac{ab^2\cosh\theta}{\sinh^2\theta}\left(1+\frac{1}{\cosh\theta}\right)(\cosh\theta-1) \\
&= \frac{ab^2\cosh\theta}{\sinh^2\theta}\left(\frac{\cosh\theta+1}{\cosh\theta}\right)(\cosh\theta-1) \\
&= \frac{ab^2\cosh\theta}{\sinh^2\theta\cosh\theta}(\cosh^2\theta-1) \\
&= \frac{ab^2\sinh^2\theta\cosh\theta}{\sinh^2\theta\cosh\theta} \\
&= \underline{\underline{ab^2}}.
\end{aligned}
$$

9. The hyperbola H has equation

$$\frac{x^2}{16}-\frac{y^2}{9}=1.$$

Find

(a) the coordinates of the foci of H, (3)

Solution

$a=4$, $b=2$, and

$$e=\sqrt{\frac{b^2}{a^2}+1}=\frac{5}{2};$$

the coordinates are $\underline{\underline{(5,0)}}$ and $\underline{\underline{(-5,0)}}$.

(b) the equations of the directrices of H. (2)

Solution

'$x = \frac{a}{e}$': $\underline{\underline{x = \pm \frac{16}{5}}}$.

10. A hyperbola H has equation
$$\frac{x^2}{a^2} - \frac{y^2}{25} = 1,$$
where a is a positive constant. The foci of H are at the points with coordinates $(13, 0)$ and $(-13, 0)$. Find

(a) the value of the constant a, (3)

Solution

$ae = 13$:
$$\begin{aligned} b^2 = a^2(e^2 - 1) &\Rightarrow b^2 = a^2e^2 - a^2 \\ &\Rightarrow 25 = 169 - a^2 \\ &\Rightarrow a^2 = 144 \\ &\Rightarrow \underline{\underline{a = 12}}. \end{aligned}$$

(b) the equation of the directrices of H. (3)

Solution
$$ae = 13 \Rightarrow e = \frac{13}{12} \Rightarrow \underline{\underline{x = \pm \frac{144}{13}}}.$$

11. The hyperbola H has foci at $(5, 0)$ and $(-5, 0)$ and directrices with equations $x = \pm\frac{9}{5}$. Find a cartesian equation for H. (7)

Solution

$ae = 5$ and $\frac{a}{e} = \frac{9}{5}$:
$$\begin{aligned} a^2 = 9 &\Rightarrow e^2 = \frac{25}{9} \\ &\Rightarrow b^2 = a^2(e^2 - 1) = 16; \end{aligned}$$
hence, the equation is
$$\underline{\underline{\frac{x^2}{9} - \frac{y^2}{16} = 1}}.$$

12. The hyperbola H is given by the equation $x^2 - y^2 = 1$.

(a) Write down the equations of the two asymptotes of H. (1)

Solution

$\underline{\underline{y = x}}$ and $\underline{\underline{y = -x}}$.

(b) Show that an equation of the tangent to H at the point $P(\cosh t, \sinh t)$ is (3)

$$y \sinh t = x \cosh t - 1.$$

Solution

$$\begin{aligned} x^2 - y^2 = 1 &\Rightarrow 2x - 2y \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}, \end{aligned}$$

and, at the point $(\cosh t, \sinh t)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cosh t}{\sinh t}.$$

Now,

$$\begin{aligned} & y - \sinh t = \frac{\cosh t}{\sinh t}(x - \cosh t) \\ \Rightarrow\quad & \sinh t(y - \sinh t) = \cosh t(x - \cosh t) \\ \Rightarrow\quad & y \sinh t - \sinh^2 t = x \cosh t - \cosh^2 t \\ \Rightarrow\quad & y \sinh t = x \cosh t - (\cosh^2 t - \sinh^2 t) \\ \Rightarrow\quad & \underline{\underline{y \sinh t = x \cosh t - 1}}. \end{aligned}$$

The tangent at P meets the asymptotes of H at the points Q and R.

