Dr Oliver Mathematics
Further Mathematics
Further Vectors
Past Examination Questions

This booklet consists of 26 questions across a variety of examination topics.
The total number of marks available is 250.

1. The points A, B, and C lie on the plane Π and, relative to a fixed origin O, they have position vectors

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k},\ \mathbf{b} = -\mathbf{i} + 2\mathbf{j},\ \text{and } \mathbf{c} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k},$$

respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (4)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix} = (9-8)\mathbf{i} - (-12+8)\mathbf{j} + (8-6)\mathbf{k}$$
$$= \underline{\underline{\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}}.$$

(b) Find an equation of Π in the form $\mathbf{r}.\mathbf{n} = p$. (2)

Solution

$$\begin{aligned} \mathbf{r}.\mathbf{n} &= (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}).(3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \\ &= 3 - 4 + 8 \\ &= \underline{\underline{7}}. \end{aligned}$$

The point D has position vector $5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(c) Calculate the volume of the tetrahedron $ABCD$. (4)

Solution

$$\begin{vmatrix} 2 & 3 & -1 \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix} = 2(9-8) - 3(-12+8) - (8-6)$$
$$= 2 + 12 - 2$$
$$= 12$$

and
$$\text{volume of the tetrahedron} = \tfrac{1}{6}\,|\mathbf{a}.(\mathbf{b}\times\mathbf{c})| = \tfrac{1}{6}\times 12 = \underline{\underline{2}}.$$

2. (a) Explain why, for any two vectors $\mathbf{a}$ and $\mathbf{a}$, $\mathbf{a}.\mathbf{b}\times\mathbf{a} = 0$. (2)

Solution

It is because there is 90° between $\mathbf{a}$ and $\mathbf{b}\times\mathbf{a}$ and
$$\mathbf{a}.(\mathbf{b}\times\mathbf{a}) = |\mathbf{a}||\mathbf{b}\times\mathbf{a}|\cos 90^\circ = \underline{\underline{0}}.$$

(b) Given vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}$, where $\mathbf{a}\neq\mathbf{0}$ and $\mathbf{b}\neq\mathbf{c}$, show that (2)
$$\mathbf{b}-\mathbf{c} = \lambda\mathbf{a},$$
where λ is a scalar.

Solution

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c} &\Rightarrow \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c} = 0\\ &\Rightarrow \mathbf{a}\times(\mathbf{b}-\mathbf{c}) = 0;\end{aligned}$$

as $\mathbf{a}\neq\mathbf{0}$ and $\mathbf{b}\neq\mathbf{c}$, $\mathbf{a}$ is parallel to $\mathbf{b}-\mathbf{c}$ and so
$$\mathbf{b}-\mathbf{c} = \underline{\underline{\lambda\mathbf{a}}},$$

where λ is a scalar.

3. The line l_1 has equation
$$\mathbf{r} = \mathbf{i} + 6\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{k})$$
and the line l_2 has equation
$$\mathbf{r} = 3\mathbf{i} + p\mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}),$$
where p is a constant. The plane Π_1 contains l_1 and l_2.

(a) Find a vector which is normal to Π_1. (2)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = (0+6)\mathbf{i} - (2-3)\mathbf{j} + (-4-0)\mathbf{k}$$
$$= \underline{\underline{6\mathbf{i} + \mathbf{j} - 4\mathbf{k}}}.$$

(b) Show that an equation for Π_1 is $6x + y - 4z = 16$. (2)

Solution

$$\begin{aligned} &\mathbf{r}.(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = (\mathbf{i} + 6\mathbf{j} - \mathbf{k}).(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \\ \Rightarrow\quad & 6x + y - 4z = 6 + 6 + 4 \\ \Rightarrow\quad & \underline{\underline{6x + y - 4z = 16}}, \end{aligned}$$

as required.

(c) Find the value of p. (1)

Solution

$\underline{\underline{p = -2}}$.

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2$.

(d) Find an equation for the line of intersection of Π_1 and Π_2, giving your answer in the form (5)

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}.$$

Solution

Well, the direction of this line is perpendicular to both normals and this is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & -4 \\ 1 & 2 & 1 \end{vmatrix} = (1+8)\mathbf{i} - (6+4)\mathbf{j} + (12-1)\mathbf{k}$$
$$= 9\mathbf{i} - 10\mathbf{j} + 11\mathbf{k}.$$

Now, the two lines are

$$6x + y - 4z = 16 \text{ and } x + 2y + z = 2;$$

put $z = 0$:

$$\begin{aligned} & 6x + y = 16,\ x + 2y = 2 \\ \Rightarrow\ & 6x + y = 16,\ 6x + 12y = 12 \\ \Rightarrow\ & -11y = 4 \\ \Rightarrow\ & y = -\tfrac{4}{11} \\ \Rightarrow\ & x = \tfrac{30}{11}. \end{aligned}$$

Hence, we have

$$\underline{\underline{[\mathbf{r} - (\tfrac{30}{11}\mathbf{i} - \tfrac{4}{11}\mathbf{j})] \times (9\mathbf{i} - 10\mathbf{j} + 11\mathbf{k}) = \mathbf{0}}}.$$

4. The plane Π passes through the points

$$P(-1, 3, -2),\ Q(4, -1, -1), \text{ and } R(3, 0, c),$$

where c is a constant.

(a) Find, in terms of c, $\overrightarrow{RP} \times \overrightarrow{RQ}$. (3)

Solution

$$\begin{aligned} & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -2-c \\ 1 & -1 & -1-c \end{vmatrix} \\ = \ & [3(-1-c) + (-2-c)]\mathbf{i} - [-4(-1-c) - (-2-c)]\mathbf{j} + [4-3]\mathbf{k} \\ = \ & (-3 - 3c - 2 - c)\mathbf{i} - (4 + 4c + 2 + c)\mathbf{j} + \mathbf{k} \\ = \ & \underline{\underline{(-5-4c)\mathbf{i} - (6+5c)\mathbf{j} + \mathbf{k}}}. \end{aligned}$$

Given that $\overrightarrow{RP} \times \overrightarrow{RQ} = 3\mathbf{i} + d\mathbf{j} + \mathbf{k}$, where d is a constant,

(b) find the value of c and show that $d = 4$, (2)

Solution

$$-5 - 4c = 3 \Rightarrow 4c = -8 \Rightarrow \underline{\underline{c = -2}}$$

and

$$d = -6 - 5 \times (-2) = -6 + 10 = \underline{\underline{4}}.$$

(c) find an equation of Π in the form $\mathbf{r}.\mathbf{n} = p$, where p is a constant. (3)

Solution

$$\begin{aligned}\mathbf{r}.(3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) &= (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}).(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \\ &= -3 + 12 - 2 \\ &= \underline{\underline{7}}.\end{aligned}$$

The point S has position vector $\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$. The point S' is the image of S under reflection in Π.

