Dr Oliver Mathematics
Mathematics: Higher
2012 Paper 2: Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. Functions f and g are defined on the set of real numbers by

$$\mathrm{f}(x) = x^2 + 3,$$
$$\mathrm{g}(x) = x + 4.$$

(a) Find expressions for: (3)

(i) $\mathrm{f}(\mathrm{g}(x))$,

Solution

$$\begin{aligned}\mathrm{f}(\mathrm{g}(x)) &= \mathrm{f}(x+4) \\ &= (x+4)^2 + 3 \\ &= (x^2 + 8x + 16) + 3 \\ &= \underline{\underline{x^2 + 8x + 19}}.\end{aligned}$$

(ii) $\mathrm{g}(\mathrm{f}(x))$.

Solution

$$\begin{aligned}\mathrm{g}(\mathrm{f}(x)) &= \mathrm{g}(x^2+3) \\ &= (x^2 + 3) + 4 \\ &= \underline{\underline{x^2 + 7}}.\end{aligned}$$

(b) Show that (3)

$$\mathrm{f}(\mathrm{g}(x)) + \mathrm{g}(\mathrm{f}(x)) = 0$$

has no real roots.

Solution

$$
\begin{aligned}
f(g(x)) + g(f(x)) = 0 &\Rightarrow (x^2 + 8x + 19) + (x^2 + 7) = 0 \\
&\Rightarrow 2x^2 + 8x + 26 = 0 \\
&\Rightarrow x^2 + 4x + 13 = 0 \\
&\Rightarrow x^2 + 4x + 4 = -9 \\
&\Rightarrow (x + 2)^2 = -9,
\end{aligned}
$$

and so the solutions of this equation have no real roots.

2. Relative to a suitable set of coordinate axes, Diagram 1 shows the line

$$2x - y + 5 = 0$$

intersecting the circle

$$x^2 + y^2 - 6x - 2y - 30 = 0$$

at the points P and Q.

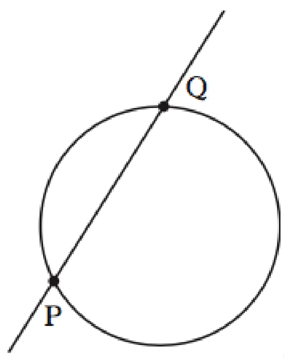


Diagram 1

(a) Find the coordinates of P and Q. (6)

Solution

$$
\begin{aligned}
2x - y + 5 = 0 &\Rightarrow y = 2x + 5 \\
&\Rightarrow x^2 + (2x + 5)^2 - 6x - 2(2x + 5) - 30 = 0 \\
&\Rightarrow x^2 + (4x^2 + 20x + 25) - 6x - 4x - 10 - 30 = 0 \\
&\Rightarrow 5x^2 + 10x - 15 = 0 \\
&\Rightarrow x^2 + 2x - 3 = 0 \\
&\Rightarrow (x + 3)(x - 1) = 0 \\
&\Rightarrow x = -3 \text{ or } x = 1 \\
&\Rightarrow y = -1 \text{ or } y = 7;
\end{aligned}
$$

hence, $\underline{\underline{P(-3, -1)}}$ and $\underline{\underline{Q(1, 7)}}$.

Diagram 2 shows the circle from (a) and a second congruent circle, which also passes through P and Q.

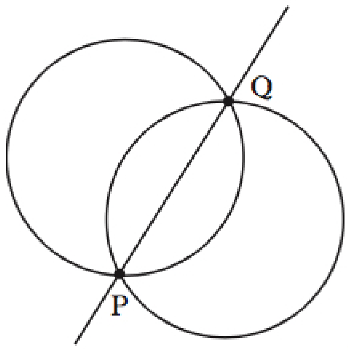


Diagram 2

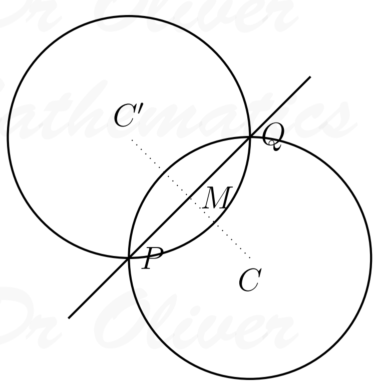
(b) Determine the equation of this second circle. (6)

