Dr Oliver Mathematics
Mathematics
Discriminant
Past Examination Questions

This booklet consists of 22 questions across a variety of examination topics.
The total number of marks available is 150.

1. Given that the equation (4)

$$kx^2 + 12x + k = 0,$$

where k is a positive constant, has equal roots, find the value of k.

Solution

$a = k$, $b = 12$, and $c = k$:

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow 12^2 - 4k^2 = 0 \\ &\Rightarrow 4k^2 = 144 \\ &\Rightarrow k^2 = 36 \\ &\Rightarrow \underline{\underline{k = 6}}. \end{aligned}$$

2. The equation (4)

$$2x^2 - 3x - (k + 1) = 0,$$

where k is a constant, has no real roots. Find the set of possible values of k.

Solution

$a = 2$, $b = -3$, and $c = -(k + 1)$:

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow (-3)^2 - 4 \times 2 \times [-(k + 1)] < 0 \\ &\Rightarrow 9 + 8(k + 1) < 0 \\ &\Rightarrow 8(k + 1) < -9 \\ &\Rightarrow k + 1 < -1\tfrac{1}{8} \\ &\Rightarrow \underline{\underline{k < -2\tfrac{1}{8}}}. \end{aligned}$$

3. The equation $x^2 + 3px + p = 0$, where p is an non-zero constant, has equal roots. Find the value of p. (4)

Solution

$a = 1$, $b = 3p$, and $c = p$:

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (3p)^2 - 4 \times 1 \times p = 0 \\ &\Rightarrow 9p^2 - 4p = 0 \\ &\Rightarrow p(9p - 4) = 0 \\ &\Rightarrow p = 0 \text{ or } \underline{\underline{p = \tfrac{4}{9}}}. \end{aligned}$$

4. Given that the equation $2qx^2 + qx - 1 = 0$, where q is a constant, has no real roots,

(a) show that $q^2 + 8q < 0$. (2)

Solution

$a = 2q$, $b = q$, and $c = -1$:

$$b^2 - 4ac < 0 \Rightarrow q^2 - 4 \times (2q) \times (-1) < 0 \Rightarrow \underline{\underline{q^2 + 8q < 0}}.$$

(b) Hence find the set of possible values of q. (3)

Solution

$$q^2 + 8q < 0 \Rightarrow q(q + 8) < 0 \Rightarrow \underline{\underline{-8 < q < 0}}.$$

5. The equation

$$x^2 + kx + (k + 3) = 0,$$

where k is a constant, has different real roots.

(a) Show that $k^2 - 4k - 12 > 0$. (2)

Solution

$a = 1$, $b = k$, and $c = k + 3$:

$$b^2 - 4ac > 0 \Rightarrow k^2 - 4 \times 1 \times (k + 3) > 0 \Rightarrow \underline{\underline{k^2 - 4k - 12 > 0}}.$$

(b) Find the set of possible values of k. (4)

Solution

$$k^2 - 4k - 12 > 0 \Rightarrow (k-6)(k+2) > 0 \Rightarrow \underline{\underline{k < -2}} \text{ or } \underline{\underline{k > 6}}.$$

6.

$$x^2 - 8x - 29 \equiv (x+a)^2 + b,$$

where a and b are constants.

(a) Find the value of a and find the value of b. (3)

Solution

$$\begin{aligned} x^2 - 8x - 29 &= (x^2 - 8x + 16) - 45 \\ &= \underline{\underline{(x-4)^2 - 45}}, \end{aligned}$$

hence, $\underline{\underline{a = -4}}$ and $\underline{\underline{b = -45}}$.

(b) Hence, or otherwise, show that the roots of (3)

$$x^2 - 8x - 29 = 0$$

are $c \pm \sqrt{5}$, where c and d are constants.