(d) Find the position vector of S'. (5)

Solution

The equation of normal to plane through S is

$$\mathbf{r} = (\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}) + t(3\mathbf{i} + 4\mathbf{j} + \mathbf{k}).$$

We need to determine where the plane and the equation of normal meets:

$$\begin{aligned}& [(1 + 3t)\mathbf{i} + (5 + 4t)\mathbf{j} + (10 + t)\mathbf{k}].(3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = 7 \\ \Rightarrow\ & 3(1 + 3t) + 4(5 + 4t) + (10 + t) = 7 \\ \Rightarrow\ & 3 + 9t + 20 + 16t + 10 + t = 7 \\ \Rightarrow\ & 26t = -26 \\ \Rightarrow\ & t = -1,\end{aligned}$$

and S' has position vector

$$(\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}) - 2(3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = \underline{\underline{-5\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}}}.$$

5. The points A, B, and C lie on the plane Π_1 and, relative to a fixed origin O, they have position vectors

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k},\ \mathbf{b} = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k},\ \text{and } \mathbf{c} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

(a) Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$. (4)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 4 & -5 & -1 \end{vmatrix} = (0-15)\mathbf{i} - (-2+12)\mathbf{j} + (-10-0)\mathbf{k}$$
$$= \underline{\underline{-15\mathbf{i} - 10\mathbf{j} - 10\mathbf{k}}}.$$

(b) Find an equation for Π_1, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. (2)

Solution

$$\begin{aligned} \mathbf{r}.(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) &= (3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}).(\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \\ &= 3 + 6 - 2 \\ &= \underline{\underline{7}}. \end{aligned}$$

The plane Π_2 has cartesian equation $x + z = 3$ and Π_1 and Π_2 intersect in the line l.

(c) Find an equation for l, giving your answer in the form $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$. (4)

Solution

Π_1 has equation

$$3x + 2y + 2z = 7.$$

If we let $x = \lambda$,

$$\begin{aligned} x = 3 - z \Rightarrow 2y &= 7 - 3x - 2z \\ \Rightarrow 2y &= 7 - 3\lambda - 2(3 - \lambda) \\ \Rightarrow 2y &= 7 - 3\lambda - 6 + 2\lambda \\ \Rightarrow 2y &= 1 - \lambda \\ \Rightarrow y &= \tfrac{1}{2} - \tfrac{1}{2}\lambda. \end{aligned}$$

The general cartesian equation

$$\lambda = \frac{x}{1} = \frac{y - \frac{1}{2}}{-\frac{1}{2}} = \frac{z - 3}{-1}$$

and an equation for l is

$$\underline{\underline{[\mathbf{r} - (\tfrac{1}{2}\mathbf{j} + 3\mathbf{k})] \times (\mathbf{i} - \tfrac{1}{2}\mathbf{j} - \mathbf{k}) = \mathbf{0}}}.$$

The point P is the point on l that is the nearest to the origin O.

(d) Find the coordinates of P. (4)

Solution

$$
\begin{aligned}
& [\lambda\mathbf{i} + (\tfrac{1}{2} - \tfrac{1}{2}\lambda)\mathbf{j} + (3-\lambda)\mathbf{k}].(\mathbf{i} - \tfrac{1}{2}\mathbf{j} - \mathbf{k}) = 0 \\
\Rightarrow\ & \lambda - \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{2}\lambda) - (3-\lambda) = 0 \\
\Rightarrow\ & \lambda - \tfrac{1}{4} + \tfrac{1}{4}\lambda - 3 + \lambda = 0 \\
\Rightarrow\ & \tfrac{9}{4}\lambda = \tfrac{13}{4} \\
\Rightarrow\ & \lambda = \tfrac{13}{9} \\
\Rightarrow\ & \underline{\underline{P(\tfrac{13}{9}, -\tfrac{2}{9}, \tfrac{14}{9})}}.
\end{aligned}
$$

6. The points A, B, and C have position vectors, relative to a fixed origin O,

$$
\begin{aligned}
\mathbf{a} &= 2\mathbf{i} - \mathbf{j}, \\
\mathbf{b} &= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \text{ and} \\
\mathbf{c} &= 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},
\end{aligned}
$$

respectively. The plane Π passes through A, B, and C.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (4)

Solution

$$
\begin{aligned}
\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 3 \\ 0 & 4 & 2 \end{vmatrix} &= (6-12)\mathbf{i} - (-2-0)\mathbf{j} + (-4-0)\mathbf{k} \\
&= \underline{\underline{-6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}}.
\end{aligned}
$$

(b) Show that a cartesian equation of Π is $3x - y + 2z = 7$. (2)

Solution

$$
\begin{aligned}
& \mathbf{r}.(-6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) = (2\mathbf{i} - \mathbf{j}).(-6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \\
\Rightarrow\ & -6x + 2y - 4z = -12 - 2 + 0 \\
\Rightarrow\ & -6x + 2y - 4z = -14 \\
\Rightarrow\ & \underline{\underline{3x - y + 2z = 7}}.
\end{aligned}
$$

The line l has equation

$$(\mathbf{r} - 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \mathbf{0}.$$

The line l and the plane Π intersect at the point T.

(c) Find the coordinates of T. (5)

Solution

$$\mathbf{r} = (5 + 2\lambda)\mathbf{i} + (5 - \lambda)\mathbf{j} + (3 - 2\lambda)\mathbf{k}$$

and so

$$\begin{aligned} 3(5 + 2\lambda) - (5 - \lambda) + 2(3 - 2\lambda) = 7 &\Rightarrow 15 + 6\lambda - 5 + \lambda + 6 - 4\lambda = 7 \\ &\Rightarrow 3\lambda = -9 \\ &\Rightarrow \lambda = -3 \\ &\Rightarrow \underline{\underline{T(-1, 8, 9)}}. \end{aligned}$$

(d) Show that A, B, and T lie on the same straight line. (3)

Solution

$$\begin{aligned} \overrightarrow{AT} &= -3\mathbf{i} + 9\mathbf{j} + 9\mathbf{k} \\ &= \tfrac{3}{2}(-2\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}) \\ &= \tfrac{3}{2}\overrightarrow{BT}; \end{aligned}$$

thus, A, B, and T are <u>collinear</u>.

