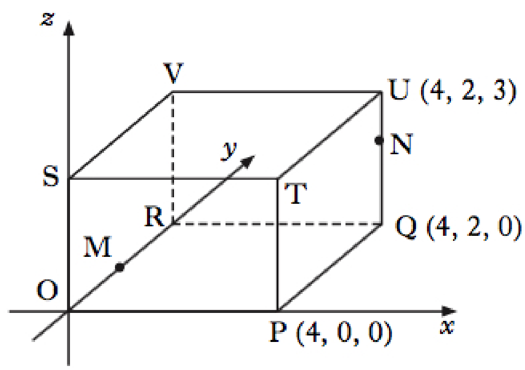
Dr Oliver Mathematics
Mathematics: Higher
2010 Paper 2: Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. The diagram shows a cuboid $OPQRSTUV$ relative to the coordinate axes.



P is the point $(4, 0, 0)$, Q is $(4, 2, 0)$, and U is $(4, 2, 3)$.
M is the midpoint of OR.
N is the point on UQ such that $UN = \frac{1}{3}UQ$.

(a) State the coordinates of M and N. (2)

Solution
$\underline{\underline{M(0, 1, 0)}}$ and $\underline{\underline{N(4, 2, 2)}}$.

(b) Express $\overrightarrow{VM}$ and $\overrightarrow{VN}$ in component form. (2)

Solution

$$
\begin{aligned}
\overrightarrow{VM} &= \overrightarrow{VO} + \overrightarrow{OM} \\
&= \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \underline{\underline{\begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}}}
\end{aligned}
$$

and

$$
\begin{aligned}
\overrightarrow{VM} &= \overrightarrow{VO} + \overrightarrow{ON} \\
&= \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} \\
&= \underline{\underline{\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}}}.
\end{aligned}
$$

(c) Calculate the size of angle MVN. (5)

Solution

$$
\begin{aligned}
\overrightarrow{VM}.\overrightarrow{VN} = |\overrightarrow{VM}||\overrightarrow{VN}| \cos MVN &\Rightarrow 0 + 0 + 3 = \sqrt{10} \cdot \sqrt{17} \cos MVN \\
&\Rightarrow \cos MVN = \frac{3}{\sqrt{170}} \\
&\Rightarrow \angle MVN = 76.697\,659\,16 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{\angle MVN = 76.7^\circ \text{ (3 sf)}}}.
\end{aligned}
$$

2.

$$12 \cos x^\circ - 5 \sin x^\circ$$

can be expressed in the form

$$k \cos(x + a)^\circ,$$

where $k > 0$ and $0 \leqslant a < 360$.

(a) Calculate the values of k and a. (4)

Solution

$$k\cos(x+a)^\circ \equiv k\cos x^\circ \cos a^\circ - k\sin x^\circ \sin a^\circ$$

and we have

$$k\cos a^\circ = 12 \text{ and } k\sin a^\circ = 5.$$

Now,

$$k = \sqrt{12^2+5^2} = \underline{\underline{13}}.$$

Finally,

$$\begin{aligned}\tan a^\circ = \frac{k\sin a^\circ}{k\cos a^\circ} &\Rightarrow \tan a^\circ = \tfrac{5}{12} \\ &\Rightarrow \underline{\underline{a = 22.619\,864\,95 \text{ (FCD)}}}.\end{aligned}$$

(b) (i) Hence state the maximum and minimum values of (3)

$$12\cos x^\circ - 5\sin x^\circ.$$

Solution
The maximum value is $\underline{\underline{13}}$ and the minimum value $\underline{\underline{-13}}$.

(ii) Determine the values of x, in the interval $0 \leqslant a < 360$, at which these maximum and minimum values occur.

Solution
Maximum:

$$\begin{aligned}\cos(x+22.614\ldots)^\circ = 1 &\Rightarrow x + 22.614\ldots = 0 \text{ (not really)}, 360 \\ &\Rightarrow x = 337.380\,135\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 337.4 \text{ (1 dp)}}}.\end{aligned}$$

Minumum:

$$\begin{aligned}\cos(x+22.614\ldots)^\circ = -1 &\Rightarrow x + 22.614\ldots = 180 \\ &\Rightarrow x = 157.380\,135\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 157.4 \text{ (1 dp)}}}.\end{aligned}$$

3. (a) (i) Show that the line with equation $y = 3 - x$ is a tangent to the circle with equation (5)

$$x^2 + y^2 + 14x + 4y - 19 = 0.$$

Solution

$$\begin{aligned}
& x^2 + y^2 + 14x + 4y - 19 = 0 \\
\Rightarrow\ & x^2 + (3 - x)^2 + 14x + 4(3 - x) - 19 = 0 \\
\Rightarrow\ & x^2 + (9 - 6x + x^2) + 14x + 12 - 4x - 19 = 0 \\
\Rightarrow\ & 2x^2 + 4x + 2 = 0 \\
\Rightarrow\ & 2(x^2 + 2x + 1) = 0 \\
\Rightarrow\ & 2(x + 1)^2 = 0 \\
\Rightarrow\ & x = -1 \text{ (twice)};
\end{aligned}$$

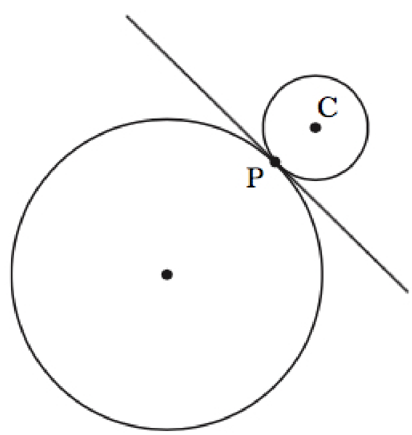
there are equal roots so that means $y = 3 - x$ is a tangent.