Solution

$$\begin{aligned} x^2 + y^2 - 6x - 2y - 30 = 0 &\Rightarrow x^2 - 6x + y^2 - 2y = 30 \\ &\Rightarrow x^2 - 6x + 9 + y^2 - 2y + 1 = 30 + 9 + 1 \\ &\Rightarrow (x-3)^2 + (y-1)^2 = 40, \end{aligned}$$

and so the original circle has centre $C(3, 1)$ and radius $\sqrt{40}$.
Now, the midpoint of PQ be M and M is

$$\left(\frac{-3+1}{2}, \frac{-1+7}{2}\right) = (-1, 3).$$



$C'MC$ is a straight line with $C'M = MC$. Next, the second circle has radius $(-5, 5)$ (why?) and radius $\sqrt{40}$.
Finally, the equation of this second circle is

$$\underline{\underline{(x+5)^2 + (y-5)^2 = 40}}.$$

3. A function f is defined on the domain $0 \leqslant x \leqslant 3$ by (7)

$$f(x) = x^3 - 2x^2 - 4x + 6.$$

Determine the maximum and minimum values of f.

Solution

$$f(x) = x^3 - 2x^2 - 4x + 6 \Rightarrow f'(x) = 3x^2 - 4x - 4$$

and

$$f'(x) = 0 \Rightarrow 3x^2 - 4x - 4 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & (+3)\times(-4) = -12 \end{array}\right\} -6,\ +2$$

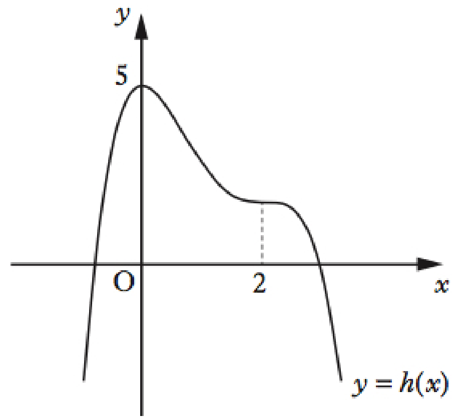
$$\begin{aligned}
&\Rightarrow 3x^2 - 6x + 2x - 4 = 0\\
&\Rightarrow 3x(x-2) + 2(x-2) = 0\\
&\Rightarrow (3x+2)(x-2) = 0\\
&\Rightarrow 3x+2 = 0 \text{ or } x-2 = 0\\
&\Rightarrow 3x+2 = 0 \text{ or } x-2 = 0\\
&\Rightarrow x = -\tfrac{2}{3} \text{ or } x = 2;
\end{aligned}$$

now, we can eliminate $x = -\frac{2}{3}$ (why?) and concentrate on $x = 0$, $x = 2$, and $x = 3$:

$$\begin{aligned}
x = 0 &\Rightarrow y = 6\\
x = 2 &\Rightarrow y = -2\\
x = 3 &\Rightarrow y = 3;
\end{aligned}$$

hence, the maximum value of f is $\underline{\underline{6}}$ and the minimum value of f is $\underline{\underline{-2}}$.

4. The diagram below shows the graph of a quartic $y = \mathrm{h}(x)$, with stationary points at $x = 0$ and $x = 2$.

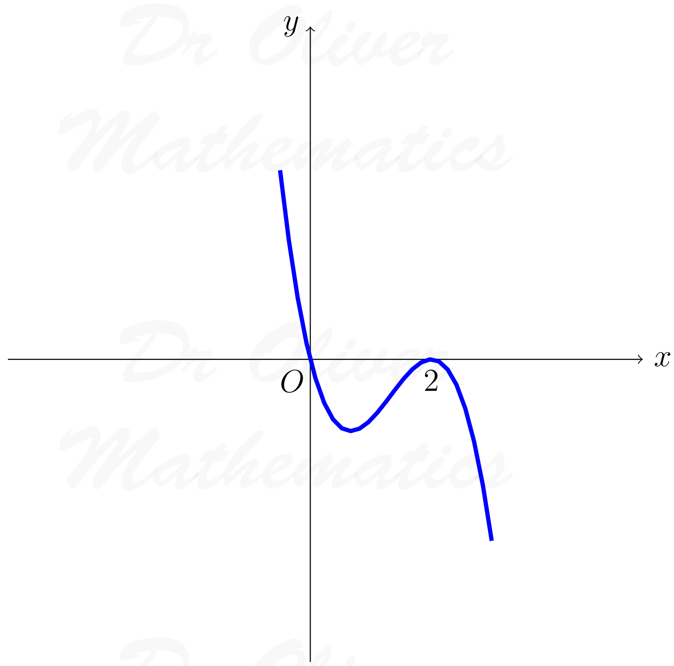


On separate diagrams sketch the graphs of:

(a) $y = \mathrm{h}'(x)$, (3)

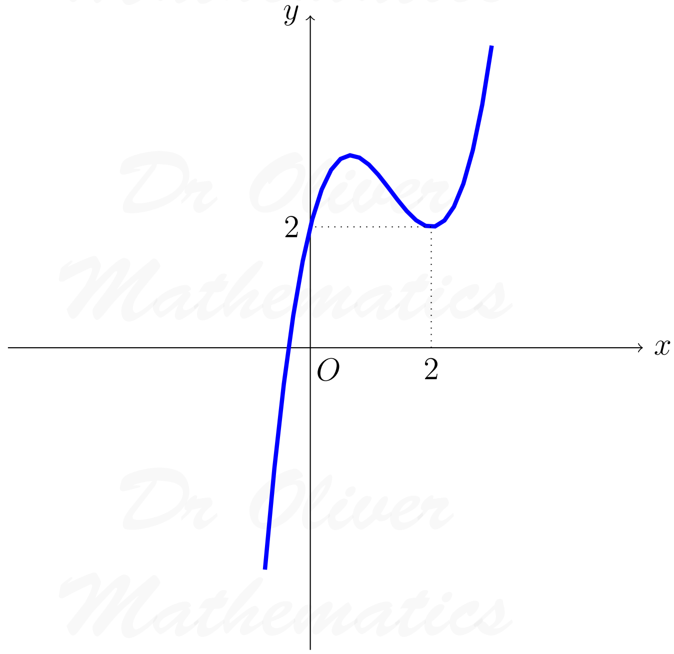
Solution

Stationary points: $\mathrm{h}'(x) = 0$ when $x = 0$ or $x = 2$ (repeated - why?) and so $y = \mathrm{h}'(x)$ looks like this:



(b) $y = 2 - \mathrm{h}'(x)$ (3)

Solution



5. A is the point $(3, -3, 0)$, B is $(2, -3, 1)$, and C is $(4, k, 0)$.

(a) (i) Express $\overrightarrow{BA}$ and $\overrightarrow{BC}$ in component form. (7)

Solution

$$\begin{aligned}\overrightarrow{BA} &= \overrightarrow{BO} + \overrightarrow{OA} \\ &= \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} \\ &= \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ k \\ 0 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 2 \\ 3+k \\ -1 \end{pmatrix}}}.\end{aligned}$$

(ii) Show that

$$\cos\angle ABC = \frac{3}{\sqrt{2(k^2+6k+14)}}.$$

Solution
Now,

$$\begin{aligned}|\overrightarrow{BA}| &= \sqrt{1^2+0+(-1)^2} \\ &= \sqrt{2}\end{aligned}$$

and

$$\begin{aligned}|\overrightarrow{BC}| &= \sqrt{2^2+(3+k)^2+(-1)^2} \\ &= \sqrt{4+(9+6k+k^2)+1} \\ &= \sqrt{k^2+6k+14}.\end{aligned}$$

Finally,

$$\begin{aligned}& \overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BA}|\cos\angle ABC \\ \Rightarrow\ & 2+0+1 = \sqrt{2}\cdot\sqrt{k^2+6k+14}\cdot\cos\angle ABC \\ \Rightarrow\ & \underline{\underline{\cos\angle ABC = \frac{3}{\sqrt{2(k^2+6k+14)}}}},\end{aligned}$$

as required.

(b) If angle $ABC = 30^\circ$, find the possible values of k. (5)

Solution

$$
\begin{aligned}
\cos 30^\circ = \frac{3}{\sqrt{2(k^2+6k+14)}} &\Rightarrow \frac{\sqrt{3}}{2} = \frac{3}{\sqrt{2(k^2+6k+14)}} \\
&\Rightarrow \sqrt{6(k^2+6k+14)} = 6 \\
&\Rightarrow 6(k^2+6k+14) = 36 \\
&\Rightarrow k^2+6k+14 = 6 \\
&\Rightarrow k^2+6k+8 = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll} \text{add to:} & +6 \\ \text{multiply to:} & +8 \end{array}\right\} +4,\ +2
$$

$$
\begin{aligned}
&\Rightarrow (k+4)(k+2) = 0 \\
&\Rightarrow k+4 = 0 \text{ or } k+2 = 0 \\
&\Rightarrow \underline{\underline{k = -4 \text{ or } k = -2}}.
\end{aligned}
$$

6. For $0 < x < \frac{1}{2}\pi$, sequences can be generated using the recurrence relation

$$u_{n+1} = (\sin x)u_n + \cos 2x, \quad \text{with } u_0 = 1.$$

(a) Why do these sequences have a limit? (2)

Solution

Since $0 < x < \frac{1}{2}\pi$, $0 < \sin x < 1$ and $-1 < \cos 2x < 1$, the sequence has a limit.