7. Figure 1 shows a pyramid $PQRST$ with base $PQRS$.

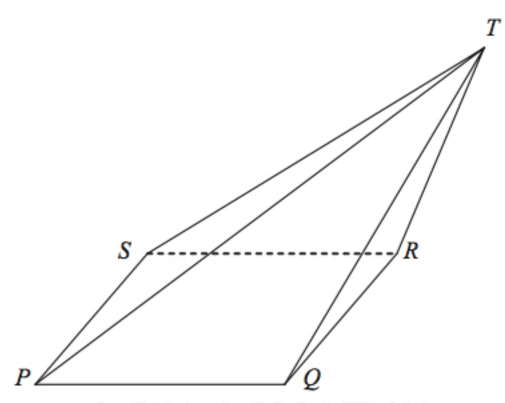

Figure 1: a pyramid $PQRST$

The coordinates of P, Q, and R are $P(1, 0, -1)$, $Q(2, -1, 1)$, and $R(3, -3, 2)$. Find

(a) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$. (3)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = (-3+6)\mathbf{i} - (3-4)\mathbf{j} + (-3+2)\mathbf{k}$$
$$= \underline{\underline{3\mathbf{i} + \mathbf{j} - \mathbf{k}}}.$$

(b) a vector equation for the plane containing the face $PQRS$, giving your answer in the form $\mathbf{r}.\mathbf{n} = d$. (2)

Solution

$$\begin{aligned} \mathbf{r}.(3\mathbf{i} + \mathbf{j} - \mathbf{k}) &= (\mathbf{i} - \mathbf{k}).(3\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &= 3 + 0 + 1 \\ &= \underline{\underline{4}}. \end{aligned}$$

The plane Π contains the face PST. The vector equation of Π is $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}) = 6$.

(c) Find cartesian equations of the line through P and S. (5)

Solution

$$3x + y - z = 4 \text{ and } x - 2y - 5z = 6.$$

Use 2 times the first equation — $6x + 2y - 2z = 8$ — and add that to the second equation:

$$\begin{aligned} 7x - 7z = 14 &\Rightarrow z = x - 2 \\ &\Rightarrow (z+2) - 2y - 5z = 6 \\ &\Rightarrow 2y + 4 = -4z \\ &\Rightarrow z = \frac{y+2}{-2}; \end{aligned}$$

hence, the line is

$$\underline{\underline{\frac{x-2}{1} = \frac{y+2}{-2} = \frac{z}{1}}}.$$

(d) Hence show that PS is parallel to QR. (2)

Solution

The direction is $\overrightarrow{PS} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Now, $\overrightarrow{QR} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ which means $\underline{\underline{PS \text{ and } QR \text{ are parallel}}}$.

Given that $PQRS$ is a parallelogram and that T has coordinates $(5, 2, -1)$,

(e) find the volume of the pyramid $PQRST$. (3)

Solution

$\overrightarrow{PT} = 4\mathbf{i} + 2\mathbf{j}$ and

$$\begin{aligned}\text{volume of the solid} &= \tfrac{1}{3}\left|\overrightarrow{PT}.(\overrightarrow{PQ} \times \overrightarrow{PR})\right| \\ &= \tfrac{1}{3}\left|(4\mathbf{i} + 2\mathbf{j}).(3\mathbf{i} + \mathbf{j} - \mathbf{k})\right| \\ &= \tfrac{1}{3}\left|12 + 2 + 0\right| \\ &= \underline{\underline{\tfrac{14}{3}}}.\end{aligned}$$

8. The points A, B, and C have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively, relative to a fixed origin O, as shown in Figure 2.

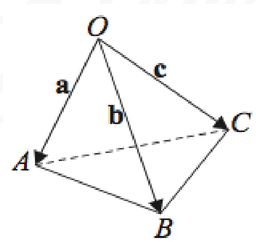

Figure 2: the points A, B, and C

It is given that

$$\mathbf{a} = \mathbf{i} + \mathbf{j},\ \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k},\ \text{and } \mathbf{c} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Calculate

(a) $\mathbf{b} \times \mathbf{c}$, (3)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = (1-1)\mathbf{i} - (-3-2)\mathbf{j} + (3+2)\mathbf{k}$$
$$= \underline{\underline{5\mathbf{j} + 5\mathbf{k}}}.$$

(b) $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$, (2)

Solution

$$(\mathbf{i} + \mathbf{j}).(5\mathbf{j} + 5\mathbf{k}) = 0 + 5 + 0 = \underline{\underline{5}}.$$

(c) the area of triangle OBC, (2)

Solution

$$\begin{aligned} \text{area of the triangle} &= \tfrac{1}{2} \left| \mathbf{b} \times \mathbf{c} \right| \\ &= \tfrac{1}{2} \times \sqrt{5^2 + 5^2} \\ &= \underline{\underline{\tfrac{5}{2}\sqrt{2}}}. \end{aligned}$$

(d) the volume of the tetrahedron $OABC$. (1)

Solution

$$\begin{aligned} \text{Volume of the tetrahedron} &= \tfrac{1}{6} \left| \mathbf{a}.(\mathbf{b} \times \mathbf{c}) \right| \\ &= \tfrac{1}{6} \times 5 \\ &= \underline{\underline{\tfrac{5}{6}}}. \end{aligned}$$

9. The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} \alpha \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

If the lines l_1 and l_2 interest,

(a) the value of α, (4)

Solution

$$\begin{aligned}\mathbf{i}:&\quad 1-\lambda=\alpha\\ \mathbf{j}:&\quad -1+3\lambda=-4+3\mu\\ \mathbf{k}:&\quad 2+4\lambda=2\mu.\end{aligned}$$

Now,

$$\begin{aligned}2+4\lambda=2\mu &\Rightarrow 1+2\lambda=\mu\\ &\Rightarrow -1+3\lambda=-4+3(1+2\lambda)\\ &\Rightarrow -1+3\lambda=-4+3+6\lambda\\ &\Rightarrow 3\lambda=0\\ &\Rightarrow \lambda=0\\ &\Rightarrow \underline{\underline{\alpha=1}}.\end{aligned}$$

