Dr Oliver Mathematics
Further Mathematics
Further Differentiation and Integration
Past Examination Questions

This booklet consists of 54 questions across a variety of examination topics.
The total number of marks available is 427.

1. Given that $\mathrm{f}(x) \equiv \dfrac{1}{\sqrt{x^2 + 4x - 12}}$,

   (a) find $\displaystyle\int \mathrm{f}(x)\,\mathrm{d}x$. (4)

   (b) Hence find the exact value of (3)

   $$\int_6^{10} \mathrm{f}(x)\,\mathrm{d}x,$$

   giving your answer as a single logarithm.

2. (a) Using the substitution $u = \mathrm{e}^x$, find (6)

   $$\int \operatorname{sech} x\,\mathrm{d}x.$$

   (b) Sketch the curve $y = \operatorname{sech} x$. (1)

   The finite region R is bounded by the curve with equation $y = \operatorname{sech} x$, the lines $x = 2$ and $x = -2$, and the x-axis.

   (c) Using your result from part (a), find the area of R, giving your answer to 3 decimal places. (3)

3. (a) Simplify $(\mathrm{e}^x + \mathrm{e}^{-x})^2 - (\mathrm{e}^x - \mathrm{e}^{-x})^2$ and hence deduce that (4)

   $$\cosh^2 x - \sinh^2 x = 1.$$

   (b) Given that $y = \operatorname{arsinh} x$, show that (4)

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 + 1}}.$$

   (c) Find $\displaystyle\int \operatorname{arsinh} x\,\mathrm{d}x$. (5)

4. (a) Express $4x^2 + 4x + 26$ in the form $(px + q)^2 + r$, where p, q, and r are constants. (3)

(b) Hence determine (3)

$$\int \frac{1}{\sqrt{4x^2 + 4x + 26}} \, \mathrm{d}x.$$

5. Find $\int x \operatorname{sech}^2 x \, \mathrm{d}x$.

6.

$$4x^2 + 4x + 5 \equiv (px + q)^2 + r.$$

(a) Find the values of p, q, and r. (3)

(b) Hence, or otherwise, find (4)

$$\int \frac{1}{4x^2 + 4x + 5} \, \mathrm{d}x.$$

(c) Show that (7)

$$\int \frac{2}{\sqrt{4x^2 + 4x + 5}} \, \mathrm{d}x = \ln\left[(2x + 1) + \sqrt{4x^2 + 4x + 5}\right] + k,$$

where k is an arbitrary constant.

7.

$$4x^2 + 4x + 5 \equiv (ax + b)^2 + c, a > 0.$$

(a) Find the values of a, b, and c. (3)

(b) Find the exact value of (4)

$$\int_{-0.5}^{1.5} \frac{1}{4x^2 + 4x + 17} \, \mathrm{d}x.$$

8. Figure 1 shows the curve with parametric equations

$$x = a \cos^3 \theta, y = a \sin^3 \theta, 0 \leqslant \theta \leqslant 2\pi.$$

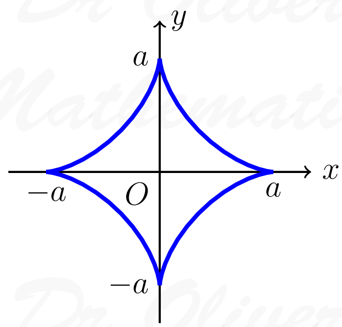

Figure 1: $x = a \cos^3 \theta, y = a \sin^3 \theta$

(a) Find the total length of this curve. (7)

The curve is rotated through π radians about the x-axis.

(b) Find the area of the surface generated. (5)

9. (a) Find (5)

$$\int \frac{1+x}{\sqrt{1-4x^2}}\,\mathrm{d}x.$$

(b) Find, to 3 decimal places, the value of (2)

$$\int_0^{0.3} \frac{1+x}{\sqrt{1-4x^2}}\,\mathrm{d}x.$$

10. Figure 2 shows the curve with parametric equations (7)

$$x = a\cos^3 t, y = a\sin^3 t, 0 \leqslant t \leqslant \tfrac{\pi}{2},$$

where a is a positive constant.

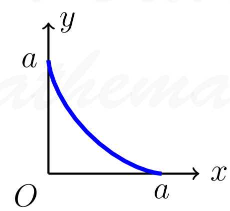

Figure 2: $x = a\cos^3 t, y = a\sin^3 t$

The curve is rotated through 2π radians about the x-axis. Find the area of the surface generated.