(c) Show that P is the midpoint of QR. (3)

Solution

$\underline{y = x}$:

$$\begin{aligned} x \sinh t = x \cosh t - 1 &\Rightarrow x(\sinh t - \cosh t) = -1 \\ &\Rightarrow x = \frac{1}{\cosh t - \sinh t} \\ &\Rightarrow y = \frac{1}{\cosh t - \sinh t} \end{aligned}$$

$\underline{y = -x}$:

$$\begin{aligned} -x \sinh t = x \cosh t - 1 &\Rightarrow -x(\sinh t + \cosh t) = -1 \\ &\Rightarrow x = \frac{1}{\cosh t + \sinh t} \\ &\Rightarrow y = -\frac{1}{\cosh t + \sinh t}. \end{aligned}$$

Hence,

$$\begin{aligned} x &= \frac{1}{2}\left(\frac{1}{\cosh t - \sinh t} + \frac{1}{\cosh t + \sinh t}\right) \\ &= \frac{(\cosh t + \sinh t) + (\cosh t - \sinh t)}{2(\cosh t - \sinh t)(\cosh t + \sinh t)} \\ &= \frac{2\cosh t}{2(\cosh^2 t - \sinh^2 t)} \\ &= \underline{\underline{\cosh t}} \end{aligned}$$

and

$$\begin{aligned} y &= \frac{1}{2}\left(\frac{1}{\cosh t - \sinh t} - \frac{1}{\cosh t + \sinh t}\right) \\ &= \frac{(\cosh t + \sinh t) - (\cosh t - \sinh t)}{2(\cosh t - \sinh t)(\cosh t + \sinh t)} \\ &= \frac{2\sinh t}{2(\cosh^2 t - \sinh^2 t)} \\ &= \underline{\underline{\sinh t}}. \end{aligned}$$

(d) Show that the area of the triangle OQR, where O is the origin, is independent of t. (3)

Solution

We have a right-angle at ROQ (why?) and

$$\begin{aligned}\text{area} &= \frac{1}{2} \times OQ \times OR \\ &= \frac{1}{2} \times \sqrt{\frac{2}{(\cosh t - \sinh t)^2}} \times \sqrt{\frac{2}{(\cosh t + \sinh t)^2}} \\ &= \frac{1}{(\cosh t - \sinh t)(\cosh t + \sinh t)} \\ &= \frac{1}{\cosh^2 t - \sinh^2 t} \\ &= \underline{\underline{1}},\end{aligned}$$

which is independent of t.

13. The hyperbola H has the equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

The point $P(4\sec\theta, 3\tan\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on H.

(a) Show that an equation of the normal to H at the point P is (5)

$$3y + 4x\sin\theta = 25\tan\theta.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{3\sec^2\theta}{4\sec\theta\tan\theta} = \frac{3}{4\sin\theta}$$

so the gradient of the normal is

$$-\frac{4\sin\theta}{3}.$$

Now,

$$\begin{aligned} y - 3\tan\theta &= -\frac{4\sin\theta}{3}(x - 4\sec\theta) \\ \Rightarrow 3y - 9\tan\theta &= -4\sin\theta(x - 4\sec\theta) \\ \Rightarrow 3y - 9\tan\theta &= -4x\sin\theta + 16\tan\theta \\ \Rightarrow \underline{\underline{3y + 4x\sin\theta = 25\tan\theta}}&,\end{aligned}$$

as required.

The line l is the directrix of H for which $x > 0$. The normal to H at P crosses the line l at the point Q. Given that $\theta = \frac{\pi}{4}$,

(b) find the y-coordinate of Q, giving your answer in the form $a + b\sqrt{2}$, where a and b are rational numbers to be found. (6)

Solution

We will find e:

$$b^2 = a^2(e^2 - 1) \Rightarrow 9 = 16(e^2 - 1) \Rightarrow e^2 = \tfrac{25}{16} \Rightarrow e = \tfrac{5}{4}.$$

Now,

$$x = \frac{a}{e} = \frac{16}{5},$$

and, $\theta = \frac{\pi}{4}$ gives

$$3y + 2\sqrt{2} \times \frac{16}{5} = 25 \Rightarrow 3y + \frac{32}{5}\sqrt{2} = 25 \Rightarrow \underline{\underline{y = \frac{25}{3} - \frac{32}{15}\sqrt{2}}}.$$