(b) an equation for the plane containing the lines l_1 and l_2, giving your answer in the form $ax+by+cz+d=0$, where a, b, c, and d are constants. (4)

Solution

$$\begin{aligned}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 3 & 4\\ 0 & 3 & 2\end{vmatrix} &= (6-12)\mathbf{i}-(-2-0)\mathbf{j}+(-3-0)\mathbf{k}\\ &= -6\mathbf{i}+2\mathbf{j}-3\mathbf{k}.\end{aligned}$$

The plane has equation

$$\begin{aligned}&\mathbf{r}.(-6\mathbf{i}+2\mathbf{j}-3\mathbf{k})=(\mathbf{i}-\mathbf{j}+2\mathbf{k}).(-6\mathbf{i}+2\mathbf{j}-3\mathbf{k})\\ \Rightarrow\quad &-6x+2y-3z=-6-2-6\\ \Rightarrow\quad &\underline{\underline{6x-2y+3z-14=0}}.\end{aligned}$$

For other values of α, the lines l_1 and l_2 do not intersect and are skew lines. Given that $\alpha=2$,

(c) find the shortest distance between the lines l_1 and l_2. (3)

Solution

$$\begin{aligned}\text{Shortest distance} &= \left|\frac{(\mathbf{i}-3\mathbf{j}-2\mathbf{k}).(-6\mathbf{i}+2\mathbf{j}-3\mathbf{k})}{|-6\mathbf{i}+2\mathbf{j}-3\mathbf{k}|}\right| \\ &= \left|\frac{-6-6+6}{\sqrt{6^2+2^2+3^2}}\right| \\ &= \underline{\underline{\tfrac{6}{7}}}.\end{aligned}$$

10. Given that

$$\mathbf{a} = \mathbf{i} + 7\mathbf{j} + 9\mathbf{k} \text{ and } \mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k},$$

(a) show that $\mathbf{a} \times \mathbf{b} = c(2\mathbf{i} + \mathbf{j} - k\mathbf{k})$, and state the value of the constant c. (2)

Solution

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 7 & 9 \\ -1 & 3 & 1 \end{vmatrix} &= (7-27)\mathbf{i} - (1+9)\mathbf{j} + (3+7)\mathbf{k} \\ &= -20\mathbf{i} - 10\mathbf{j} + 10\mathbf{k} \\ &= \underline{\underline{-10(2\mathbf{i}+\mathbf{j}-\mathbf{k})}},\end{aligned}$$

and so $\underline{\underline{c = -10}}$.

The plane Π_1 passes through the point $(3, 1, 3)$ and the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to Π_1.

(b) Find a cartesian equation for the plane Π_1. (2)

Solution

$$\begin{aligned} & \mathbf{r}.(2\mathbf{i}+\mathbf{j}-\mathbf{k}) = (3\mathbf{i}+\mathbf{j}+3\mathbf{k}).(2\mathbf{i}+\mathbf{j}-\mathbf{k}) \\ \Rightarrow\ & 2x + y - z = 6 + 1 - 3 \\ \Rightarrow\ & \underline{\underline{2x + y - z = 4}}.\end{aligned}$$

The line l_1 has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda\mathbf{a}$.

(c) Show that the line l_1 lies in the plane Π_1. (2)

Solution

The line l_1 passes through the point $(1, 0, -2)$ and this lies in the plane. l_1 has direction $\mathbf{a}$ which is perpendicular to $\mathbf{a} \times \mathbf{b}$, so l_1 is parallel to the plane. Hence, $\underline{\underline{l_1 \text{ lies in the plane}}}$.

The line l_2 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \mu\mathbf{b}$. The line l_2 lies in a plane Π_2, which is parallel to the plane Π_1.

(d) Find a cartesian equation of the plane Π_2. (2)

Solution

$$\begin{aligned} & \mathbf{r}.(2\mathbf{i} + \mathbf{j} - \mathbf{k}) = (\mathbf{i} + \mathbf{j} + \mathbf{k}).(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ \Rightarrow \quad & 2x + y - z = 2 + 1 - 1 \\ \Rightarrow \quad & \underline{\underline{2x + y - z = 2}}. \end{aligned}$$

(e) Find the distance between the planes Π_1 and Π_2. (3)

Solution

$$\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

and

$$\text{distance between the planes} = \frac{4 - 2}{\sqrt{6}} = \underline{\underline{\frac{\sqrt{3}}{6}}}.$$

11. The plane Π has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j}) + \mu(6\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

(a) Find an equation of Π in the form $\mathbf{r}.\mathbf{n} = p$, where $\mathbf{n}$ is a vector perpendicular to Π and p is a constant. (5)

Solution

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 0 \\ 6 & -2 & 1 \end{vmatrix} &= (1 - 0)\mathbf{i} - (-4 - 0)\mathbf{j} + (8 - 6)\mathbf{k} \\ &= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}, \end{aligned}$$

and so

$$\begin{aligned} \mathbf{r}.(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) &= (3\mathbf{i} + \mathbf{k}).(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) \\ \Rightarrow \mathbf{r}.(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) &= 3 + 0 + 2 \\ \Rightarrow \underline{\underline{\mathbf{r}.(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = 5}}. \end{aligned}$$

The point P has coordinates $(6, 13, 5)$. The line l passes through P and is perpendicular to Π. The line l intersects Π at the point N.

(b) Show that the coordinates of N are $(3, 1, -1)$. (4)

Solution

The line l is

$$\mathbf{r} = 6\mathbf{i} + 13\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}).$$

The intersection is

$$\begin{aligned} (6 + t) + 4(13 + 4t) + 2(5 + 2t) = 5 &\Rightarrow 6 + t + 52 + 16t + 10 + 4t = 5 \\ &\Rightarrow 21t = -63 \\ &\Rightarrow t = -3 \\ &\Rightarrow \underline{\underline{N(3, 1, -1)}}. \end{aligned}$$

The point R lies on Π and has coordinates $(1, 0, 2)$.

