Dr Oliver Mathematics
AQA Further Maths Level 2
June 2023 Paper 1
1 hour 45 minutes

The total number of marks available is 80.
You must write down all the stages in your working.
You are **not** permitted to use a scientific or graphical calculator in this paper.

1. The function f is given by
$$f(x) = 2x + 1.$$

   (a) Work out x when (2)
$$f(x) = -5.$$

Solution

$$\begin{aligned} f(x) = -5 &\Rightarrow 2x + 1 = -5 \\ &\Rightarrow 2x = -6 \\ &\Rightarrow \underline{\underline{x = -3}}. \end{aligned}$$

   The function g is given by
$$g(x) = x^2.$$

   (b) Work out (2)
$$f\,g(3).$$

Solution

$$\begin{aligned} f\,g(3) &= f(g(3)) \\ &= f(3^2) \\ &= f(9) \\ &= 2(9) + 1 \\ &= \underline{\underline{19}}. \end{aligned}$$

2. Factorise fully (2)

$$6x^2y + 21xy.$$

Solution

$$6x^2y + 21xy = \underline{\underline{3xy(2x + 7)}}.$$

3. (a) Circle the transformation matrix that represents a reflection in the line $y = -x$. (1)

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Solution

$$\underline{\underline{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) Show that (2)

$$\begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -3 & -4 \\ 1 & 2 \end{pmatrix} = k\mathbf{I},$$

where k is an integer.

Solution

Well,

$$\begin{aligned} \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -3 & -4 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \\ &= \underline{\underline{-2\mathbf{I}}}; \end{aligned}$$

hence, $\underline{\underline{k = -2}}$.

4. $S(7, 2)$ and $T(5, -4)$ are points on a straight line.

(a) Work out the gradient of the line. (2)

Solution

$$\begin{aligned} m &= \frac{-4-2}{5-7} \\ &= \frac{-6}{-2} \\ &= \underline{\underline{3}}. \end{aligned}$$

(b) Work out the distance between S and T. (3)
Give your answer in the form $a\sqrt{b}$ where a and b are both integers greater than 1.

Solution

$$\begin{aligned} ST &= \sqrt{(5-7)^2 + (-4-2)^2} \\ &= \sqrt{(-2)^2 + (-6)^2} \\ &= \sqrt{4+36} \\ &= \sqrt{40} \\ &= \sqrt{4 \times 10} \\ &= \sqrt{4} \times \sqrt{10} \\ &= \underline{\underline{2\sqrt{10}}}; \end{aligned}$$

so, $\underline{\underline{a = 2}}$ and $\underline{\underline{b = 10.}}$

5. X_n and Y_n are the nth terms of two sequences. (3)

$$\begin{aligned} X_n &= (n-1)(n+1) \\ Y_n &= (n+1)(n+2). \end{aligned}$$

Prove that every term of the sequence with nth term

$$Y_n - X_n$$

is a multiple of 3.

Solution

$$\begin{aligned} Y_n - X_n &= (n+1)(n+2) - (n-1)(n+1) \\ &= (n+1)[(n+2) - (n-1)] \\ &= (n+1)(n+2-n+1) \\ &= 3(n+1); \end{aligned}$$

hence, every term of the sequence is $\underline{\underline{\text{a multiple of 3}}}$.

6. The equation of a curve is (4)

$$y = x^6 + 4x^2 - 7.$$

Work out the equation of the tangent to the curve at the point $(1, -2)$.
Give your answer in the form $y = mx + c$.