11. Figure 3 shows a sketch of the curve with equation

$$y = x \operatorname{arcosh} x,\ 1 \leqslant x \leqslant 2.$$

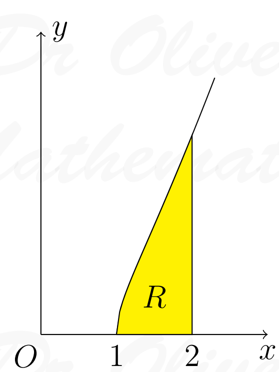

Figure 3: $y = x \operatorname{arcosh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$. Show that the area of R is

$$\tfrac{7}{4}\ln(2+\sqrt{3}) - \tfrac{\sqrt{3}}{2}.$$

12. (a) Show that, for $0 < x \leqslant 1$, (3)

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

(b) Using the definitions of $\cosh x$ or $\operatorname{sech} x$ in terms of exponentials, for $0 < x \leqslant 1$, (5)

$$\operatorname{arsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

13. Evaluate

$$\int_1^4 \frac{1}{\sqrt{x^2-2x+17}}\,\mathrm{d}x,$$

giving your answer as an exact logarthim.

14. (a) Show that $\operatorname{artanh}(\sin\frac{\pi}{4}) = \ln(1+\sqrt{2})$. (3)

(b) Given that $y = \operatorname{artanh}(\sin x)$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec x$. (2)

(c) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \operatorname{artanh}(\sin x)\,\mathrm{d}x$. (5)

15. Figure 4 shows a sketch of the curve with equation (10)

$$y = x^2 \operatorname{arsinh} x.$$

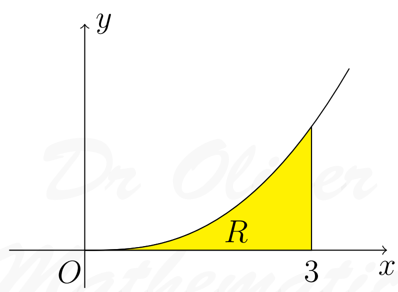

Figure 4: $y = x^2 \operatorname{arsinh} x$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 3$. Show that the area of R is

$$9\ln(3+\sqrt{10}) - \tfrac{1}{9}(2+7\sqrt{10}).$$

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16. The curve C has parametric equations

$$x = t - \ln t, y = 4\sqrt{t}, 1 \leqslant t \leqslant 4.$$

(a) Show that the length of C is $3 + \ln 4$. (7)

The curve is rotated through 2π radians about the x-axis.

(b) Find the area of the curved surface generated. (4)

17. Evaluate $\displaystyle\int_1^3 \frac{1}{\sqrt{x^2+4x-5}}\,\mathrm{d}x$, giving your answer as a natural logarithm. (5)

18. The curve C has equation (7)

$$y = \tfrac{1}{4}(2x^2 - \ln x), x > 0.$$

Find the length of C from $x = 0.5$ to $x = 2$, giving your answer in the form $a + b\ln 2$, where a and b are rational numbers.

19. Figure 5 shows a sketch of the curve with equation

$$y = \operatorname{arsinh}\sqrt{x},\ x \geqslant 0.$$

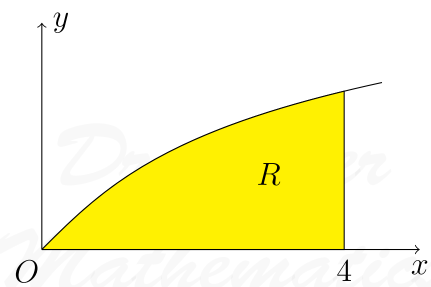

Figure 5: $y = \operatorname{arsinh}\sqrt{x}$

(a) Find the gradient of C at the point where $x = 4$. (3)

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 4$.

(b) Using the substitution $x = \sinh^2\theta$, or otherwise, show that the area of R is (10)

$$k\ln(2+\sqrt{5}) - \sqrt{5},$$

where k is a constant to be found.