(c) Find the perpendicular distance from N to the line PR. Give your answer to 3 significant figures. (5)

Solution

$$\begin{aligned} \overrightarrow{PR}.\overrightarrow{PN} &= (-5\mathbf{i} - 13\mathbf{j} - 3\mathbf{k}).(-3\mathbf{i} - 12\mathbf{j} - 6\mathbf{k}) \\ &= 15 + 156 + 18 \\ &= 189. \end{aligned}$$

Now,

$$
\begin{aligned}
\cos NPR &= \frac{189}{\sqrt{5^2+13^2+3^3}\sqrt{3^2+12^2+6^3}} \\
\Rightarrow \quad \cos NPR &= \frac{189}{\sqrt{203}\times\sqrt{189}} \\
\Rightarrow \quad \cos NPR &= \frac{189}{\sqrt{203}\times\sqrt{189}} \\
\Rightarrow \quad \cos NPR &= \frac{3\sqrt{87}}{29},
\end{aligned}
$$

and

$$
\begin{aligned}
\text{distance} &= NP \sin NPR \\
&= 3.610\,330\,007 \text{ (FCD)} \\
&= \underline{\underline{3.61 \text{ (3 sf)}}}.
\end{aligned}
$$

12. The plane P has equation

$$
\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.
$$

(a) Find a vector perpendicular to the plane P. (2)

Solution

$$
\begin{aligned}
\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ 3 & 2 & 2 \end{vmatrix} &= (4+2)\mathbf{i} - (0+3)\mathbf{j} + (0-6)\mathbf{k} \\
&= 6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k};
\end{aligned}
$$

hence, $\underline{\underline{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}}$ is a vector.

The line l passes through the point $A(1, 3, 3)$ and meets P at $(3, 1, 2)$. The acute angle between the plane P and the line l is α.

(b) Find α to the nearest degree. (4)

Solution

The line l has direction $2\mathbf{i}-2\mathbf{j}-\mathbf{k}$ and angle between line l and normal is given by

$$\begin{aligned} \sin\alpha &= \frac{(2\mathbf{i}-\mathbf{j}-2\mathbf{k}).(2\mathbf{i}-2\mathbf{j}-\mathbf{k})}{\sqrt{2^2+1^2+2^2}\sqrt{2^2+2^2+1^2}} \\ \Rightarrow \quad \sin\alpha &= \frac{4+2+2}{3\times 3} \\ \Rightarrow \quad \sin\alpha &= \frac{8}{9} \\ \Rightarrow \quad \alpha &= 62.733\,955\,55 \text{ (FCD)} \\ \Rightarrow \quad \alpha &= \underline{\underline{63^\circ \text{ (nearest degree)}}}. \end{aligned}$$

(c) Find the perpendicular distance from A to the plane P. (4)

Solution

Now, the plane P has equation

$$\mathbf{r}.(2\mathbf{i}-\mathbf{j}-2\mathbf{k}) = 6-1-4 = 1$$

and

$$\text{distance between the planes} = \frac{1-(-7)}{3} = \underline{\underline{\tfrac{8}{3}}}.$$

13. The straight line l_1 is mapped onto the straight line l_2 by the transformation represented (5)

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}.$$

The equation of l_2 is $(\mathbf{r}-\mathbf{a})\times\mathbf{b} = \mathbf{0}$, where $\mathbf{a} = 4\mathbf{i}+\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

Find a vector equation for the line l_1.

Solution

Let (x, y, z) be on l_1. Now, equation of l_2 can be written as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}.$$

We need the inverse of the matrix.
Determinant:

$$\det = 2(0-2) + (1+3) + (-2-0) = -2.$$

Matrix of minors:

$$\begin{pmatrix} -2 & 4 & -2 \\ 1 & -1 & -1 \\ 1 & -3 & 1 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} -2 & -4 & -2 \\ -1 & -1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} -2 & -1 & 1 \\ -4 & -1 & 3 \\ -2 & 1 & 1 \end{pmatrix}$$

Inverse:

$$-\tfrac{1}{2}\begin{pmatrix} -2 & -1 & 1 \\ -4 & -1 & 3 \\ -2 & 1 & 1 \end{pmatrix}.$$

Next,

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= -\tfrac{1}{2}\begin{pmatrix} -2 & -1 & 1 \\ -4 & -1 & 3 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4+4\lambda \\ 1+\lambda \\ 7+3\lambda \end{pmatrix} \\
&= -\tfrac{1}{2}\begin{pmatrix} -2-6\lambda \\ 4-8\lambda \\ -4\lambda \end{pmatrix} \\
&= \begin{pmatrix} 1+3\lambda \\ -2+4\lambda \\ 2\lambda \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix},
\end{aligned}$$

and the vector equation is

$$[\mathbf{r} - (\mathbf{i} - 2\mathbf{j})] \times (3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = \mathbf{0}.$$

14. The position vectors of the points A, B, and C relative to an origin O are $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $7\mathbf{i} - 3\mathbf{k}$, and $4\mathbf{i} + 4\mathbf{j}$. Find

(a) $\overrightarrow{AC} \times \overrightarrow{BC}$, (4)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 6 & 2 \\ -3 & 4 & 3 \end{vmatrix} = (18-8)\mathbf{i} - (9+6)\mathbf{j} + (12+18)\mathbf{k}$$
$$= \underline{\underline{10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}}}.$$

(b) the area of triangle ABC, (2)

Solution

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BC}\right| \\ &= \tfrac{1}{2} \times \sqrt{10^2 + 15^2 + 30^2} \\ &= \tfrac{1}{2} \times 35 \\ &= \underline{\underline{17.5}}. \end{aligned}$$

(c) an equation of the plane ABC in the form $\mathbf{r}.\mathbf{n} = p$. (2)

Solution

$$\begin{aligned} & \mathbf{r}.(10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}) = (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}).(10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}) \\ \Rightarrow\ & \mathbf{r}.(10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}) = 10 + 30 - 60 \\ \Rightarrow\ & \underline{\underline{\mathbf{r}.(10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}) = -20}}. \end{aligned}$$

15. The straight line l_1 is mapped onto the straight line l_2 by the transformation represented by the matrix $\mathbf{M}$, where (5)

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

The equation of l_1 is $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, where $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Find a vector equation for the line l_2.