(b) The limit of one sequence generated by this recurrence relation is $\frac{1}{2}\sin x$. Find the value(s) of x. (7)

Solution

Suppose the limit was $\frac{1}{2}\sin x$. Then

$$
\begin{aligned}
\tfrac{1}{2}\sin x = (\sin x)(\tfrac{1}{2}\sin x) + \cos 2x &\Rightarrow \tfrac{1}{2}\sin x = \tfrac{1}{2}\sin^2 x + (1 - 2\sin^2 x) \\
&\Rightarrow \tfrac{1}{2}\sin x = -\tfrac{3}{2}\sin^2 x + 1 \\
&\Rightarrow \tfrac{3}{2}\sin^2 x + \tfrac{1}{2}\sin x - 1 = 0 \\
&\Rightarrow 3\sin^2 x + \sin x - 2 = 0
\end{aligned}
$$

$$\left.\begin{array}{ll}\text{add to:} & +1 \\ \text{multiply to:} & (+3)\times(-2) = -6\end{array}\right\} +3,\ -2$$

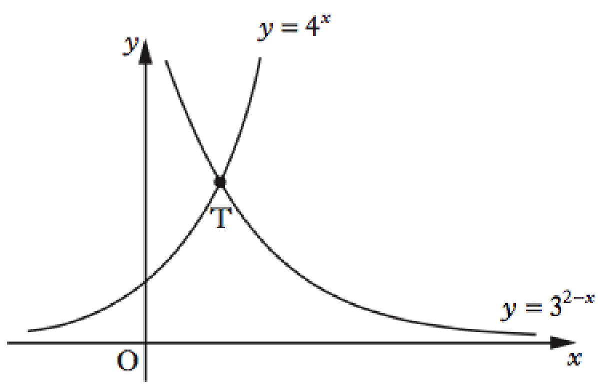
$$\begin{aligned}
&\Rightarrow 3\sin^2 x + 3\sin x - 2\sin x - 2 = 0 \\
&\Rightarrow 3\sin x(\sin x + 1) - 2(\sin x - 1) = 0 \\
&\Rightarrow (3\sin x - 2)(\sin x + 1) = 0 \\
&\Rightarrow 3\sin x - 2 = 0 \text{ or } \sin x + 1 = 0 \\
&\Rightarrow \sin x = \tfrac{2}{3} \text{ or } \sin x = -1.
\end{aligned}$$

Now, $\sin x = -1$ does not have a solution in this range ($\sin x = -1 \Rightarrow x = \frac{3}{2}\pi$) and hence

$$\begin{aligned}
\sin x = \tfrac{2}{3} &\Rightarrow x = 0.729\,727\,656\,2 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{x = 0.730 \text{ (3 sf)}}}.
\end{aligned}$$

7. The diagram shows the curves with equations

$$y = 4^x \text{ and } y = 3^{2-x}.$$



The graphs intersect at the point T.

(a) Show that the x-coordinate of T can be written in the form (6)

$$\frac{\log_a p}{\log_a q},$$

for all $a > 1$.

Solution

For all $a > 1$,

$$\begin{aligned} 4^T = 3^{2-T} &\Rightarrow \log_a 4^T = \log_a 3^{2-T} \\ &\Rightarrow T \log_a 4 = (2-T) \log_a 3 \\ &\Rightarrow T \log_a 4 = 2 \log_a 3 - T \log_a 3 \\ &\Rightarrow T \log_a 4 + T \log_a 3 = \log_a 3^2 \\ &\Rightarrow T(\log_a 4 + \log_a 3) = \log_a 9 \\ &\Rightarrow T \log_a 12 = \log_a 9 \\ &\Rightarrow T = \underline{\underline{\frac{\log_a 9}{\log_a 12}}}; \end{aligned}$$

hence, $\underline{\underline{p = 9}}$ and $\underline{\underline{q = 12}}$.

(b) Calculate the y-coordinate of T, giving your answer to 1 decimal place. (2)

Solution

$$\begin{aligned} y &= 4^{\frac{\log_a 9}{\log_a 12}} \\ &= 4^{\log_{12} 9} \\ &= 3.406\,892\,521 \text{ (FCD)} \\ &= \underline{\underline{3.4 \text{ (1 dp)}}}. \end{aligned}$$