20. Show that (4)

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tanh x)] = 2\operatorname{cosech} 2x,\ x > 0.$$

21. Show that (7)

$$\int_5^6 \frac{3+x}{\sqrt{x^2-9}}\,\mathrm{d}x = 3\ln\left(\frac{2+\sqrt{3}}{3}\right) + 3\sqrt{3} - 4.$$

22. The curve C has equation $y = \operatorname{arsinh}(x^3)$, $x \geqslant 0$. The point P on C has x-coordinate $\sqrt{2}$. Show that an equation of the tangent to C at P is (5)

$$y = 2x - 2\sqrt{2} + \ln(3 + 2\sqrt{2}).$$

23. Figure 6 shows a sketch of the curve with equation

$$y = \tfrac{1}{10}\cosh x \arctan(\sinh x),\ x \geqslant 0.$$

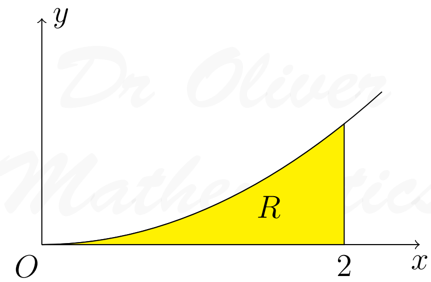

Figure 6: $y = \frac{1}{10}\cosh x \arctan(\sinh x)$

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, and the $x = 2$.

(a) Find $\displaystyle\int \cosh x \arctan(\sinh x)\,\mathrm{d}x$. (5)

(b) Hence show that, to 2 significant figures, the area of R is 0.34. (2)

24. The curve C has parametric equations

$$x = 3(t + \sin t), y = 3(1 - \cos t), 0 \leqslant t < \pi.$$

(a) Show that (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\frac{t}{2}.$$

The arc length s of C is measured from the origin O.

(b) Show that $s = 12\sin\frac{t}{2}$. (4)

The point P lies on C and the arc OP of C has length L. The arc OP is rotated through 2π radians about the x-axis.

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(c) Show that the area of the curved surface generated is given by $\dfrac{\pi L^3}{36}$. (7)

25. (5)

$$y = (\operatorname{arsinh} 2x)^2.$$

Find the exact value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{1}{2}$, giving your answer in the form $a \ln b$, where a and b are real numbers.

26. A curve has parametric equations (9)

$$x = 2t^3, y = 3t^2, 0 \leqslant t \leqslant 1.$$

The curve is rotated through 2π radians about the x-axis.
Prove that the area of the curved surface generated is $\dfrac{24\pi}{5}(\sqrt{2} + 1)$.

27. Using the substitution $u = \cosh\theta$, find the value of

$$\int_{\ln 2}^{\ln 4} \frac{\cosh\theta + 1}{\sinh\theta(\cosh\theta - 1)^2}\,\mathrm{d}\theta,$$

giving your answer as an exact fraction.

28. The curve C, with equation $y = \cosh 3x - 4x$, has a minimum point, as shown in Figure 7.

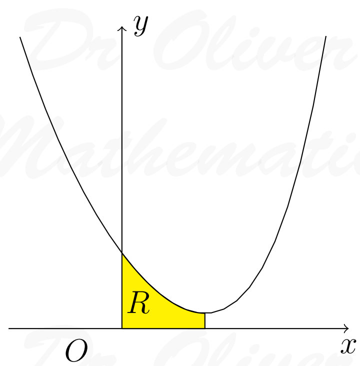

Figure 7: $y = \cosh 3x - 4x$

(a) Use calculus to find the x-coordinate of A. Give your answer in terms of natural logarithm. (5)

The region R, as shown shaded in the figure, is bounded by the curve, the x-axis, the y-axis, and the line through A parallel to the y-axis.

(b) Show that the area of R is $\frac{2}{9}[2 - (\ln 3)^2]$. (6)

29. (a) Using the substitution $x = \dfrac{a}{u}$, or otherwise, find (6)

$$\int \frac{1}{x\sqrt{a^2 - x^2}}\,\mathrm{d}x.$$

(b) Hence find (5)

$$\int_3^4 \frac{1}{x\sqrt{25 - x^2}}\,\mathrm{d}x,$$

giving your answer in the form $a \ln b$, where a and b are rational numbers.

30. Given that $y = \operatorname{arsinh}(\sqrt{x})$, $x > 0$,

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer as a simplified fraction. (3)

(b) Hence, or otherwise, find (6)

$$\int_{\frac{1}{4}}^4 \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x,$$

giving your answer in the form $\ln\left(\frac{a+b\sqrt{5}}{2}\right)$, where a and b are integers.

