Dr Oliver Mathematics
Applied Mathematics: Integration

The total number of marks available is 103.
You must write down all the stages in your working.

1. (a) Express (3)

$$\frac{x^2+3}{x(1+x^2)}$$

in partial fractions.

Solution

$$\begin{aligned}\frac{x^2+3}{x(1+x^2)} &\equiv \frac{A}{x} + \frac{B+Cx}{1+x^2} \\ &\equiv \frac{A(1+x^2)+x(B+Cx)}{x(1+x^2)}\end{aligned}$$

and so

$$x^2+3 \equiv A(1+x^2)+x(B+Cx).$$

$\underline{x=0}$: $3 = A$.
$\underline{x=1}$: $4 = 2A + B + C \Rightarrow B + C = -2 \quad (1)$.
$\underline{x=-1}$: $4 = 2A - B + C \Rightarrow -B + C = -2 \quad (2)$.
Add (1) + (2):

$$\begin{aligned}2C = -4 &\Rightarrow C = -2 \\ &\Rightarrow B = 0.\end{aligned}$$

Hence,

$$\frac{x^2+3}{x(1+x^2)} \equiv \underline{\underline{\frac{3}{x} - \frac{2x}{1+x^2}}}.$$

(b) Hence obtain (3)

$$\int_{\frac{1}{2}}^{1} \frac{x^2+3}{x(1+x^2)}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}
\int_{\frac{1}{2}}^{1} \frac{x^2+3}{x(1+x^2)}\,\mathrm{d}x &= \int_{\frac{1}{2}}^{1} \left(\frac{3}{x} - \frac{2x}{1+x^2}\right)\,\mathrm{d}x \\
&= \left[3\ln|x| - \ln|1+x^2|\right]_{x=\frac{1}{2}}^{1} \\
&= (3\ln 1 - \ln 2) - \left(3\ln\tfrac{1}{2} - \ln\tfrac{5}{4}\right) \\
&= -\ln 2 - \ln\tfrac{1}{8} + \ln\tfrac{5}{4} \\
&= \ln\left(\frac{\frac{5}{4}}{2\cdot\frac{1}{8}}\right) \\
&= \ln\left(\frac{\frac{5}{4}}{\frac{1}{4}}\right) \\
&= \underline{\underline{\ln 5}}.
\end{aligned}$$

2. Use the substitution $u = 1 + x^2$ to obtain (5)

$$\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}
u = 1 + x^2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \\
&\Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x.
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x &= \tfrac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}\,2x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int \frac{(1+x^2)-1}{\sqrt{1+x^2}}\,2x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u \\
&= \tfrac{1}{2}\int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\mathrm{d}u \\
&= \tfrac{1}{2}\left(\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) + c \\
&= \tfrac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} + c \\
&= \underline{\underline{\tfrac{1}{3}(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + c}}.
\end{aligned}$$

3. (a) Evaluate (4)

$$\int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x}$$

Now,

$$\begin{aligned}\int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x &= \left[\tfrac{1}{2}x\mathrm{e}^{2x}\right]_{x=0}^1 - \tfrac{1}{2}\int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x \\ &= \left(\tfrac{1}{2}\mathrm{e}^2 - 0\right) - \tfrac{1}{2}\left[\tfrac{1}{2}\mathrm{e}^{2x}\right]_{x=0}^1 \\ &= \tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{2}\left(\tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{2}\right) \\ &= \tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{4}\tfrac{1}{2}\mathrm{e}^2 + \tfrac{1}{4} \\ &= \tfrac{1}{4}\mathrm{e}^2 + \tfrac{1}{4} \\ &= \underline{\underline{\tfrac{1}{4}(\mathrm{e}^2 + 1)}}.\end{aligned}$$