Solution

Now,

$$y = x^6 + 4x^2 - 7 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5 + 8x$$

and

$$\begin{aligned} x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 6 + 8 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 14. \end{aligned}$$

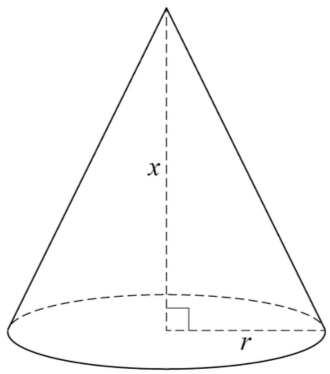
Hence, the equation of the tangent is

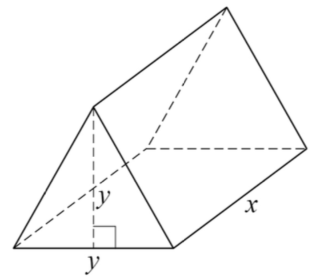
$$\begin{aligned} y + 2 = 14(x-1) &\Rightarrow y + 2 = 14x - 14 \\ &\Rightarrow \underline{\underline{y = 14x - 16}}; \end{aligned}$$

so, $\underline{\underline{m = 14}}$ and $\underline{\underline{c = -16}}$.

7. The diagram below shows a cone and a prism. (4)
All measurements are in cm.

- The cone has base radius r and perpendicular height x.
- The prism has a triangular cross section with base y and perpendicular height y.
- The length of the prism is x.





Volume of a cone = $\frac{1}{3}$ × area of base × perpendicular height

The volume of the cone is **four** times the volume of the prism.

Express r in terms of y.

Solution

Well,

$$\begin{aligned} V_{\text{cone}} &= \tfrac{1}{3} \times (\pi \times 3^2) \times x \\ &= \tfrac{1}{3}\pi r^2 x \end{aligned}$$

and

$$\begin{aligned} V_{\text{prism}} &= (\tfrac{1}{2} \times y^2) \times x \\ &= \tfrac{1}{2}xy^2. \end{aligned}$$

Now,

$$\begin{aligned} V_{\text{cone}} = 4 \times V_{\text{prism}} &\Rightarrow \tfrac{1}{3}\pi r^2 x = 4 \times \tfrac{1}{2}xy^2 \\ &\Rightarrow \tfrac{1}{3}\pi r^2 = 2y^2 \\ &\Rightarrow r^2 = \frac{6y^2}{\pi} \\ &\Rightarrow \underline{\underline{r = y\sqrt{\frac{6}{\pi}}}}. \end{aligned}$$

8. A circle has centre $(0, 0)$ and radius 5. (6)

A straight line has equation

$$2y = x + 5.$$

Work out the coordinates of the **two** points where the circle and straight line intersect.
Do **not** use trial and improvement.
You **must** show your working.

Solution

Well, the circle has equation

$$x^2 + y^2 = 25.$$

Now,

$$2y = x + 5 \Rightarrow x = 2y - 5$$

and insert this into the equation of the circle:

$$x^2 + y^2 = 25 \Rightarrow (2y - 5)^2 + y^2 = 25$$

$\times$	$2y$	-5
$2y$	$4y^2$	$-10y$
-5	$-10y$	$+25$

$$\begin{aligned}
&\Rightarrow (4y^2 - 20y + 25) + y^2 = 25 \\
&\Rightarrow 5y^2 - 20y = 0 \\
&\Rightarrow 5y(y - 4) = 0 \\
&\Rightarrow y = 0 \text{ or } y = 4 \\
&\Rightarrow x = -5 \text{ or } x = 3;
\end{aligned}$$

hence, the two points have coordinates $\underline{\underline{(-5, 0)}}$ and $\underline{\underline{(3, 4)}}$.

9. Rearrange (4)

$$w = \frac{y^2 + 5}{y^2 - 2}$$

to make y the subject.

Solution

$$\begin{aligned} w = \frac{y^2+5}{y^2-2} &\Rightarrow w(y^2-2) = y^2+5 \\ &\Rightarrow wy^2 - 2w = y^2 + 5 \\ &\Rightarrow wy^2 - y^2 = 2w + 5 \\ &\Rightarrow y^2(w-1) = 2w+5 \\ &\Rightarrow y^2 = \frac{2w+5}{w-1} \\ &\Rightarrow \underline{\underline{y = \pm\sqrt{\frac{2w+5}{w-1}}}}, \end{aligned}$$

as we are given no information about w and y.