31. A curve, which is part of an ellipse, has parametric equations

$$x = 3\cos\theta, y = 5\sin\theta, 0 \leqslant \theta \leqslant \tfrac{\pi}{2}.$$

The curve is rotated through 2π radians about the x-axis.

(a) Show that the area of the surface generated is given by the integral (6)

$$k\pi \int_0^a \sqrt{16c^2 + 9}\,\mathrm{d}c,$$

where $c = \cos\theta$ and where k and a are constants to be found.

(b) Using the substitution $\cos\theta = \frac{3}{4}\sinh u$, or otherwise, evaluate the integral, showing all of your working and giving the final answer to 3 significant figures. (5)

32. Use calculus to find the exact value of $\displaystyle\int_{-2}^{1} \frac{1}{x^2 + 4x + 13}\,\mathrm{d}x$. (5)

33. The curve C has equation $y = 2x^3$, $0 \leqslant x \leqslant 2$. (5)
The curve C is rotated through 2π radians about the x-axis.
Using calculus, find the area of the surface generated, giving your answer to 3 significant figures.

34. Show that

(a) $\displaystyle\int_5^8 \frac{1}{x^2 - 10x + 34}\,\mathrm{d}x = k\pi$, giving the value of the fraction k. (5)

(b) $\displaystyle\int_5^8 \frac{1}{\sqrt{x^2-10x+34}}\,\mathrm{d}x = \ln(A+\sqrt{n})$, giving the values of the integers A and n. (4)

35. The curve C, as shown in Figure 8, has equation (6)

$$y = \tfrac{1}{3}\cosh 3x,\ 0 \leqslant x \leqslant \ln a,$$

where a is a constant and $a > 1$.

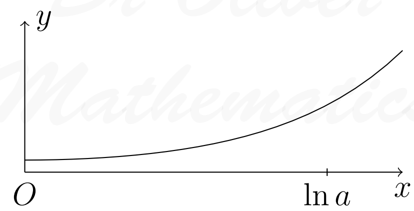

Figure 8: $y = \frac{1}{3}\cosh 3x$

Using calculus, show that the length of curve C is

$$k\left(a^3 - \frac{1}{a^3}\right)$$

and state the value of the constant k.

36. (a) Differentiate $x \operatorname{arsinh} 2x$ with respect to x. (3)

(b) Hence, or otherwise, find the exact value of (7)

$$\int_0^{\sqrt{2}} x \operatorname{arsinh} 2x \,\mathrm{d}x,$$

giving your answer in the form $A\ln B + C$, where A, B, C are real numbers.

37.

$$\mathrm{f}(x) = 5\cosh x - 4\sinh x.$$

(a) Show that $\mathrm{f}(x) = \frac{1}{2}(\mathrm{e}^x + 9\mathrm{e}^{-x})$. (2)

Hence

(b) solve $\mathrm{f}(x) = 5$, (4)

(c) show that (5)

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x = \tfrac{\pi}{18}.$$

38. (a) Find (2)

$$\int \frac{1}{\sqrt{4x^2+9}}\,\mathrm{d}x.$$

(b) Use your answer to part (a) to find the exact value of (3)

$$\int_{-3}^{3} \frac{1}{\sqrt{4x^2+9}}\,\mathrm{d}x,$$

giving your answer in the form $k\ln(a+b\sqrt{5})$, where a and b are integers and k is a constant.

39. The curve with parametric equations (7)

$$x=\cosh 2\theta, y=4\sinh\theta, 0\leqslant\theta\leqslant 1,$$

is rotated through 2π radians about the x-axis.
Show that the area of of the surface generated is $\lambda(\cosh^3 1-1)$, where λ is a constant to be found.