(b) Use part (a) to evaluate (3)

$$\int_0^1 x^2\mathrm{e}^{2x}\,\mathrm{d}x.$$

Solution

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x}$$

Now,

$$\begin{aligned}\int_0^1 x^2\mathrm{e}^{2x}\,\mathrm{d}x &= \left[\tfrac{1}{2}x^2\mathrm{e}^{2x}\right]_{x=0}^1 - \int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x \\ &= \left(\tfrac{1}{2}\mathrm{e}^2 - 0\right) - \left(\tfrac{1}{4}\mathrm{e}^2 + \tfrac{1}{4}\right) \\ &= \tfrac{1}{4}\mathrm{e}^2 - \tfrac{1}{4} \\ &= \underline{\underline{\tfrac{1}{4}(\mathrm{e}^2 - 1)}}.\end{aligned}$$

(c) Hence obtain (2)

$$\int_0^1 (3x^2 + 2x)e^{2x}\,dx.$$

Solution

$$\begin{aligned}\int_0^1 (3x^2 + 2x)e^{2x}\,dx &= 3\int_0^1 x^2 e^{2x}\,dx + 2\int_0^1 x e^{2x}\,dx \\ &= \tfrac{3}{4}(e^2 - 1) + \tfrac{1}{2}(e^2 + 1) \\ &= (\tfrac{3}{4}e^2 - \tfrac{3}{4}) + (\tfrac{1}{2}e^2 + \tfrac{1}{2}) \\ &= \tfrac{5}{4}e^2 - \tfrac{1}{4} \\ &= \underline{\underline{\tfrac{1}{4}(5e^2 - 1)}}.\end{aligned}$$

4. Find the exact value of (5)

$$\int_0^{\frac{1}{6}\pi} x \sin 3x\,dx.$$

Solution

$$u = x \Rightarrow \frac{du}{dx} = 1$$
$$\frac{dv}{dx} = \sin 3x \Rightarrow v = -\tfrac{1}{3}\cos 3x$$

$$\begin{aligned}\int x \sin 3x\,dx &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\int \cos 3x\,dx \\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + c\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{1}{6}\pi} x \sin 3x\,dx &= \left[-\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x\right]_{x=0}^{\frac{1}{6}\pi} \\ &= \left(0 + \tfrac{1}{9}\right) - (0 + 0) \\ &= \underline{\underline{\tfrac{1}{9}}}.\end{aligned}$$

5. (a) Express (4)

$$\frac{8}{x(x+2)(x+4)}$$

in partial fractions.

Solution

$$\frac{8}{x(x+2)(x+4)} \equiv \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4}$$
$$\equiv \frac{A(x+2)(x+4) + Bx(x+4) + Cx(x+2)}{x(x+2)(x+4)}$$

and so

$$8 \equiv A(x+2)(x+4) + Bx(x+4) + Cx(x+2).$$

$\underline{x=0}$: $8 = 8A \Rightarrow A = 1$.
$\underline{x=-2}$: $8 = -4B \Rightarrow B = -2$.
$\underline{x=-4}$: $8 = 8C \Rightarrow B = 1$.
Hence,

$$\frac{8}{x(x+2)(x+4)} \equiv \underline{\underline{\frac{1}{x} - \frac{2}{x+2} + \frac{1}{x+4}}}.$$

(b) Calculate the area under the curve (5)

$$y = \frac{8}{x^3 + 6x^2 + 8x}$$

between $x = 1$ and $x = 2$.
Express your answer in the form $\ln \frac{a}{b}$, where a and b are positive integers.

Solution

$$
\begin{aligned}
\int_1^2 \frac{8}{x^3+6x^2+8x}\,\mathrm{d}x &= \int_1^2 \frac{8}{x(x+2)(x+4)}\,\mathrm{d}x \\
&= \int_1^2 \left(\frac{1}{x} - \frac{2}{x+2} + \frac{1}{x+4}\right)\,\mathrm{d}x \\
&= \left[\ln|x| - 2\ln|x+2| + \ln|x+4|\right]_{x=1}^2 \\
&= (\ln 2 - 2\ln 4 + \ln 6) - (\ln 1 - 2\ln 3 + \ln 5) \\
&= \ln 2 - \ln 4^2 + \ln 6 + \ln 3^2 - \ln 5 \\
&= \ln\left(\frac{2\times 6\times 9}{16\times 5}\right) \\
&= \ln\left(\frac{108}{80}\right) \\
&= \underline{\underline{\ln\left(\tfrac{27}{20}\right)}};
\end{aligned}
$$

hence, $\underline{\underline{a = 27}}$ and $\underline{\underline{b = 20}}$.