Solution

$$\begin{aligned} x^2 - 8x - 29 = 0 &\Rightarrow (x-4)^2 - 45 = 0 \\ &\Rightarrow (x-4)^2 = 45 \\ &\Rightarrow x - 4 = \pm\sqrt{45} \\ &\Rightarrow x - 4 = \pm 3\sqrt{5} \\ &\Rightarrow \underline{\underline{x = 4 \pm 3\sqrt{5}}}. \end{aligned}$$

7. The equation $x^2 + 2px + (3p+4) = 0$, where p is a positive constant, has equal roots.

(a) Find the value of p. (4)

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Solution

$a = 1$, $b = 2p$, and $c = 3p + 4$:

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (2p)^2 - 4 \times 1 \times (3p+4) = 0 \\ &\Rightarrow 4p^2 = 12p + 16 \\ &\Rightarrow 4p^2 - 12p - 16 = 0 \\ &\Rightarrow 4(p^2 - 3p - 4) = 0 \\ &\Rightarrow 4(p-4)(p+1) = 0, \\ &\Rightarrow p = 4 \text{ or } p = -1, \end{aligned}$$

and hence $\underline{\underline{p = 4}}$.

(b) For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$. (2)

Solution

$$x^2 + 8x + 16 = 0 \Rightarrow (x+4)^2 = 0 \Rightarrow \underline{\underline{x = -4}}.$$

8. (a) Show that $x^2 + 6x + 11$ can be written as (2)

$$(x+p)^2 + q,$$

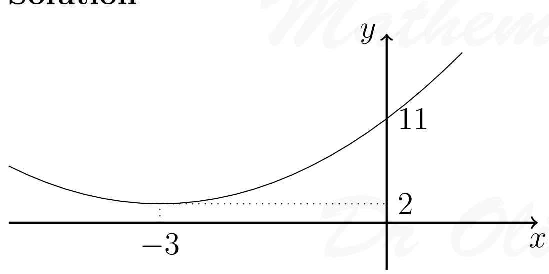
where p and q are constants.

Solution

$$x^2 + 6x + 11 = (x^2 + 6x + 9) + 2 = \underline{\underline{(x+3)^2 + 2}}.$$

(b) Sketch the graph of $y = x^2 + 6x + 11$, indicating clearly the coordinates of any intersections with the coordinate axes. (2)

Solution



(c) Find the value of the discriminant of $x^2 + 6x + 11$. (2)

> **Solution**
> $a = 1$, $b = 6$, and $c = 11$:
>
> $$b^2 - 4ac = 6^2 - 4 \times 1 \times (-11) = 36 - 44 = \underline{\underline{-8}}.$$

9.

$$f(x) = x^2 + (k + 3)x + k,$$

where k is a real constant.

(a) Find the discriminant of $f(x)$ in terms of k. (2)

> **Solution**
> $a = 1$, $b = k + 3$, and $c = k$:
>
> $$b^2 - 4ac = (k + 3)^2 - 4 \times 1 \times k = (k^2 + 6k + 9) - 4k = \underline{\underline{k^2 + 2k + 9}}.$$

(b) Show that the discriminant of $f(x)$ can be expressed in the form (2)

$$(k + a)^2 + b,$$

where a and b are integers to be found.

> **Solution**
>
> $$k^2 + 2k + 9 = (k^2 + 2k + 1) + 8 = \underline{\underline{(k + 1)^2 + 8}}.$$

(c) Show that, for all values of k, the equation $f(x) = 0$ has real roots. (2)

> **Solution**
> For all k, $(k + 1)^2 + 8 > 0$; this is what we need to establish $\underline{\underline{\text{real roots}}}$.

10. The equation

$$x^2 + kx + 8 = k$$

has no real solutions for x.