10. Rationalise the denominator and simplify fully (4)

$$\frac{1+\sqrt{5}}{3-\sqrt{5}}.$$

Solution

$\times$	3	$-\sqrt{5}$
3	9	$-3\sqrt{5}$
$+\sqrt{5}$	$+3\sqrt{5}$	-5

Now,

$$\begin{aligned} \frac{1+\sqrt{5}}{3-\sqrt{5}} &= \frac{1+\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} \\ &= \frac{(1+\sqrt{5})(3+\sqrt{5})}{9-5} \end{aligned}$$

$\times$	3	$+\sqrt{5}$
1	3	$+\sqrt{5}$
$+\sqrt{5}$	$+3\sqrt{5}$	$+5$

$$= \frac{8 + 4\sqrt{5}}{4}$$
$$= \underline{\underline{2 + \sqrt{5}}}.$$

11. (3)

$$y = \tfrac{1}{12}x^4 + 3x^2 + 4.$$

Work out the **positive** value of x for which

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 55.$$

Solution

Well,

$$\begin{aligned} y = \tfrac{1}{12}x^4 + 3x^2 + 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{1}{3}x^3 + 6x \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= x^2 + 6 \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 55 &\Rightarrow x^2 + 6 = 55 \\ &\Rightarrow x^2 = 49 \\ &\Rightarrow \underline{\underline{x = 7}}, \end{aligned}$$

as we want the positive value.

12. (a) Write down the value of x for $0^\circ \leqslant x \leqslant 360^\circ$ when (1)

$$\sin x = -1.$$

Solution
$\underline{\underline{x = 270^\circ}}$.

(b) Work out the values of y for $0^\circ \leqslant y \leqslant 360^\circ$ when (3)

$$\sqrt{3}\tan y = 1.$$

Solution

$$\begin{aligned}\sqrt{3}\tan y = 1 &\Rightarrow \tan y = \tfrac{1}{\sqrt{3}} \\ &\Rightarrow \underline{\underline{y = 30^\circ,\ 210^\circ}}.\end{aligned}$$

13. Write (3)

$$\frac{2x-3}{x} - \frac{1}{3x} + 1$$

as a single fraction.
Give your answer in its simplest form.

Solution

$$\begin{aligned}\frac{2x-3}{x} - \frac{1}{3x} + 1 &= 2 - \frac{3}{x} - \frac{1}{3x} + 1 \\ &= 3 - \frac{9}{3x} - \frac{1}{3x} \\ &= 3 - \frac{10}{3x} \\ &= \underline{\underline{\frac{9x-10}{3x}}}\end{aligned}$$

14. Solve (4)

$$\frac{8}{x} + 3x \leqslant 10,$$

where x is positive.

Solution

As x is positive,

$$\begin{aligned}\frac{8}{x} + 3x \leqslant 10 &\Rightarrow 8 + 3x^2 \leqslant 10x \\ &\Rightarrow 3x^2 - 10x + 8 \leqslant 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -10 \\ \text{multiply to:} & (+3) \times (+8) = -24\end{array}\right\} -6,\ -4$$

e.g.,

$$\begin{aligned}&\Rightarrow 3x^2 - 6x - 4x + 8 \leqslant 0 \\ &\Rightarrow 3x(x-2) - 4(x-2) \leqslant 0 \\ &\Rightarrow (3x-4)(x-2) \leqslant 0.\end{aligned}$$

We need a 'table of signs':

	$x < \frac{4}{3}$	$x = \frac{4}{3}$	$\frac{4}{3} < x < 2$	$x = 2$	$x > 2$
$3x - 4$	$-$	0	$+$	$+$	$+$
$x - 2$	$-$	$-$	$-$	0	$+$
$(3x-4)(x-2)$	$+$	0	$-$	0	$+$