6. (a) Use integration by parts to show that (2)

$$\int \ln x\,\mathrm{d}x = x\ln x - x + c.$$

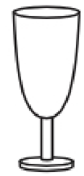
Solution

$$
\begin{aligned}
u = \ln x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \\
\frac{\mathrm{d}v}{\mathrm{d}x} = 1 &\Rightarrow v = x
\end{aligned}
$$

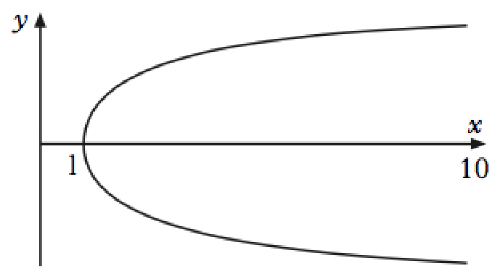
$$
\begin{aligned}
\int \ln x\,\mathrm{d}x &= \int \ln x \cdot 1\,\mathrm{d}x \\
&= x\ln x - \int 1\,\mathrm{d}x \\
&= \underline{\underline{x\ln x - x + c}},
\end{aligned}
$$

as required.

A goblet consists of a bowl and a short stem.



The diagram below shows the bowl section of the goblet (on its side).



The equation of the upper half of the curve is

$$y = 2\sqrt{\ln x}$$

for $1 \leqslant x \leqslant 10$.

(b) Given that the stem has length 1 and the overall height is 10, what is the capacity of the bowl? (4)

Solution

$$\begin{aligned}\text{Volume} &= \int_1^{10} \pi(2\sqrt{\ln x})^2 \,\mathrm{d}x \\ &= 4\pi \int_1^{10} \ln x \,\mathrm{d}x \\ &= 4\pi \left[x \ln x - x\right]_{x=1}^{10} \\ &= 4\pi \left[(10 \ln 10 - 10) - (0 - 1)\right] \\ &= \underline{\underline{4\pi\,(10 \ln 10 - 9)}}.\end{aligned}$$

7. Newton's law of cooling states that a body loses heat at a rate which is proportional to the difference in temperature between itself and its surroundings. So, in a room with constant temperature 22°C, the temperature T°C of a body after a time t minutes

satisfies

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - 22),$$

where k is a negative constant.

(a) Hence show that T can be expressed in the form (4)

$$T = A\mathrm{e}^{kt} + 22$$

for some arbitrary constant A.

Solution

$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} = k(T-22) &\Rightarrow \frac{1}{(T-22)}\,\mathrm{d}T = k\,\mathrm{d}t \\
&\Rightarrow \int \frac{1}{(T-22)}\,\mathrm{d}T = \int k\,\mathrm{d}t \\
&\Rightarrow \ln(T-22) = kt + c \\
&\Rightarrow T - 22 = \mathrm{e}^{kt+c} \\
&\Rightarrow T - 22 = \mathrm{e}^{kt}\,\mathrm{e}^{c} \\
&\Rightarrow T - 22 = A\mathrm{e}^{kt} \text{ (for some constant } A) \\
&\Rightarrow \underline{\underline{T = A\mathrm{e}^{kt} + 22}},
\end{aligned}$$

as required.

In a restaurant, where the temperature remains constant at 22°C, a freshly baked roll, with temperature 82°C, is placed on a cooling tray. After 5 minutes, the temperature of the roll has fallen by 20°C.