(a) Show that k satisfies $k^2 + 4k - 32 < 0$. (3)

Solution

$a = 1$, $b = k$, and $c = 8 - k$:

$$b^2 - 4ac < 0 \Rightarrow k^2 - 4 \times 1 \times (8 - k) < 0 \Rightarrow \underline{\underline{k^2 + 4k - 32 < 0}}.$$

(b) Hence find the set of possible values of k. (4)

Solution

$$k^2 + 4k - 32 < 0 \Rightarrow (k + 8)(k - 4) < 0 \Rightarrow \underline{\underline{-8 < k < 4}}.$$

11. The equation $kx^2 + 4x + (5 - k) = 0$, where k is a constant, has 2 different real solutions for x.

(a) Show that k satisfies $k^2 - 5k + 4 > 0$. (3)

Solution

$a = k$, $b = 4$, and $c = 5 - k$:

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow 4^2 - 4 \times k \times (5 - k) > 0 \\ &\Rightarrow 16 - 4k(5 - k) > 0 \\ &\Rightarrow 16 - 20k + 5k^2 > 0 \\ &\Rightarrow 5(k^2 - 5k + 4) > 0 \\ &\Rightarrow \underline{\underline{k^2 - 5k + 4 > 0}}. \end{aligned}$$

(b) Hence find the set of possible values of k. (4)

Solution

$$k^2 - 5k + 4 < 0 \Rightarrow (k - 1)(k - 4) < 0 \Rightarrow \underline{\underline{k < 1}} \text{ or } \underline{\underline{k > 4}}.$$

12. The equation

$$x^2 + (k - 3)x + (3 - 2k) = 0,$$

where k is a constant, has two distinct real roots.

(a) Show that k satisfies (3)

$$k^2 + 2k - 3 > 0.$$

Solution

$a = 1$, $b = k - 3$, and $c = 3 - 2k$:

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow (k-3)^2 - 4 \times 1 \times (3 - 2k) > 0 \\ &\Rightarrow (k^2 - 6k + 9) - (12 - 8k) > 0 \\ &\Rightarrow \underline{\underline{k^2 + 2k - 3 > 0}}. \end{aligned}$$

(b) Hence find the set of possible values of k. (4)

Solution

$$k^2 + 2k - 3 < 0 \Rightarrow (k+3)(k-1) < 0 \Rightarrow \underline{\underline{k < -3}} \text{ or } \underline{\underline{k > 1}}.$$

13. Given the simultaneous equations

$$\begin{aligned} 2x + y &= 1 \\ x^2 - 4ky + 5k &= 0, \end{aligned}$$

where k is a non-zero constant,

(a) show that (2)

$$x^2 + 8kx + k = 0.$$

Solution

$$y = 1 - 2x \Rightarrow x^2 - 4k(1 - 2x) + 5k = 0 \Rightarrow \underline{\underline{x^2 + 8kx + k = 0}}.$$

Given that $x^2 + 8kx + k = 0$ has equal roots,

(b) find the value of k. (3)

Solution

$a = 1$, $b = 8k$, and $c = k$:

$$b^2 - 4ac = 0 \Rightarrow (8k)^2 - 4 \times 1 \times k = 0 \Rightarrow 64k^2 - 4k = 0 \Rightarrow 4k(16k - 1) = 0.$$

Since $k \neq 0$, $\underline{\underline{k = \frac{1}{16}}}$.

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(c) For this value of k, find the solution of the simultaneous equations. (3)

Solution

$k = \frac{1}{16}$:

$$x^2 + \tfrac{1}{2}x + \tfrac{1}{16} = 0 \Rightarrow (x + \tfrac{1}{4})^2 = 0 \Rightarrow \underline{\underline{x = -\tfrac{1}{4}}}$$

and

$$y = 1 - 2 \times (-\tfrac{1}{4}) = \underline{\underline{1\tfrac{1}{2}}}.$$

14.

$$x^2 + 2x + 3 \equiv (x + a)^2 + b,$$

where a and b are constants.

(a) Find the value of a and find the value of b. (2)

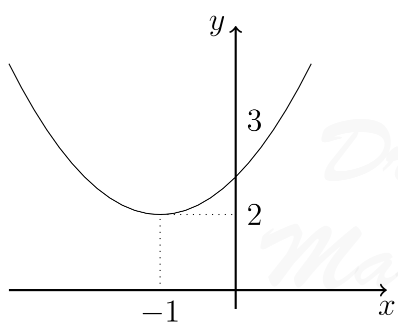
Solution

$$\begin{aligned} x^2 + 2x + 3 &= (x^2 + 2x + 1) + 2 \\ &= \underline{\underline{(x + 1)^2 + 2}}, \end{aligned}$$

hence, $\underline{\underline{a = 1}}$ and $\underline{\underline{b = 2}}$.

