

Dr Oliver Mathematics
Mathematics: Advanced Higher
2022 Paper 1: Non-Calculator
1 hour

The total number of marks available is 36.

You must write down all the stages in your working.

1. (a) Given

$$y = \frac{1 - 3x}{x^2 + 4},$$

(3)

find $\frac{dy}{dx}$.

Simplify your answer.

Solution

$$u = 1 - 3x \Rightarrow \frac{du}{dx} = -3$$

$$v = x^2 + 4 \Rightarrow \frac{dv}{dx} = 2x$$

Now,

$$\begin{aligned} y = \frac{1 - 3x}{x^2 + 4} &\Rightarrow \frac{dy}{dx} = \frac{(x^2 + 4)(-3) - (1 - 3x)(2x)}{(x^2 + 4)^2} \\ &\Rightarrow \frac{dy}{dx} = \frac{(-3x^2 - 12) - (2x - 6x^2)}{(x^2 + 4)^2} \\ &\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 2x - 12}{(x^2 + 4)^2}. \end{aligned}$$

[NB: it does not factorise - $(-2)^2 - 4 \times 3 \times (-12) = 148$]

(b) Given

$$f(x) = \operatorname{cosec} 5x,$$

(2)

find $f'(x)$.

Solution

$$f(x) = \operatorname{cosec} 5x \Rightarrow \underline{\underline{f'(x) = -5 \operatorname{cosec} 5x \cot 5x.}}$$

2. Use Gaussian elimination to solve the following system of equations:

(4)

$$\begin{aligned}x - 2y + z &= 4 \\2x + y - 3z &= 3 \\x - 7y - 4z &= 9.\end{aligned}$$

Solution

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 2 & 1 & -3 & 3 \\ 1 & -7 & -4 & 9 \end{array} \right)$$

Do $R_2 - 2R_1$ and $R_3 - R_1$:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -5 \\ 0 & -5 & -5 & 5 \end{array} \right)$$

Do $R_3 + R_2$:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -5 \\ 0 & 0 & -10 & 0 \end{array} \right)$$

Hence,

$$\underline{\underline{x = 2, y = -1, z = 0.}}$$

3. Given that

(3)

$$z_1 = 5 + 3i \text{ and } z_2 = 6 + 2i,$$

express $z_1 \bar{z}_2$ in the form $a + ib$, where a and b are real numbers.

Solution

$$\begin{array}{r|l} \times & 5 \quad +3i \\ \hline 6 & 30 \quad +18i \\ -2i & -10i \quad +6 \\ \hline \end{array}$$

Hence,

$$z_1 \bar{z}_2 = \underline{\underline{36 + 8i}};$$

so $\underline{\underline{a = 36}}$ and $\underline{\underline{b = 8}}$.

4. A curve is defined by the equation

$$y^3 + 4y = 2xy + 1.$$

- (a) Use implicit differentiation to find an expression for $\frac{dy}{dx}$. (3)

Solution

$$\begin{aligned}y^3 + 4y = 2xy + 1 &\Rightarrow 3y^2 \frac{dy}{dx} + 4 \frac{dy}{dx} = 2y + 2x \frac{dy}{dx} \\&\Rightarrow 3y^2 \frac{dy}{dx} + 4 \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y \\&\Rightarrow \frac{dy}{dx} (3y^2 + 4 - 2x) = 2y \\&\Rightarrow \frac{dy}{dx} = \frac{2y}{3y^2 + 4 - 2x}.\end{aligned}$$

- (b) Find the gradient of the tangent to the curve when $y = -1$. (1)

Solution

$$\begin{aligned}y = -1 &\Rightarrow (-1)^3 + 4(-1) = 2x(-1) + 1 \\&\Rightarrow -1 - 4 = -2x + 1 \\&\Rightarrow -6 = -2x \\&\Rightarrow x = 3\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{2(-1)}{3(-1)^2 + 4 - 2(3)} \\&= \frac{-2}{3 + 4 - 6} \\&= \underline{\underline{-2}}.\end{aligned}$$

- (c) Show that the curve has no stationary point. (2)

Solution

Stationary points occur when $\frac{dy}{dx} = 0$. Now,

$$\begin{aligned}\frac{dy}{dx} = 0 &\Rightarrow \frac{2y}{3y^2 + 4 - 2x} = 0 \\ &\Rightarrow y = 0;\end{aligned}$$

substitute that and we get

$$0 + 0 = 0 + 1$$

which is a contradiction! Hence, the curve has no stationary point.

5. (a) Find, and simplify, the Maclaurin expansion for e^{-4x} , up to and including the term in x^3 . (2)

Solution

Let $f(x) = e^{-4x}$. Then

$$\begin{aligned}f(x) = e^{-4x} &\Rightarrow f(0) = 1 \\ f'(x) = -4e^{-4x} &\Rightarrow f'(0) = -4 \\ f''(x) = 16e^{-4x} &\Rightarrow f''(0) = 16 \\ f'''(x) = -64e^{-4x} &\Rightarrow f'''(0) = -64\end{aligned}$$

and

$$\begin{aligned}f(x) &= 1 + \frac{1}{1!} f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 \dots \\ &= 1 + (-4)x + \frac{1}{2}(16)x^2 + \frac{1}{6}(-64)x^3 + \dots \\ &= \underline{\underline{1 - 4x + 8x^2 + \frac{32}{3}x^3 + \dots}}\end{aligned}$$