(b) (i) Calculate the values of A and k. (2)

Solution

$$\begin{aligned}
t = 0, T = 82 &\Rightarrow 82 = A + 22 \\
&\Rightarrow \underline{\underline{A = 60}} \\
t = 5, T = 62 &\Rightarrow 62 = 60\mathrm{e}^{5k} + 22 \\
&\Rightarrow 40 = 60\mathrm{e}^{5k} \\
&\Rightarrow \mathrm{e}^{5k} = \tfrac{2}{3} \\
&\Rightarrow 5k = \ln\tfrac{2}{3} \\
&\Rightarrow \underline{\underline{k = \tfrac{1}{5}\ln\tfrac{2}{3}}}.
\end{aligned}$$

(ii) Write down an expression for the temperature of the roll after t minutes. (2)

> **Solution**
>
> $$\underline{\underline{T = 60\mathrm{e}^{(\frac{1}{5}\ln\frac{2}{3})t} + 22}}.$$

(iii) Supposing the roll remains uneaten after a further 5 minutes, what will its temperature be? (1)

> **Solution**
>
> $$\begin{aligned} t = 10 \Rightarrow T &= 60\mathrm{e}^{2\ln\frac{2}{3}} + 22 \\ \Rightarrow \underline{\underline{T}} &\underline{\underline{= 48\tfrac{2}{3}{}^{\circ}\mathrm{C}}}. \end{aligned}$$

8. Obtain (4)

$$\int_0^{\frac{1}{3}\pi} \cos^5 x \sin x \, \mathrm{d}x$$

by using the substitution $u = \cos x$ or otherwise.

> **Solution**
>
> We will use the suggested substitution:
>
> $$\begin{aligned} u = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} &= -\sin x \\ \Rightarrow \mathrm{d}u &= -\sin x \, \mathrm{d}x \end{aligned}$$
>
> and
>
> $$\begin{aligned} x = 0 &\Rightarrow u = 1 \\ x = \tfrac{1}{3}\pi &\Rightarrow u = \tfrac{1}{2}. \end{aligned}$$

Now,

$$\begin{aligned}\int_0^{\frac{1}{3}\pi} \cos^5 x \sin x \, \mathrm{d}x &= -\int_0^{\frac{1}{3}\pi} \cos^5 x(-\sin x \, \mathrm{d}x) \\ &= -\int_1^{\frac{1}{2}} u^5 \, \mathrm{d}u \\ &= -\left[\tfrac{1}{6}u^6\right]_{u=1}^{\frac{1}{2}} \\ &= -\left(\tfrac{1}{384} - \tfrac{1}{6}\right) \\ &= \underline{\underline{\tfrac{21}{128}}}.\end{aligned}$$

9. (a) Express (3)

$$\frac{3x}{(x+1)^2}$$

in partial fractions.

Solution

$$\begin{aligned}\frac{3x}{(x+1)^2} &\equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \\ &\equiv \frac{A(x+1)+B}{(x+1)^2}\end{aligned}$$

and so

$$3x \equiv A(x+1) + B.$$

$\underline{x=-1}$: $-3 = B$
$\underline{x=0}$: $0 = A - 3 \Rightarrow A = 3$
Hence,

$$\frac{3x}{(x+1)^2} \equiv \underline{\underline{\frac{3}{(x+1)} - \frac{3}{(x+1)^2}}}.$$

(b) Hence obtain (2)

$$\int \frac{3x}{(x+1)^2} \, \mathrm{d}x.$$

Solution

$$\int \frac{3x}{(x+1)^2}\,\mathrm{d}x = \int \left(\frac{3}{(x+1)} - \frac{3}{(x+1)^2}\right)\mathrm{d}x$$
$$= \underline{\underline{3\ln|x+1| + 3(x+1)^{-1} + c}}.$$

10. Use the substitution $u = \ln x$ to obtain (4)

$$\int \frac{2}{x\ln x}\,\mathrm{d}x,$$

where $x > 1$.