(b) Sketch the graph of $y = x^2 + 2x + 3$, indicating clearly the coordinates of any intersections with the coordinate axes. (3)

Solution



(c) Find the value of the discriminant of $x^2 + 2x + 3$. Explain how the sign of the discriminant relates to your sketch in part (b). (2)

Solution

$a = 1$, $b = 2$, and $c = 3$:

$$b^2 - 4ac = 2^2 - 4 \times 1 \times 3 = \underline{\underline{-8}}.$$

There is no intercept between the x-axis and graph.

The equation $x^2 + kx + 3 = 0$, where k is a constant, has no real roots.

(d) Find the set of possible values of k, giving your answer in surd form. (4)

Solution

$a = 1$, $b = k$, and $c = 3$:

$$b^2 - 4ac < 0 \Rightarrow k^2 - 4 \times 1 \times 3 < 0 \Rightarrow k^2 < 12 \Rightarrow \underline{\underline{|k| < 2\sqrt{3}}}.$$

15. The equation

$$(k + 3)x^2 + 6x + k = 6,$$

where k is a constant, has two distinct real solutions for x.

(a) Show that k satisfies (4)

$$k^2 - 2k - 24 < 0.$$

Solution

$a = k + 3$, $b = 6$, and $c = k - 5$:

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow 6^2 - 4 \times (k + 3) \times (k - 5) > 0 \\ &\Rightarrow 36 - 4(k^2 - 2k - 15) > 0 \\ &\Rightarrow k^2 - 2k - 15 < 9 \\ &\Rightarrow \underline{\underline{k^2 - 2k - 24 < 0}}. \end{aligned}$$

(b) Hence find the set of possible values of k. (3)

Solution

$$k^2 - 2k - 24 < 0 \Rightarrow (k - 6)(k + 4) < 0 \Rightarrow \underline{\underline{-4 < k < 6}}.$$

16. The equation

$$(p-1)x^2 + 4x + (p-5) = 0,$$

where p is a constant, has no real roots.

(a) Show that p satisfies $p^2 - 6p + 1 > 0$. (3)

Solution

$a = p - 1$, $b = 4$, and $c = p - 5$:

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow 4^2 - 4 \times (p-1) \times (p-5) < 0 \\ &\Rightarrow 0 < 4(p-1)(p-5) - 16 \\ &\Rightarrow 0 < (p-1)(p-5) - 4 \\ &\Rightarrow 0 < (p^2 - 6p + 5) - 4 \\ &\Rightarrow \underline{\underline{p^2 - 6p + 1 > 0}}. \end{aligned}$$

(b) Hence find the set of possible values of p. (4)

Solution

$$\begin{aligned} p^2 - 6p + 1 > 0 &\Rightarrow p^2 - 6p + 9 > 8 \\ &\Rightarrow (p-3)^2 > 8 \\ &\Rightarrow p - 3 < -2\sqrt{2} \text{ or } p - 3 > 2\sqrt{2} \\ &\Rightarrow \underline{\underline{p < 3 - 2\sqrt{2}}} \text{ or } \underline{\underline{p > 3 + 2\sqrt{2}}}. \end{aligned}$$

17.

$$4x - 5 - x^2 = q - (x+p)^2,$$

where p and q are integers.

(a) Find the value of p and the value of q. (3)

Solution

$$4x - 5 - x^2 = -1 - (x^2 - 4x + 4) = \underline{\underline{-1 - (x-2)^2}};$$

hence, $\underline{\underline{p = -2}}$ and $\underline{\underline{q = -1}}$.