40. Figure 9 shows a sketch of the curve with equation (7)

$$y=40\,\text{arcosh}\,x-9x,\ x\geqslant 1.$$

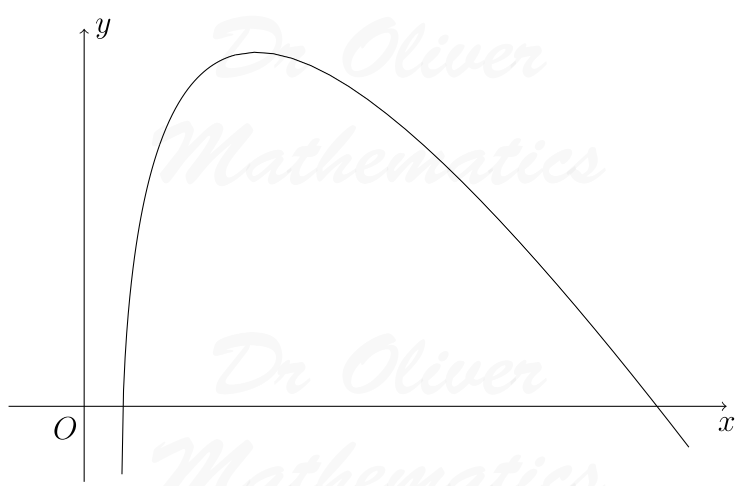

Figure 9: $y=40\,\text{arcosh}\,x-9x$

Use calculus to find the exact coordinates of the turning point of the curve, giving your answer in the form $\left(\frac{p}{q}, r\ln 3+s\right)$, where p, q, r, and s are integers.

41. Figure 10 shows a sketch of the curve with equations

$$y = 6\cosh x \text{ and } y = 9 - 2\sinh x.$$

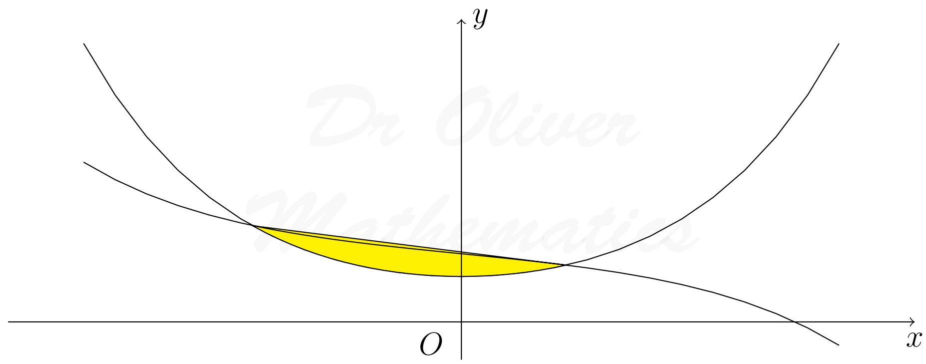

Figure 10: $y = 6\cosh x$ and $y = 9 - 2\sinh x$

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect. (6)

The finite region between the two curves is shown shaded in the figure.

(b) Using calculus, find the area of the shaded region, giving your answer in the form $a\ln b + c$, where a, b, and c are integers. (6)

42. The curve C, shown in Figure 11, has equation

$$y = 2x^{\frac{1}{2}},\ 1 \leqslant x \leqslant 8.$$

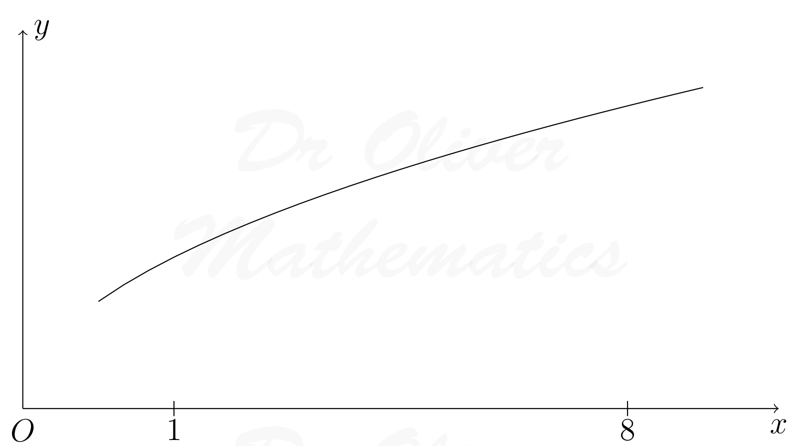

Figure 11: $y = 2x^{\frac{1}{2}}$

(a) Show that the length s of the curve C is given by the equation (2)

$$s = \int_1^8 \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x.$$

(b) Using the substitution $x = \sinh^2 u$, or otherwise, find an exact value for s. Give your answer in the form $a\sqrt{2} + \ln(b + c\sqrt{2})$, where a, b, and c are integers. (9)