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\Rightarrow \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$

and

$$\int \frac{2}{x\ln x}\,\mathrm{d}x = \int \frac{2}{u}\,\mathrm{d}u$$
$$= 2\ln u + c$$
$$= \underline{\underline{2\ln(\ln x) + c}}.$$

11. (a) Express (3)

$$\frac{1}{x^2+x}$$

in partial fractions, where x is neither 0 nor -1.

Solution

$$\begin{aligned}\frac{1}{x^2+x} &\equiv \frac{1}{x(x+1)} \\ &\equiv \frac{A}{x} + \frac{B}{x+1} \\ &\equiv \frac{A(x+1)+Bx}{x(x+1)}\end{aligned}$$

and so

$$1 \equiv A(x+1) + Bx.$$

$\underline{x=0}$: $1 = A$.
$\underline{x=-1}$: $1 = -B \Rightarrow B = -1$.
Hence,

$$\frac{1}{x^2+x} \equiv \underline{\underline{\frac{1}{x} - \frac{1}{x+1}}}.$$

A region is enclosed by the curve with equation

$$y = \frac{1}{\sqrt{x^2+x}},$$

the x-axis, and the lines $x = 1$ and $x = 3$.

(b) Calculate the volume of the solid of revolution formed by rotating this region through 360° about the x-axis. (4)

Solution

$$\begin{aligned}\text{Volume} &= \int_1^3 \pi \left(\frac{1}{\sqrt{x^2+x}}\right)^2 \, \mathrm{d}x \\ &= \pi \int_1^3 \frac{1}{x^2+x} \, \mathrm{d}x \\ &= \pi \int_1^3 \left(\frac{1}{x} - \frac{1}{x+1}\right) \mathrm{d}x \\ &= \pi \left[\ln|x| - \ln|x+1|\right]_{x=1}^{3} \\ &= \pi \left[(\ln 3 - \ln 4) - (0 - \ln 2)\right] \\ &= \pi \ln\left(\frac{3 \times 2}{4}\right) \\ &= \underline{\underline{\pi \ln \tfrac{3}{2}}}.\end{aligned}$$

12. Use integration by parts to obtain (4)

$$\int \frac{\ln x}{x^3}\,\mathrm{d}x,$$

where $x > 0$.

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = x^{-1}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{-3} \Rightarrow v = -\tfrac{1}{2}x^{-2}$$

Now,

$$\begin{aligned}\int \frac{\ln x}{x^3}\,\mathrm{d}x &= \int \left(\ln x \cdot \frac{1}{x^3}\right)\,\mathrm{d}x \\ &= -\tfrac{1}{2}x^{-2}\ln x - \int (x^{-1})(-\tfrac{1}{2}x^{-2})\,\mathrm{d}x \\ &= -\tfrac{1}{2}x^{-2}\ln x + \tfrac{1}{2}\int x^{-3}\,\mathrm{d}x \\ &= \underline{\underline{-\tfrac{1}{2}x^{-2}\ln x - \tfrac{1}{4}x^{-2} + c}}.\end{aligned}$$

13. Find the exact value of (5)

$$\int_0^{2\pi} x\sin 3x\,\mathrm{d}x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 3x \Rightarrow v = -\tfrac{1}{3}\cos 3x$$

Now,

$$\begin{aligned}\int_0^{2\pi} x\sin 3x\,\mathrm{d}x &= \left[-\tfrac{1}{3}x\cos 3x\right]_{x=0}^{2\pi} + \tfrac{1}{3}\int_0^{2\pi}\cos 3x\,\mathrm{d}x \\ &= \left(-\tfrac{2}{3}\pi - 0\right) + \tfrac{1}{3}\left[\tfrac{1}{3}\sin 3x\right]_{x=0}^{2\pi} \\ &= -\tfrac{2}{3}\pi + \tfrac{1}{3}(0-0) \\ &= \underline{\underline{-\tfrac{2}{3}\pi}}.\end{aligned}$$

14. A flu-like virus starts to spread through the 20 000 inhabitants of Dumbarton.
The situation can be modelled by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000},$$

where N is the number of people infected after t days and $0 < N < 20\,000$.