Solution

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 7 & -1 \\ -11 & 7 \end{pmatrix},$$

and the vector equation is

$$\underline{\underline{[\mathbf{r} - (8\mathbf{i} + 7\mathbf{j} - 11\mathbf{k})] \times (\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = \mathbf{0}}}.$$

16. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5.$$

(a) Find the perpendicular distance from the point $(6, 2, 12)$ to the plane Π_1. (3)

Solution

$$\begin{aligned} \text{The perpendicular distance} &= \left| \frac{(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}).(6\mathbf{i} + 2\mathbf{j} + 12\mathbf{k}) - 5}{\sqrt{3^2 + 4^2 + 2^2}} \right| \\ &= \left| \frac{(18 - 8 + 24) - 5}{\sqrt{29}} \right| \\ &= \left| \frac{29}{\sqrt{29}} \right| \\ &= \underline{\underline{\sqrt{29}}}. \end{aligned}$$

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k}),$$

where λ and μ are scalar parameters.

(b) find the acute angle between Π_1 and Π_2 giving your answer to the nearest degree. (5)

Solution

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 5 \\ 1 & -1 & -2 \end{vmatrix} &= (-2 + 5)\mathbf{i} - (-4 - 5)\mathbf{j} + (-2 - 1)\mathbf{k} \\ &= 3\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}, \end{aligned}$$

and

$$\begin{aligned}
&\cos\theta = \frac{(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}).(3\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})}{\sqrt{3^2 + 9^2 + 3^2}\sqrt{3^2 + 4^2 + 2^2}} \\
\Rightarrow\quad &\cos\theta = \frac{9 - 36 - 6}{\sqrt{99}\sqrt{29}} \\
\Rightarrow\quad &\cos\theta = \frac{-33}{\sqrt{99}\sqrt{29}} \\
\Rightarrow\quad &\theta = 128.016\,018\,7 \text{ (FCD)},
\end{aligned}$$

and the angle is

$$180 - 128.016\ldots = \underline{\underline{52^\circ \text{ (nearest degree)}}}.$$

(c) Find an equation of the line of intersection of the two planes in the form $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. (6)

Solution

$$\begin{aligned}
\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 1 & 3 & -1 \end{vmatrix} &= (4 - 6)\mathbf{i} - (-3 - 5)\mathbf{j} + (9 + 4)\mathbf{k} \\
&= -2\mathbf{i} + 8\mathbf{j} + 13\mathbf{k}.
\end{aligned}$$

Now, we have

$$3x - 4y + 2z = 5 \text{ and } x + 3y - z = 0$$

and we set $x = 0$:

$$\begin{aligned}
3y - z = 0 &\Rightarrow z = 3y \\
&\Rightarrow -4y + 2(3y) = 5 \\
&\Rightarrow 2y = 5 \\
&\Rightarrow y = \tfrac{5}{2} \\
&\Rightarrow z = \tfrac{15}{2},
\end{aligned}$$

and the equation of the line of intersection is

$$\underline{\underline{\mathbf{r} \times (\tfrac{5}{2}\mathbf{j} + \tfrac{15}{2}\mathbf{k}) = -2\mathbf{i} + 8\mathbf{j} + 13\mathbf{k}}}.$$

17. Two skew lines l_1 and l_2 have equations

$$l_1: \quad \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}),$$
$$l_2: \quad \mathbf{r} = (3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) + \mu(-4\mathbf{i} + 6\mathbf{j} + \mathbf{k}),$$

respectively, where λ and μ are real parameters.

(a) Find a vector in the direction of the common perpendicular to l_1 and l_2. (2)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 2 \\ -4 & 6 & 1 \end{vmatrix} = (3 - 12)\mathbf{i} - (4 + 8)\mathbf{j} + (24 + 12)\mathbf{k}$$
$$= \underline{\underline{-9\mathbf{i} - 12\mathbf{j} + 36\mathbf{k}}}.$$

(b) Find the shortest distance between these two lines. (5)

Solution

$$(\mathbf{i} - \mathbf{j} + \mathbf{k}) - (3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) = -2\mathbf{i} - 8\mathbf{j} - \mathbf{k}$$

and the

$$\begin{aligned} \text{shortest distance} &= \left| \frac{(-2\mathbf{i} - 8\mathbf{j} - \mathbf{k}).(-9\mathbf{i} - 12\mathbf{j} + 36\mathbf{k})}{\sqrt{9^2 + 12^2 + 36^2}} \right| \\ &= \left| \frac{18 + 96 - 36}{39} \right| \\ &= \left| \frac{78}{39} \right| \\ &= \underline{\underline{2}}. \end{aligned}$$

18. The plane Π_1 has vector equation (9)

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

where s and t are real parameters. The plane Π_1 is transformed to the plane Π_2 by the transformation represented by the matrix $\mathbf{T}$, where

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Find an equation of the plane Π_2 in the form $\mathbf{r}.\mathbf{n} = p$.

Solution

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 2 & -4 \\ -4 & 2 & 6 \\ 3 & 1 & -2 \end{pmatrix}$$

and

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -4 & 6 & -2 \end{vmatrix} = (-4-6)\mathbf{i} - (-4+4)\mathbf{j} + (12+8)\mathbf{k}$$
$$= -10\mathbf{i} + 20\mathbf{k}.$$

Finally,

$$\begin{aligned} \mathbf{r}.(\mathbf{i} - 2\mathbf{k}) &= (8\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}).(\mathbf{i} - 2\mathbf{k}) \\ &= 8 + 0 - 6 \\ &= \underline{\underline{2}}. \end{aligned}$$

19. The line l passes through the point $P(2, 1, 3)$ and is perpendicular to the plane Π whose vector equation is

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 3.$$

Find

(a) a vector equation of the line l, (2)

Solution

$$\underline{\underline{\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})}}.$$

(b) the position vector of the point where l meets Π. (4)

Solution

$$\mathbf{r} = (2+t)\mathbf{i} + (1-2t)\mathbf{j} + (3-t)\mathbf{k}$$

and

$$\begin{aligned} & [(2+t)\mathbf{i}+(1-2t)\mathbf{j}+(3-t)\mathbf{k}].(\mathbf{i}-2\mathbf{j}-\mathbf{k}) = 3 \\ \Rightarrow\quad & (2+t)-2(1-2t)-(3-t) = 3 \\ \Rightarrow\quad & 2+t-2+4t-3+t = 3 \\ \Rightarrow\quad & 6t = 6 \\ \Rightarrow\quad & t = 1, \end{aligned}$$

and the position vector is $\underline{\underline{3\mathbf{i}-\mathbf{j}+2\mathbf{k}}}$.