Hence,

$$\frac{8}{x} + 3x \leqslant 10 \Rightarrow \underline{\underline{\tfrac{4}{3} \leqslant x \leqslant 2}}.$$

15. Solve (4)

$$\left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)^2 = x^2 + x.$$

Solution

$\times$	$x^{\frac{1}{2}}$	$-x^{\frac{3}{2}}$
$x^{\frac{1}{2}}$	x	$-x^2$
$-x^{\frac{3}{2}}$	$-x^2$	$+x^3$

so

$$\begin{aligned}\left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)^2 = x^2 + x &\Rightarrow x - 2x^2 + x^3 = x^2 + x \\ &\Rightarrow x^3 - 3x^2 = 0 \\ &\Rightarrow x^2(x-3) = 0 \\ &\Rightarrow \underline{\underline{x = 0 \text{ (repeated twice) or } x = 3}}.\end{aligned}$$

16. The expansions of (4)

$$(1 + 12x)^4$$

and

$$(a + 4x)^3$$

have the same coefficient of x^2.

Work out the value of a.

Solution

Well,

$$\begin{aligned}(1 + 12x)^4 &= 1^5 + \binom{4}{1}(1)^3(12x)^1 + \binom{4}{2}(1)^2(12x)^2 + \binom{4}{3}(1)^1(12x)^3 + (12x)^4 \\ &= 1 + 48x + 864x^2 + 6\,912x^3 + 20\,736x^4\end{aligned}$$

and

$$\begin{aligned}(a + 4x)^3 &= a^3 + \binom{3}{1}(a)^2(4x)^1 + \binom{3}{2}(a)^1(4x)^2 + (4x)^3 \\ &= a^3 + 12a^2x + 48ax^2 + 64x^3.\end{aligned}$$

Same coefficient:

$$48a = 864 \Rightarrow \underline{\underline{a = 18}}.$$

17. The curve (5)

$$y = ax^3 + bx^2 + 7$$

has a stationary point at $(-2, 11)$.

Work out the values of a and b.

Solution

Well,

$$y = ax^3 + bx^2 + 7 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3a[(-2)^2] + 2b(-2) = 0 \\ &\Rightarrow \boxed{12a - 4b = 0 \quad (1).}\end{aligned}$$

Next,

$$\begin{aligned}x = -2, y = 11 &\Rightarrow a[(-2)^3] + b[(-2)^2] + 7 = 11 \\ &\Rightarrow \boxed{-8a + 4b = 4 \quad (2).}\end{aligned}$$

Now, do (1) + (2):

$$\begin{aligned}4a = 4 &\Rightarrow \underline{\underline{a = 1}} \\ &\Rightarrow 12(1) - 4b = 0 \\ &\Rightarrow 4b = 12 \\ &\Rightarrow \underline{\underline{b = 3}}.\end{aligned}$$

18. Solve the simultaneous equations (5)

$$\begin{aligned}2x + y &= 13 \\ x + 3z &= 2 \\ z - 2y &= -7.\end{aligned}$$

Do **not** use trial and improvement.
You **must** show your working.

Solution

$$2x + y = 13 \quad (1)$$
$$x + 3z = 2 \quad (2)$$
$$-2y + z = -7 \quad (3)$$

Do $2 \times (2)$:

$$2x + 6z = 4 \quad (4)$$

then do $(1) - (4)$:

$$y - 6z = 9 \quad (5).$$

Now, do $2 \times (5)$:

$$2y - 12z = 18 \quad (6)$$

and do $(3) + (6)$:

$$\begin{aligned} -11z = 11 &\Rightarrow \underline{\underline{z = -1}} \\ &\Rightarrow y - 6(-1) = 9 \\ &\Rightarrow y + 6 = 9 \\ &\Rightarrow \underline{\underline{y = 3}} \\ &\Rightarrow 2x + 3 = 13 \\ &\Rightarrow 2x = 10 \\ &\Rightarrow \underline{\underline{x = 5}}. \end{aligned}$$

19. (3)

$$8x^2 + 20x + n \equiv c(x + d)^2 + 3,$$

where c, d, and n are constants.