(a) How many people are infected when the infection is spreading most rapidly? (1)

Solution

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000} &\Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} = \tfrac{1}{10\,000}(20\,000N - N^2) \\ &\Rightarrow \frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \tfrac{1}{10\,000}(20\,000 - 2N) \\ &\Rightarrow \frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \tfrac{1}{5\,000}(10\,000 - N)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = 0 &\Rightarrow \tfrac{1}{5\,000}(10\,000 - N) \\ &\Rightarrow \underline{\underline{N = 10\,000}}.\end{aligned}$$

(b) Express (5)

$$\frac{10\,000}{N(20\,000 - N)}$$

in partial fractions and show that

$$\ln\left(\frac{N}{20\,000 - N}\right) = 2t + c,$$

for some constant c.

Solution

$$\frac{10\,000}{N(20\,000-N)} \equiv \frac{A}{N} + \frac{B}{20\,000-N}$$
$$\equiv \frac{A(20\,000-N)+BN}{N(20\,000-N)}$$

which means

$$10\,000 \equiv A(20\,000-N)+BN.$$

$\underline{N=0}$: $10\,000 = 20\,000A \Rightarrow A = \frac{1}{2}$.
$\underline{N=20\,000}$: $10\,000 = 20\,000B \Rightarrow B = \frac{1}{2}$.
Hence,

$$\frac{10\,000}{N(20\,000-N)} \equiv \underline{\underline{\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}}}$$

and

$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000-N)}{10\,000} &\Rightarrow \frac{10\,000}{N(20\,000-N)}\,\mathrm{d}N = \mathrm{d}t \\
&\Rightarrow \left(\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}\right)\mathrm{d}N = \mathrm{d}t \\
&\Rightarrow \int \left(\frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{20\,000-N}\right)\mathrm{d}N = \int \mathrm{d}t \\
&\Rightarrow \tfrac{1}{2}\ln N - \tfrac{1}{2}\ln(20\,000-N) = t + a \\
&\Rightarrow \tfrac{1}{2}\ln\left(\frac{N}{20\,000-N}\right) = t + a \\
&\Rightarrow \underline{\underline{\ln\left(\frac{N}{20\,000-N}\right) = 2t + c,}}
\end{aligned}$$

where $c = 2a$.

Initially there were 100 people infected.

(c) Show that (4)

$$N = \frac{20\,000\,\mathrm{e}^{2t}}{199+\mathrm{e}^{2t}}.$$

Solution

$$t = 0, N = 100 \Rightarrow \ln\left(\frac{100}{20\,000 - 100}\right) = 0 + c$$
$$\Rightarrow c = \ln \tfrac{1}{199}$$

and

$$\begin{aligned}
\ln\left(\frac{N}{20\,000 - N}\right) = 2t + \ln \tfrac{1}{199} &\Rightarrow \frac{N}{20\,000 - N} = \mathrm{e}^{2t + \ln \frac{1}{199}} \\
&\Rightarrow \frac{N}{20\,000 - N} = \mathrm{e}^{2t}\,\mathrm{e}^{\ln \frac{1}{199}} \\
&\Rightarrow \frac{N}{20\,000 - N} = \tfrac{1}{199}\mathrm{e}^{2t} \\
&\Rightarrow N = \tfrac{1}{199}\mathrm{e}^{2t}(20\,000 - N) \\
&\Rightarrow N = \tfrac{20\,000}{199}\mathrm{e}^{2t} - \tfrac{1}{199}\mathrm{e}^{2t}N \\
&\Rightarrow N + \tfrac{1}{199}\mathrm{e}^{2t}N = \tfrac{20\,000}{199}\mathrm{e}^{2t} \\
&\Rightarrow \tfrac{1}{199}N\left(199 + \mathrm{e}^{2t}\right) = \tfrac{20\,000}{199}\mathrm{e}^{2t} \\
&\Rightarrow N\left(199 + \mathrm{e}^{2t}\right) = 20\,000\mathrm{e}^{2t} \\
&\Rightarrow \underline{\underline{N = \frac{20\,000\mathrm{e}^{2t}}{199 + \mathrm{e}^{2t}}}},
\end{aligned}$$

as required.