- (b) Hence find the first four terms of the Maclaurin expansion of (2)

$$\frac{3 + 2x}{e^{4x}}.$$

Solution

$$\begin{aligned}\frac{3 + 2x}{e^{4x}} &= (3 + 2x)e^{-4x} \\ &= (3 + 2x)(1 - 4x + 8x^2 + \frac{32}{3}x^3 + \dots)\end{aligned}$$

$$\begin{array}{r|rrrr} \times & 1 & -4x & +8x^2 & -\frac{32}{3}x^3 \\ \hline 3 & 3 & -12x & +24x^2 & -32x^3 \\ +2x & +2x & -8x^2 & +16x^3 & \dots \\ \hline \end{array}$$

$$= \underline{\underline{3 - 10x + 16x^2 - 16x^3 + \dots}}$$

6. (a) Consider the statement:

(1)

For all odd numbers n , $n^2 + 4$ is prime.

Find a counterexample to show that the statement is false.

Solution

$$n = 1 \Rightarrow 1^2 + 4 = 5, \text{ which is prime,}$$

$$n = 3 \Rightarrow 3^2 + 4 = 13, \text{ which is prime,}$$

$$n = 5 \Rightarrow 5^2 + 4 = 29, \text{ which is prime,}$$

$$n = 7 \Rightarrow 7^2 + 4 = 53, \text{ which is prime,}$$

$$n = 9 \Rightarrow 9^2 + 4 = 85 = 5 \times 17 \text{ which is not prime .}$$

(b) Prove directly that the difference between the cubes of any two consecutive integers is not divisible by 3.

(3)

Solution

Let $n \in \mathbb{N}$. Then

$$\begin{aligned} (n+1)^3 - n^3 &= (n^3 + 3n^2 + 3n + 1) - n^3 \\ &= 3n^2 + 3n + 1 \\ &= 3(n^2 + n) + 1 \\ &= 3 \times \text{some natural number} + 1; \end{aligned}$$

hence, the number is not divisible by 3.

7. (a) Use the substitution $u = y^2 + 1$, or otherwise, to find the exact value of

(4)

$$\int_0^5 \frac{4y}{\sqrt{y^2 + 1}} dy.$$

Solution

$$\begin{aligned} u = y^2 + 1 &\Rightarrow \frac{du}{dy} = 2y \\ &\Rightarrow du = 2y dy \end{aligned}$$

and

$$\begin{aligned} y = 0 &\Rightarrow u = 1 \\ y = 5 &\Rightarrow u = 26. \end{aligned}$$

Now,

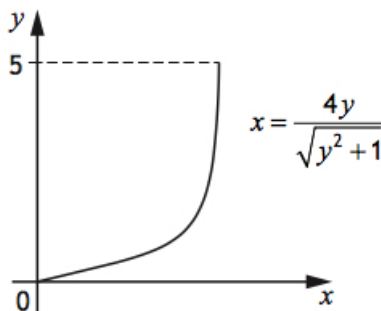
$$\begin{aligned} \int_0^5 \frac{4y}{\sqrt{y^2 + 1}} dy &= \int_1^{26} \frac{2}{\sqrt{u}} du \\ &= \left[4u^{\frac{1}{2}} \right]_{u=1}^{26} \\ &= \underline{\underline{4\sqrt{26} - 4}}. \end{aligned}$$

Student engineers are using a 3D printer to make a model.

Relative to a suitable set of axes, the cross-section of the model is **symmetrical about the y -axis** and is represented **in the first quadrant** by the curve with equation

$$x = \frac{4y}{\sqrt{y^2 + 1}}, \quad 0 \leq y \leq 5,$$

as shown in the diagram.



(b) State the area of the cross-section. (1)

Solution

$$2(4\sqrt{26} - 4) = \underline{\underline{8\sqrt{26} - 8.}}$$

(c) Express (1)

$$\frac{y^2}{y^2 + 1}$$

in the form

$$a + \frac{b}{y^2 + 1},$$

where a and b are real numbers.

Solution

$$\begin{aligned} \frac{y^2}{y^2 + 1} &= \frac{(y^2 + 1) - 1}{y^2 + 1} \\ &= \frac{y^2 + 1}{y^2 + 1} - \frac{1}{y^2 + 1} \\ &= \underline{\underline{1 - \frac{1}{y^2 + 1}}}; \end{aligned}$$

hence, $a = 1$ and $b = -1$.

The curve

$$x = \frac{4y}{\sqrt{y^2 + 1}}, \quad 0 \leq y \leq 5,$$

will be rotated through 2π radians about the y -axis to make the model.

(d) Find the volume of the model.

(4)

Solution

$$\begin{aligned} \text{Volume} &= \int_0^5 \pi \left(\frac{4y}{\sqrt{y^2 + 1}} \right)^2 dy \\ &= \pi \int_0^5 \frac{16y^2}{y^2 + 1} dy \\ &= 16\pi \int_0^5 \left(1 - \frac{1}{y^2 + 1} \right) dy \\ &= 16\pi [y - \arctan y]_{y=0}^5 \\ &= 16\pi \{(5 - \arctan 5) - (0 - 0)\} \\ &= \underline{\underline{16\pi(5 - \arctan 5)}}. \end{aligned}$$