(ii) Find the coordinates of the point of contact, P.

Solution

$P(-1, 4)$.

Relative to a suitable set of coordinate axes, the diagram below shows the circle from (a) and a second smaller circle with centre C.



The line $y = 3 - x$ is a common tangent at the point P.
The radius of the larger circle is three times the radius of the smaller circle.

(b) Find the equation of the smaller circle. (6)

Solution

$$\begin{aligned} & x^2 + y^2 + 14x + 4y - 19 = 0 \\ \Rightarrow\ & x^2 + 14x + y^2 + 4y = 19 \\ \Rightarrow\ & (x^2 + 14x + 49) + (y^2 + 4y + 4) = 19 + 49 + 4 \\ \Rightarrow\ & (x+7)^2 + (y+2)^2 = 72; \end{aligned}$$

the radius is $6\sqrt{2}$ and the centre is $(-7, -2)$.

The original centre and P: you go along 6 units to the right and up by 6 units. So, P and C: you go along 2 units to the right and up by 2 units.

Finally, the smaller circle has radius of

$$\sqrt{2^2 + 2^2} = \underline{\underline{2\sqrt{2}}}.$$

4. Solve (5)

$$2\cos 2x - 5\cos x - 4 = 0$$

for $0 \leqslant x < 2\pi$.

Solution

$$\begin{aligned} 2\cos 2x - 5\cos x - 4 = 0 &\Rightarrow 2(2\cos^2 x - 1) - 5\cos x - 4 = 0 \\ &\Rightarrow 4\cos^2 x - 5\cos x - 6 = 0 \end{aligned}$$

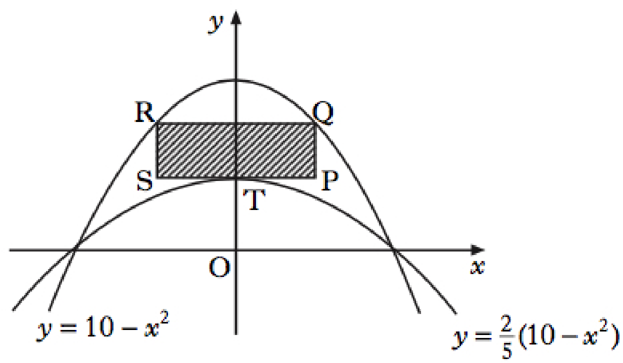
$$\left.\begin{array}{ll} \text{add to:} & -5 \\ \text{multiply to:} & (+4) \times (-6) = -24 \end{array}\right\} -8,\ +3$$

$$\begin{aligned} &\Rightarrow 4\cos^2 x - 8\cos x + 3\cos x - 6 = 0 \\ &\Rightarrow 4\cos x(\cos x - 2) + 3(\cos x - 2) = 0 \\ &\Rightarrow (4\cos x + 3)(\cos x - 2) = 0 \\ &\Rightarrow \cos x = -\tfrac{3}{4} \text{ (and the other has no solutions)} \\ &\Rightarrow x = 2.418\,858\,406, 3.864\,326\,901 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 2.42, 3.86 \text{ (3 sf)}}}. \end{aligned}$$

5. The parabolas with equations

$$y = 10 - x^2 \text{ and } y = \tfrac{2}{5}(10 - x^2)$$

are shown in diagram below.



A rectangle $PQRS$ is placed between the two parabolas as shown, so that Q and R lie on the upper parabola, RQ and SP are parallel to the x-axis, and T, the turning point of the lower parabola, lies on SP.

(a) (i) If $TP = x$ units, find an expression for the length of PQ. (3)

Solution

$$x = 0 \Rightarrow y = 4$$

and so $T(0, 4)$. Finally,

$$PQ = (10 - x^2) - 4 = \underline{\underline{6 - x^2}}.$$

$$TP = x \Rightarrow PS = 2x$$

(ii) Hence show that the area, A, of rectangle $PQRS$ is given by

$$A(x) = 12x - 2x^3.$$

Solution

$$\begin{aligned} A(x) &= 2 \times x \times (6 - x^2) \\ &= 2x(6 - x^2) \\ &= \underline{\underline{12x - 2x^3}}, \end{aligned}$$

as required.