(c) Hence find the perpendicular distance of P from Π. (2)

Solution

$$\text{Distance} = 1 \times \sqrt{1^2+2^2+1^2} = \underline{\underline{\sqrt{6}}}.$$

20. (4)

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}.$$

The transformation $M : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{M}$. Find a cartesian equation of the image, under this transformation, of the line

$$x = \frac{y}{2} = \frac{z}{-1}.$$

Solution

We will rewrite the line as

$$\frac{x-0}{1} = \frac{y-0}{2} = \frac{z-0}{-1}$$

and

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 7 \\ 0 & 10 \end{pmatrix}$$

giving us

$$\underline{\underline{\frac{x}{-1} = \frac{y}{7} = \frac{z}{10}}}.$$

21. The position vectors of the points A, B, and C from a fixed origin O are

$$\mathbf{a} = \mathbf{i} - \mathbf{j},\ \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k},\ \text{and } \mathbf{c} = 2\mathbf{j} + \mathbf{k},$$

respectively.

(a) Using vector products, find the area of the triangle ABC. (4)

Solution

$$\overrightarrow{AB} = 2\mathbf{j} + \mathbf{k} \text{ and } \overrightarrow{AC} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

Now,

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{vmatrix} &= (2-3)\mathbf{i} - (0+1)\mathbf{j} + (0+2)\mathbf{k} \\ &= -\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \end{aligned}$$

and

$$\begin{aligned} \text{area of the triangle} &= \tfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| \\ &= \tfrac{1}{2}|-\mathbf{i} - \mathbf{j} + 2\mathbf{k}| \\ &= \tfrac{1}{2} \times \sqrt{1^2 + 1^2 + 2^2} \\ &= \underline{\underline{\tfrac{\sqrt{6}}{2}}}. \end{aligned}$$

(b) Show that (3)

$$\tfrac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0.$$

Solution

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} &= (1-2)\mathbf{i} - (1-0)\mathbf{j} + (2-0)\mathbf{k} \\ &= -\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c}) &= \tfrac{1}{6}(\mathbf{i} - \mathbf{j}).(-\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \tfrac{1}{6}(-1 + 1 + 0) \\ &= \underline{\underline{0}}, \end{aligned}$$

as required.

(c) Hence or otherwise, state what can be deduced about the vectors **a**, **b**, and **c**. (1)

Solution

The position vectors **a**, **b**, and **c** lie in the same plane.

22. The plane Π_1 has vector equation

$$\mathbf{r}.(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 5.$$

The plane Π_2 has vector equation

$$\mathbf{r}.(-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = 7.$$

(a) Find a vector equation for the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where **a** and **b** are constant vectors and λ is a scalar parameter. (6)

Solution

The equations are

$$2x + y + 3z = 5 \text{ and } -x + 2y + 4z = 7.$$

Well, we want the first equation plus two times the second:

$$\begin{aligned} 5y + 11z = 19 \Rightarrow 5y &= -11z + 19 \\ \Rightarrow y &= \frac{-11z + 19}{5} \\ \Rightarrow y &= \frac{z - \frac{19}{11}}{-\frac{5}{11}}. \end{aligned}$$

Now,

$$\begin{aligned} &\Rightarrow -x + 2y + \frac{4(5y - 19)}{-11} = 7 \\ &\Rightarrow x = 2y + \frac{4(5y - 19)}{-11} - 7 \\ &\Rightarrow x = \frac{-22y + (20y - 76) + 77}{-11} \\ &\Rightarrow x = \frac{-2y + 1}{-11} \\ &\Rightarrow y = \frac{-11x - 1}{-2} \\ &\Rightarrow y = \frac{x - (-\frac{1}{11})}{\frac{2}{11}}; \end{aligned}$$

hence, the equation of the line is

$$\frac{x-(-\frac{1}{11})}{\frac{2}{11}} = \frac{y-0}{1} = \frac{z-\frac{19}{11}}{-\frac{5}{11}}$$

which gives

$$\underline{\underline{\mathbf{r} = -\tfrac{1}{11}\mathbf{i} + \tfrac{19}{11}\mathbf{k} + \lambda(\tfrac{2}{11}\mathbf{i} + \mathbf{j} - \tfrac{5}{11}\mathbf{k})}}.$$

The plane Π_3 has cartesian equation $x - y + 2z = 31$.

(b) Using your answer to part (a), or otherwise, find the coordinates of the point of intersection of the planes Π_1, Π_2, and Π_3. (3)

Solution

$$\begin{aligned}
&\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} -\frac{1}{11} + \frac{2}{11}\lambda \\ \lambda \\ \frac{19}{11} - \frac{5}{11}\lambda \end{pmatrix} = 31 \\
\Rightarrow\quad & (-\tfrac{1}{11} + \tfrac{2}{11}\lambda) - \lambda + 2(\tfrac{19}{11} - \tfrac{5}{11}\lambda) = 31 \\
\Rightarrow\quad & -\tfrac{1}{11} + \tfrac{2}{11}\lambda - \lambda + \tfrac{38}{11} - \tfrac{10}{11}\lambda = 31 \\
\Rightarrow\quad & -\tfrac{19}{11}\lambda = \tfrac{304}{11} \\
\Rightarrow\quad & \lambda = -16,
\end{aligned}$$

and hence the point is $\underline{\underline{(-3, -16, 9)}}$.

23. The points A, B, and C have position vectors

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},$$

respectively.

(a) Find a vector equation of the straight line AB. (2)

Solution

e.g.,

$$\underline{\underline{\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}}.$$

(b) Find a cartesian form of the equation of the straight line AB. (2)

Solution

$$\underline{\underline{\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-2}{1}}}.$$

The plane Π contains the points A, B, and C.

(c) Find a vector equation of Π in the form $\mathbf{r}.\mathbf{n} = p$. (4)

Solution

$$\overrightarrow{AB} = -2\mathbf{i} - 3\mathbf{j} - \mathbf{k} \text{ and } \overrightarrow{AC} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

Now,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & -1 \\ 1 & -2 & -2 \end{vmatrix} = (6-2)\mathbf{i} - (4+1)\mathbf{j} + (4+3)\mathbf{k}$$
$$= 4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}.$$

Finally,

$$\begin{aligned} & \mathbf{r}.(4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}) = (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}).(4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}) \\ \Rightarrow\ & \mathbf{r}.(4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}) = 4 - 15 + 14 \\ \Rightarrow\ & \underline{\underline{\mathbf{r}.(4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}) = 3}}. \end{aligned}$$

(d) Find the perpendicular distance from the origin to Π. (2)

Solution

$$\sqrt{4^2 + 5^2 + 7^2} = 3\sqrt{10}$$

and

$$\text{the perpendicular distance} = \frac{3}{3\sqrt{10}} = \underline{\underline{\frac{\sqrt{10}}{10}}}.$$

24. The plane Π_1 has equation

$$x - 5y - 2z = 3.$$

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k}),$$

where λ and μ are scalar parameters.