15. Find the general solution, in the form $y = \mathrm{f}(x)$, of the differential equation (6)

$$\frac{1}{\cos x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = \tan x,\ 0 < x < \pi.$$

Solution

$$\begin{aligned}
\frac{1}{\cos x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = \tan x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y \cos x \tan x = \cos x \tan x \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x = \sin x
\end{aligned}$$

$$
\begin{aligned}
\text{IF} &= \mathrm{e}^{\int \sin x \, \mathrm{d}x} \\
&= \mathrm{e}^{-\cos x}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \mathrm{e}^{-\cos x}\frac{\mathrm{d}y}{\mathrm{d}x} + y\mathrm{e}^{-\cos x}\sin x = \mathrm{e}^{-\cos x}\sin x \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-\cos x}y) = \sin x \, \mathrm{e}^{-\cos x} \\
&\Rightarrow \mathrm{e}^{-\cos x}y = \int \sin x \, \mathrm{e}^{-\cos x} \, \mathrm{d}x \\
&\Rightarrow \mathrm{e}^{-\cos x}y = \mathrm{e}^{-\cos x} + c \\
&\Rightarrow \underline{\underline{y = 1 + c\,\mathrm{e}^{\cos x}}}.
\end{aligned}
$$

16. (a) Express (3)

$$\frac{1}{1-y^2}$$

in partial fractions.

Solution

$$
\left.\begin{array}{ll} \text{add to:} & 0 \\ \text{multiply to:} & -1 \end{array}\right\} -1, +1
$$

$$
\begin{aligned}
\frac{1}{1-y^2} &\equiv \frac{1}{(1+y)(1-y)} \\
&\equiv \frac{A}{1+y} + \frac{B}{1-y} \\
&\equiv \frac{A(1-y) + B(1+y)}{(1+y)(1-y)}
\end{aligned}
$$

which means

$$1 \equiv A(1-y) + B(1+y).$$

$\underline{y = 1}$: $1 = 2B \Rightarrow B = \frac{1}{2}$.
$\underline{y = -1}$: $1 = 2A \Rightarrow A = \frac{1}{2}$.
Hence,

$$\frac{1}{1-y^2} \equiv \underline{\underline{\frac{\frac{1}{2}}{1+y} + \frac{\frac{1}{2}}{1-y}}}.$$

(b) Use the substitution $u = \sqrt{1-x}$ to obtain (6)

$$\int \frac{1}{x\sqrt{1-x}}\,\mathrm{d}x,\ 0 < x < 1.$$

Solution

$$\begin{aligned} u = \sqrt{1-x} &\Rightarrow u = (1-x)^{\frac{1}{2}} \\ &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \\ &\Rightarrow \mathrm{d}u = -\tfrac{1}{2}(1-x)^{-\frac{1}{2}}\,\mathrm{d}x \\ &\Rightarrow \mathrm{d}u = -\frac{1}{2\sqrt{1-x}}\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} u = \sqrt{1-x} &\Rightarrow u^2 = 1-x \\ &\Rightarrow x = 1-u^2. \end{aligned}$$

Finally,

$$\begin{aligned} \int \frac{1}{x\sqrt{1-x}}\,\mathrm{d}x &= \int \frac{2}{x(2\sqrt{1-x})}\,\mathrm{d}x \\ &= \int \frac{-2}{1-u^2}\,\mathrm{d}u \\ &= \int -2\left(\frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}\right)\mathrm{d}u \\ &= \int \left(-\frac{1}{1+u} - \frac{1}{1-u}\right)\mathrm{d}u \\ &= -\ln(1+u) + \ln(1-u) + c \\ &= \ln\left(\frac{1-u}{1+u}\right) + c \\ &= \underline{\underline{\ln\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right) + c.}} \end{aligned}$$