43. Using calculus, find the exact value of

(a) $\displaystyle\int_1^2 \frac{1}{\sqrt{x^2 - 2x + 3}}\,\mathrm{d}x$, (4)

(b) $\displaystyle\int_0^1 \mathrm{e}^{2x} \sinh x \,\mathrm{d}x$. (4)

44. A circle C with centre O and radius r has cartesian equation $x^2 + y^2 = r^2$, where r is a positive constant.

(a) Show that $1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{r^2}{r^2 - x^2}$. (3)

(b) Show that the surface area of the sphere generated by rotating C through π radians about the x-axis is $4\pi r^2$. (5)

(c) Write down the arc length of the arc of the curve $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$. (1)

45.

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c.$$

(a) Find the values of the constants a, b, and c. (3)

Hence, or otherwise, find

(b) $\displaystyle\int \frac{1}{9x^2 + 6x + 5}\,\mathrm{d}x$, (2)

(c) $\displaystyle\int \frac{1}{\sqrt{9x^2 + 6x + 5}}\,\mathrm{d}x$. (2)

46. The curve C has equation

$$y = \mathrm{e}^{-x},\ x \in \mathbb{R}.$$

The part of the curve C between $x = 0$ and $x = \ln 3$ is rotated through 2π radians about the x-axis.

(a) Show that the area S of the curved surface generated is given by (3)

$$S = 2\pi \int_0^{\ln 3} \mathrm{e}^{-x}\sqrt{1 + \mathrm{e}^{-2x}}\,\mathrm{d}x.$$

(b) Use the substitution $e^{-x} = \sinh u$ to show that (5)

$$S = 2\pi \int_{\operatorname{arsinh}\alpha}^{\operatorname{arsinh}\beta} \cosh^2 u \, du,$$

where α and β are constants to be determined.

(c) Show that (2)

$$2\int \cosh^2 u \, du = \tfrac{1}{2}\sinh 2u + u + k,$$

where k is an arbitrary constant.

(d) Hence find the value of S, giving your answer to 3 decimal places. (2)

47. A curve has equation (5)

$$y = \cosh x,\ 1 \leqslant x \leqslant \ln 5.$$

Find the length of this curve. Give your answer in terms of e.

48. The curve C has equation

$$y = \frac{1}{\sqrt{x^2+2x-3}},\ x > 1.$$

(a) Find $\int y \, dx$. (3)

The region R is bounded by the curve C, the x-axis, and the lines with equations $x = 2$ and $x = 3$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. Give your answer in the form $p\pi \ln q$, where p and q are rational numbers to be found. (4)

49. The curve C has equation (7)

$$y = \frac{x^2}{8} - \ln x, 2 \leqslant x \leqslant 3.$$

Find the length of the curve C giving your answer in the form $p + lnq$, where p and q are rational numbers to be found.

50. (a) Prove that (3)

$$\frac{d}{dx}(\operatorname{arcoth} x) = \frac{1}{1-x^2}.$$

Given that $y = (\operatorname{arcoth} x)^2$,

(b) show that (5)

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = \frac{k}{1-x^2},$$

where k is a constant to be determined.

51. (a) Find, without using a calculator, (5)

$$\int_3^5 \frac{1}{\sqrt{15+2x-x^2}}\,\mathrm{d}x,$$

giving your answer as a multiple of π.

(b) Show that (3)

$$5\cosh x - 4\sinh x = \frac{\mathrm{e}^{2x}+9}{2\mathrm{e}^x}.$$

(c) Hence, using the substitution $u = \mathrm{e}^x$ or otherwise, find (4)

$$\int \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x.$$

52. Given that $y = \operatorname{arsinh}(\tanh x)$, show that (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\tanh^2 x}}.$$

53. Use the substitution $x + 2 = u^2$, where $u > 0$, to show that (9)

$$\int_{-1}^{7} \frac{(x+2)^{\frac{1}{2}}}{x+5}\,\mathrm{d}x = a + b\pi\sqrt{3},$$

where a and b are rational numbers to be found.

54. The curve C has equation

$$y = \ln\left(\frac{\mathrm{e}^x+1}{\mathrm{e}^x-1}\right),\ \ln 2 \leqslant x \leqslant \ln 3.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\mathrm{e}^x}{\mathrm{e}^{2x}-1}.$$

(b) Find the length of the curve C, giving your answer in the form $\ln a$, where a is a rational number. (6)