(b) Find the maximum area of this rectangle. (6)

Solution

$$\begin{aligned} A(x) = 12x - 2x^3 &\Rightarrow A'(x) = 12 - 6x^2 \\ &\Rightarrow A''(x) = -12x. \end{aligned}$$

Now,

$$\begin{aligned} A'(x) = 0 &\Rightarrow 12 - 6x^2 = 0 \\ &\Rightarrow 6x^2 = 12 \\ &\Rightarrow x^2 = 2 \\ &\Rightarrow x = \pm\sqrt{2}. \end{aligned}$$

We take $x = \sqrt{2}$ (why?):

$$A(\sqrt{2}) = \underline{\underline{8\sqrt{2}}}.$$

What kind of turning point is it?

$$A''(\sqrt{2}) = -12\sqrt{2} < 0$$

and so it is a maximum.

6. A curve has equation

$$y = (2x - 9)^{\frac{1}{2}}.$$

(a) Show that the equation of the tangent to this curve at the point where $x = 9$ is $y = \frac{1}{3}x$. (5)

Solution

$$\begin{aligned} (2x - 9)^{\frac{1}{2}} = \tfrac{1}{3}x &\Rightarrow 2x - 9 = (\tfrac{1}{3}x)^2 \\ &\Rightarrow 2x - 9 = \tfrac{1}{9}x^2 \\ &\Rightarrow 18x - 81 = x^2 \\ &\Rightarrow x^2 - 18x + 81 = 0 \\ &\Rightarrow (x - 9)^2 = 0 \\ &\Rightarrow x = 9 \text{ (repeated twice)}. \end{aligned}$$

Hence, as we have equal roots, it means that we have a tangent.

Diagram 1 shows part of the curve and the tangent.

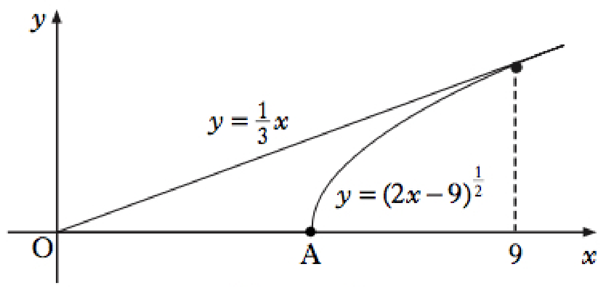


Diagram 1

The curve cuts the x-axis at the point A.

(b) Find the coordinates of point A. (1)

Solution

$$\begin{aligned} y = 0 &\Rightarrow (2x-9)^{\frac{1}{2}} = 0 \\ &\Rightarrow 2x - 9 = 0 \\ &\Rightarrow 2x = 9 \\ &\Rightarrow \underline{\underline{x = 4\tfrac{1}{2}}}. \end{aligned}$$

(c) Calculate the shaded area shown in diagram 2. (7)

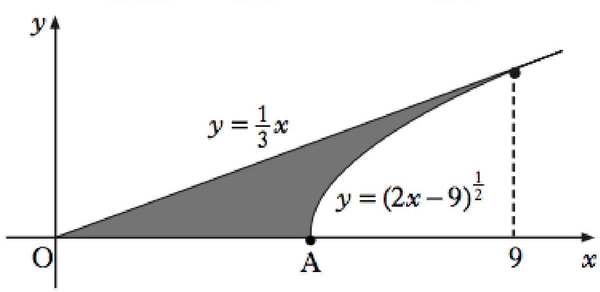


Diagram 2

Solution

$$\begin{aligned}\text{Area} &= \int_0^9 \tfrac{1}{3}x \,\mathrm{d}x - \int_{4\frac{1}{2}}^9 (2x-9)^{\frac{1}{2}} \,\mathrm{d}x \\ &= \left[\tfrac{1}{6}x^2\right]_{x=0}^9 - \left[\tfrac{1}{3}(2x-9)^{\frac{3}{2}}\right]_{x=4\frac{1}{2}}^9 \\ &= \left(13\tfrac{1}{2} - 0\right) - (9-0) \\ &= \underline{\underline{4\tfrac{1}{2}}}.\end{aligned}$$

7. (a) Given that $\log_4 x = P$, show that (3)

$$\log_{16} x = \tfrac{1}{2}P.$$

Solution

$$\begin{aligned}\log_{16} x &= \frac{\log_4 x}{\log_4 16} \\ &= \frac{\log_4 x}{\log_4 4^2} \\ &= \frac{\log_4 x}{2\log_4 4} \\ &= \underline{\underline{\tfrac{1}{2}P}},\end{aligned}$$

as required.

(b) Solve (3)

$$\log_3 x + \log_9 x = 12.$$

Solution

$$\begin{aligned}
\log_3 x + \log_9 x = 12 &\Rightarrow \frac{\log_9 x}{\log_9 3} + \log_9 x = 12 \\
&\Rightarrow \frac{\log_9 x}{\frac{1}{2}} + \log_9 x = 12 \\
&\Rightarrow 2\log_9 x + \log_9 x = 12 \\
&\Rightarrow 3\log_9 x = 12 \\
&\Rightarrow \log_9 x = 4 \\
&\Rightarrow x = 9^4 \\
&\Rightarrow \underline{\underline{x = 6\,561}}.
\end{aligned}$$