(b) Calculate the discriminate of $4x - 5 - x^2$. (2)

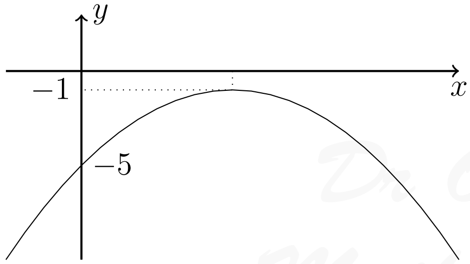
Solution

$a = -1$, $b = 4$, and $c = -5$:

$$b^2 - 4ac = 4^2 - 4 \times (-1) \times (-5) = 16 - 20 = \underline{\underline{-4}}.$$

(c) Sketch the graph of $y = 4x - 5 - x^2$, showing the coordinates of any point at which the graph crosses the coordinate axes. (3)

Solution



18. The straight line with equation $y = 3x - 7$ does not cross or touch the curve with equation $y = 2px^2 - 6px + 4p$, where p is a constant.

(a) Show that $4p^2 - 20p + 9 < 0$. (4)

Solution

$$2px^2 - 6px + 4p = 3x - 7 \Rightarrow 2px^2 - (6p + 3)x + (4p + 7) = 0.$$

Now, $a = 2p$, $b = -(6p + 3)$, and $c = 4p + 7$:

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow (6p + 3)^2 - 4 \times 2p \times (4p + 7) < 0 \\ &\Rightarrow (36p^2 + 36p + 9) - 8p(4p + 7) < 0 \\ &\Rightarrow 36p^2 + 36p + 9 - 32p^2 - 56p < 0 \\ &\Rightarrow \underline{\underline{4p^2 - 20p + 9 < 0}}. \end{aligned}$$

(b) Hence find the set of possible values of p. (4)

Solution

$$4p^2 - 20p + 9 < 0 \Rightarrow (2p - 9)(2p - 1) < 0 \Rightarrow \underline{\underline{\tfrac{1}{2} < p < 4\tfrac{1}{2}}}.$$

19.

$$f(x) = x^2 + 4kx + (3 + 11k), \text{ where } k \text{ is a constant.}$$

(a) Express $f(x)$ in the form $(x + p)^2 + q$, where p and q are constants to be found in terms of k. (3)

Solution

$$\begin{aligned} x^2 + 4kx + (3 + 11k) &= (x^2 + 4kx + 4k^2) + (3 + 11k - 4k^2) \\ &= (x + 2k)^2 + (3 + 11k - 4k^2); \end{aligned}$$

hence, $p = 2k$ and $q = 3 + 11k - 4k^2$.

Given that the equation $f(x) = 0$ has no real roots,

(b) find the set of possible values of k. (4)

Solution

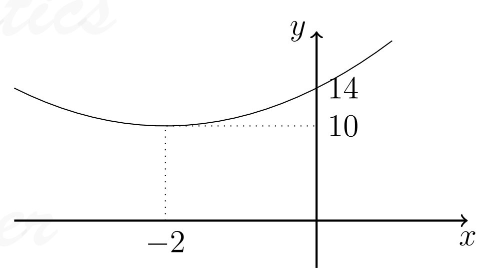
$a = 1$, $b = 4k$, and $c = 3 + 11k$:

$$\begin{aligned} b^2 - 4ac < 0 &\Rightarrow (4k)^2 - 4 \times 1 \times (3 + 11k) < 0 \\ &\Rightarrow 16k^2 - 44k - 12 < 0 \\ &\Rightarrow 4(4k^2 - 11k - 3) < 0 \\ &\Rightarrow 4k^2 - 11k - 3 < 0 \\ &\Rightarrow (4k + 1)(k - 3) < 0 \\ &\Rightarrow -\tfrac{1}{4} < k < 3. \end{aligned}$$

Given that $k = 1$,

(c) sketch the graph of $y = f(x)$, showing the coordinates of any point at which the graph crosses a coordinate axis. (3)

Solution



The graph is $y = x^2 + 4x + 14 = (x + 2)^2 + 10$.