Work out the values of c, d, and n.

Solution

$\times$	x	$+d$
x	x^2	$+dx$
$+d$	$+dx$	$+d^2$

so

$$\begin{aligned} 8x^2 + 20x + n &\equiv c(x+d)^2 + 3 \\ &\equiv c(x^2 + 2dx + d^2) + 3 \\ &\equiv cx^2 + 2cdx + (cd^2 + 3). \end{aligned}$$

Well, $\underline{\underline{c = 8}}$ and

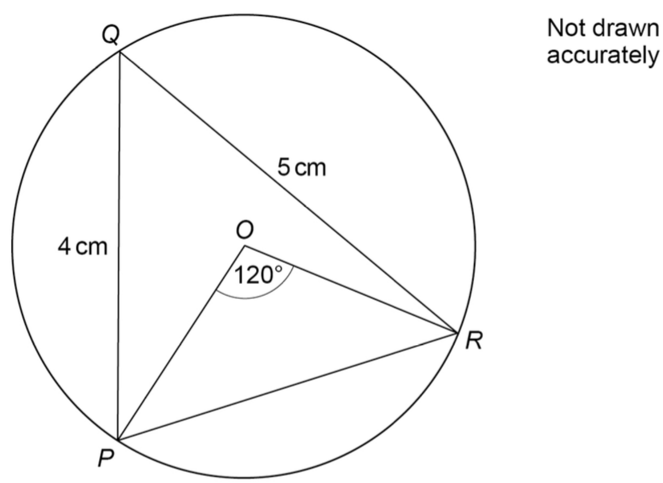
$$\begin{aligned} 2(8)d = 20 &\Rightarrow 16d = 20 \\ &\Rightarrow \underline{\underline{d = \tfrac{5}{4}}}. \end{aligned}$$

Finally,

$$\begin{aligned} n &= (8)(\tfrac{5}{4})^2 + 3 \\ &= (8)(\tfrac{25}{16}) + 3 \\ &= \tfrac{25}{2} + 3 \\ &= \underline{\underline{\tfrac{31}{2}}}. \end{aligned}$$

20. P, Q, and R are points on a circle, centre O. (6)

- Angle $POR = 120°$.
- $PQ = 4$ cm.
- $QR = 5$ cm.



Work out the radius of the circle.
Give your answer in the form $\sqrt{k}$ where k is an integer.

Solution

Well, $\angle PQR = \frac{1}{2} \times 120 = 60^\circ$ (angle at the centre is twice the angle at the circumference).

Now, cosine rule:

$$\begin{aligned} & PR^2 = PQ^2 + QR^2 - 2 \times PQ \times QR \times \cos PQR \\ \Rightarrow \quad & PR^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos 60^\circ \\ \Rightarrow \quad & PR^2 = 16 + 25 - 20 \\ \Rightarrow \quad & PR^2 = 21 \\ \Rightarrow \quad & PR = \sqrt{21}. \end{aligned}$$

Next, $\angle OPR = \angle ORP = \frac{1}{2}(180 - 120) = 30^\circ$ (completing the triangle).

Finally, sine rule:

$$\begin{aligned} \frac{OP}{\sin OPR} = \frac{PR}{\sin POR} & \Rightarrow \frac{OP}{\sin 30^\circ} = \frac{\sqrt{21}}{\sin 120^\circ} \\ & \Rightarrow OP = \frac{\sqrt{21}\sin 30^\circ}{\sin 120^\circ} \\ & \Rightarrow OP = \frac{\frac{1}{2}\sqrt{21}}{\frac{\sqrt{3}}{2}} \\ & \Rightarrow OP = \frac{\sqrt{21}}{\sqrt{3}} \\ & \Rightarrow OP = \sqrt{\frac{21}{3}} \\ & \Rightarrow \underline{\underline{OP = \sqrt{7}\text{ cm}}}. \end{aligned}$$