(a) Show that Π_1 is perpendicular to Π_2. (4)

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ 2 & -1 & 1 \end{vmatrix} = (4+3)\mathbf{i} - (1-6)\mathbf{j} + (-1-8)\mathbf{k}$$
$$= 7\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}.$$

Now,

$$(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}).(7\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}) = 7 - 25 + 18 = 0,$$

and Π_1 is perpendicular to Π_2.

(b) Find a cartesian equation for Π_2. (2)

Solution

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(7\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}) = (7\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}).(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$
$$\Rightarrow \quad 7x + 5y - 9z = 7 + 10 - 9$$
$$\Rightarrow \quad 7x + 5y - 9z = 8.$$

(c) Find an equation for the line of intersection of Π_1 and Π_2 giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found. (6)

Solution

Add Π_1 plus Π_2:

$$8x - 11z = 11 \Rightarrow x = \frac{11z + 11}{8}$$
$$\Rightarrow x = \frac{z+1}{\frac{8}{11}}.$$

Now,

$$\begin{aligned}
x - 5y - \frac{2(8x-11)}{11} = 3 &\Rightarrow \frac{11x - 2(8x-11)}{11} = 5y + 3 \\
&\Rightarrow \frac{-5x + 22}{11} = 5y + 3 \\
&\Rightarrow -5x + 22 = 55y + 33 \\
&\Rightarrow -5x = 55y + 11 \\
&\Rightarrow x = \frac{55y + 11}{-5} \\
&\Rightarrow x = \frac{y + \frac{1}{5}}{-\frac{1}{11}}.
\end{aligned}$$

Hence, the line in question is

$$\frac{x-0}{1} = \frac{y - (-\frac{1}{5})}{-\frac{1}{11}} = \frac{z - (-1)}{\frac{8}{11}}$$

which gives

$$\underline{\underline{[\mathbf{r} - (-\tfrac{1}{5}\mathbf{j} - \mathbf{k})] \times (\mathbf{i} - \tfrac{1}{11}\mathbf{j} + \tfrac{8}{11}\mathbf{k}) = \mathbf{0}}}.$$

25. The plane Π_1 has equation

$$x - 2y - 3z = 5$$

and the plane Π_2 has equation

$$6x + y - 4z = 7.$$

(a) Find, to the nearest degree, the acute angle between Π_1 and Π_2. (3)

Solution

$$\begin{aligned}
\cos\theta &= \left|\frac{(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}).(6\mathbf{i} + \mathbf{j} - 4\mathbf{k})}{\sqrt{1^2 + 2^2 + 3^2}\sqrt{6^2 + 1^2 + 4^2}}\right| \\
\Rightarrow \quad \cos\theta &= \left|\frac{6 - 2 + 12}{\sqrt{14}\sqrt{53}}\right| \\
\Rightarrow \quad \cos\theta &= \left|\frac{16}{\sqrt{14}\sqrt{53}}\right| \\
\Rightarrow \quad \theta &= 54.028\,803\,06 \text{ (FCD)} \\
\Rightarrow \quad \theta &= 54 \text{ (nearest degree)}.
\end{aligned}$$

The point P has coordinates $(2, 3, -1)$. The line l is perpendicular to Π_1 and passes through the point P. The line l intersects Π_2 at the point Q.

(b) Find the coordinates of Q. (4)

Solution

$$\begin{aligned}\overrightarrow{PQ} &= (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + t(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \\ &= (2 + t)\mathbf{i} + (3 - 2t)\mathbf{j} + (-1 - 3t)\mathbf{k}.\end{aligned}$$

Now,

$$\begin{aligned}6(2 + t) + (3 - 2t) - 4(-1 - 3t) = 7 &\Rightarrow 12 + 6t + 3 - 2t + 4 + 12t = 7 \\ &\Rightarrow 16t = 12 \\ &\Rightarrow t = -\tfrac{3}{4},\end{aligned}$$

and

$$\underline{\underline{Q(1\tfrac{1}{4}, 4\tfrac{1}{2}, 1\tfrac{1}{4})}}.$$

The plane Π_3 passes through the point Q and is perpendicular to Π_1 and Π_2.

(c) Find an equation of the plane Π_3 in the form $\mathbf{r}.\mathbf{n} = p$. (4)

Solution

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ 6 & 1 & -4 \end{vmatrix} &= (8 + 3)\mathbf{i} - (-4 + 18)\mathbf{j} + (1 + 12)\mathbf{k} \\ &= 11\mathbf{i} - 14\mathbf{j} + 13\mathbf{k}.\end{aligned}$$

Finally,

$$\begin{aligned} & \mathbf{r}.(11\mathbf{i} - 14\mathbf{j} + 13\mathbf{k}) = (11\mathbf{i} - 14\mathbf{j} + 13\mathbf{k}).(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \\ \Rightarrow\ & \mathbf{r}.(11\mathbf{i} - 14\mathbf{j} + 13\mathbf{k}) = 22 - 42 - 13 \\ \Rightarrow\ & \underline{\underline{\mathbf{r}.(11\mathbf{i} - 14\mathbf{j} + 13\mathbf{k}) = -33}}.\end{aligned}$$

26. The straight line l_2 is mapped onto the straight line l_1 by the transformation represented by the matrix (6)

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ -6 & -1 & -2 \end{pmatrix}.$$

Given that l_2 has cartesian equation

$$\frac{x-1}{5} = \frac{y+2}{2} = \frac{z-3}{1},$$

find a cartesian equation of the line l_1.

Solution

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ -6 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ -2 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -3 & -13 \\ -10 & -34 \end{pmatrix};$$

hence,

$$\underline{\underline{\frac{x-1}{5} = \frac{y+3}{-13} = \frac{z+10}{-34}}}.$$