20. Given that

$$f(x) = 2x^2 + 8x + 3,$$

(a) find the value of the discriminant of $f(x)$. (2)

Solution

$a = 2$, $b = 8$, and $c = 3$:

$$b^2 - 4ac = 8^2 - 4 \times 2 \times 3 = 64 - 24 = \underline{\underline{40}}.$$

(b) Express $f(x)$ in the form $p(x+q)^2 + r$, where p, q, and r are integers to be found. (3)

Solution

$$2x^2 + 8x + 3 = 2(x^2 + 4x + 4) - 5 = \underline{\underline{2(x+2)^2 - 5}}.$$

The line $y = 4x + c$, where c is a constant, is a tangent to the curve with equation $y = f(x)$.

(c) Calculate the value of c. (5)

Solution

$$2x^2 + 8x + 3 = 4x + c \Rightarrow 2x^2 + 4x + (3 - c) = 0.$$

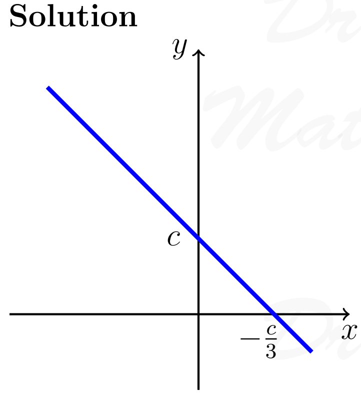
If it is a tangent, $b^2 - 4ad = 0$: $a = 2$, $b = 4$, and $d = 3 - c$ and we have

$$4^2 - 4 \times 2 \times (3 - c) = 0 \Rightarrow 16 - 8(3 - c) = 0 \Rightarrow 8c = 8 \Rightarrow \underline{\underline{c = 1}}.$$

21. (a) On separate axes sketch the graphs of (4)

(i) $y = -3x + c$, where c is a positive constant,

Solution



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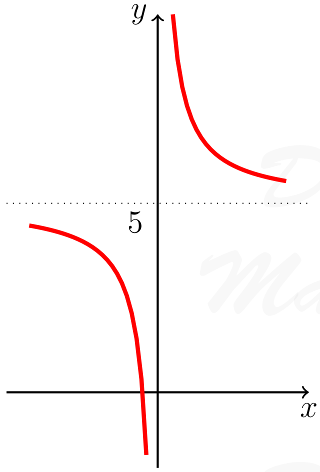
(ii) $y = \frac{1}{x} + 5$.

Solution



$\underline{\underline{y = 5}}$

On each sketch show the coordinates of any point at which the graph crosses the y-axis and the equation of any horizontal asymptote.

Given that $y = -3x + c$, where c is a positive constant, meets the curve $y = \frac{1}{x} + 5$ at two distinct points,

(b) show that $(5 - c)^2 > 12$. (3)

Solution

$$-3x + c = \frac{1}{x} + 5 \Rightarrow -3x^2 + cx = 1 + 5x$$
$$\Rightarrow 3x^2 + (5 - c)x + 1 = 0.$$

$a = 3$, $b = 5 - c$, and $c = 1$:

$$b^2 - 4ac > 0 \Rightarrow (5 - c)^2 - 4 \times 3 \times 1 > 0$$
$$\Rightarrow \underline{\underline{(5 - c)^2 > 12}}.$$

(c) Hence find the range of possible values for c. (4)

Solution

$$(5-c)^2 > 12 \Rightarrow 5-c < -2\sqrt{3} \text{ or } 5-c > 2\sqrt{3}$$
$$\Rightarrow \underline{\underline{5+2\sqrt{3} < c}} \text{ or } \underline{\underline{5-2\sqrt{3} > c > 0}}$$

as c is a positive constant.

22. (2)

$$\mathrm{f}(x) = x^2 - 8x + 13.$$

Express $\mathrm{f}(x)$ in the form $(x+a)^2 + b$, where a and b are constants.

Solution

$$x^2 - 8x + 13 = (x^2 - 8x + 16) + 3$$
$$= \underline{\underline{(x-4)^2 + 3}},$$

hence, $\underline{\underline{a = -4}}$ and $\underline{\underline{b = 3}}$.

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