Dr Oliver Mathematics
Mathematics: Advanced Higher
2007 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Express the binomial expansion of (4)

$$\left(x - \frac{2}{x}\right)^4$$

in the form

$$ax^4 + bx^2 + c + \frac{d}{x^2} + \frac{e}{x^4}$$

for integers a, b, c, d, and e.

2. Obtain the derivative of each of the following functions:
   (a) $f(x) = \exp(\sin 2x)$, (3)
   (b) $y = 4^{(x^2+1)}$. (3)

3. Show that (4)

$$z = 3 + 3\mathrm{i}$$

is a root of the equation

$$z^3 - 18z + 108 = 0$$

and obtain the remaining roots of the equation.

4. (a) Express (3)

$$\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)}$$

in partial fractions.

   (b) Given that (3)

$$\int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \ln \frac{m}{n},$$

determine values for the integers m and n.

5. Matrices $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

(a) Find the product $\mathbf{AB}$. (2)

(b) Obtain the determinants of $\mathbf{A}$ and of $\mathbf{AB}$. (2)

(c) Hence, or otherwise, obtain an expression for $\det \mathbf{B}$. (1)

6. (a) Find the Maclaurin series for $\cos x$ as far as the term in x^4. (2)

(b) Deduce the Maclaurin series for $\mathrm{f}(x) = \frac{1}{2}\cos 2x$ as far as the term in x^4. (2)

(c) Hence write down the first three non-zero terms of the series for $\mathrm{f}(3x)$. (1)

7. Use the Euclidean algorithm to find integers p and q such that (4)

$$599p + 53q = 1.$$

8. Obtain the general solution of the equation (6)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \mathrm{e}^{2x}.$$

9. (a) Show that (2)

$$\sum_{r=1}^{n}(4-6r) = n - 3n^2.$$

(b) Hence write down a formula for (1)

$$\sum_{r=1}^{2q}(4-6r).$$

(c) Show that (2)

$$\sum_{r=q+1}^{2n}(4-6r) = q - 9q^2.$$

10. (a) Use the substitution $u = 1 + x^2$ to obtain (5)

$$\int_0^1 \frac{x^3}{(1+x^2)^4}\,\mathrm{d}x.$$

A solid is formed by rotating the curve

$$y = \frac{x^{\frac{3}{2}}}{(1+x^2)^2}$$

between $x = 0$ and $x = 1$ through 360° about the x-axis.

(b) Write down the volume of this solid. (1)

11. (a) Given that (3)

$$|z-2| = |z+\mathrm{i}|,$$

where $z = x + \mathrm{i}y$, show that

$$ax + by + c = 0$$

for suitable values of a, b, and c.

(b) Indicate on an Argand diagram the locus of complex numbers z which satisfy (1)

$$|z-2| = |z+\mathrm{i}|.$$

12. Prove by induction that, for $a > 0$, (5)

$$(1+a)^n \geqslant 1 + na$$

for all positive integers n.

13. A curve is defined by the parametric equations

$$x = \cos 2t,\ y = \sin 2t,\ 0 < t < \tfrac{\pi}{2}.$$

(a) Use parametric differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (3)

(b) Hence find the equation of the tangent when $t = \frac{\pi}{8}$. (2)

(c) Obtain an expression for $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and hence show that (5)

$$\sin 2t\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = k,$$

where k is an integer.
State the value of k.

14. A garden centre advertises young plants to be used as hedging. After planting, the growth G metres (i.e., the increase in height) after t years is modelled by the differential equation

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{25k - G}{25},$$

where k is a constant and $G = 0$ when $t = 0$.

(a) Express G in terms of t and k. (4)

(b) Given that a plant grows 0.6 metres by the end of 5 years, find the value of k correct to 3 decimal places. (2)

(c) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified? (2)

(d) Given that the initial height of the plants was 0.3 m, what is the likely long-term height of the plants? (2)

15. Lines L_1 and L_2 are given by the parametric equations

$$L_1: \quad x = 2 + s,\ y = -s,\ z = 2 - s$$
$$L_2: \quad x = -1 - 2t,\ y = t,\ z = 2 + 3t.$$

(a) Show that L_1 and L_2 do not intersect. (3)

The line L_3 passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both L_1 and L_2.

(b) Obtain parametric equations for L_3. (3)

(c) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1. (3)

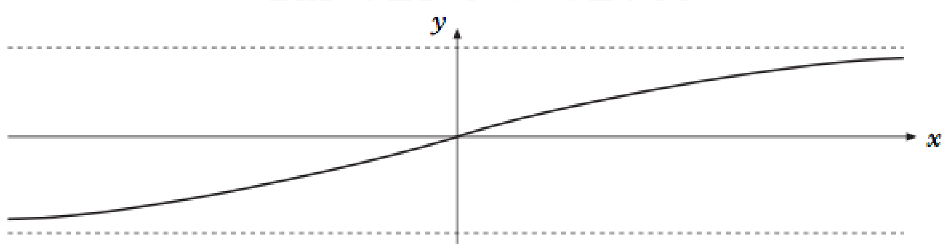
PQ is the shortest distance between the lines L_1 and L_2.

(d) Calculate PQ. (1)

16. (a) The diagram shows part of the graph of (2)

$$\mathrm{f}(x) = \tan^{-1} 2x$$

and its asymptotes.



State the equations of these asymptotes.

(b) Use integration by parts to find the area between $\mathrm{f}(x)$, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$. (5)

(c) Sketch the graph of $y = |f(x)|$ and calculate the area between this graph, the x axis, and the lines $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. (3)