Dr Oliver Mathematics
Mathematics: Advanced Higher
2008 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. The first term of an arithmetic sequence is 2 and the 20th term is 97. (4)
   Obtain the sum of the first 50 terms.

2. (a) Differentiate (2)

$$f(x) = \cos^{-1}(3x)$$

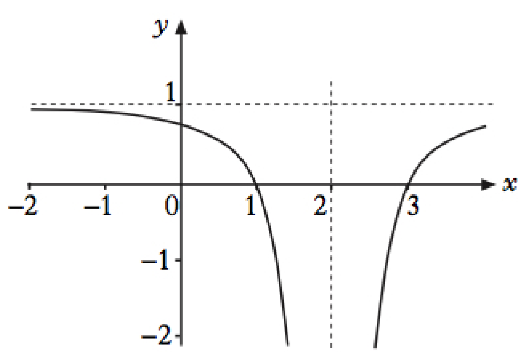
   where $-\frac{1}{3} < x < \frac{1}{3}$.

   (b) Given (3)

$$x = 2\sec\theta, y = 3\sin\theta,$$

   use parametric differentiation to find $\dfrac{dy}{dx}$ in terms of θ.

3. Part of the graph $y = f(x)$ is shown below, where the dotted lines indicate asymptotes. (4)



   Sketch the graph $y = -f(x+1)$ showing its asymptotes.
   Write down the equations of the asymptotes.

4. (a) Express (3)

$$\frac{12x^2 + 20}{x(x^2 + 5)}$$

   in partial fractions.

(b) Hence evaluate (3)

$$\int_1^2 \frac{12x^2 + 20}{x(x^2 + 5)} \, \mathrm{d}x.$$

5. A curve is defined by the equation

$$xy^2 + 3x^2y = 4$$

for $x > 0$ and $y > 0$.

(a) Use implicit differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (3)

(b) Hence find an equation of the tangent to the curve where $x = 1$. (3)

6. Let the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}.$$

(a) Obtain the value(s) of x for which $\mathbf{A}$ is singular. (2)

(b) When $x = 2$, show that (1)

$$\mathbf{A}^2 = p\mathbf{A}$$

for some constant p.

(c) Determine the value of q such that (2)

$$\mathbf{A}^4 = q\mathbf{A}.$$

7. Use integration by parts to obtain (5)

$$\int 8x^2 \sin 4x \, \mathrm{d}x.$$

8. (a) Write down and simplify the general term in the expansion of (3)

$$\left(x^2 + \frac{1}{x}\right)^{10}.$$

(b) Hence, or otherwise, obtain the term in x^{14}. (2)

9. (a) Write down the derivative of $\tan x$. (1)

(b) Show that (1)

$$1 + \tan^2 x = \sec^2 x.$$

(c) Hence obtain (2)

$$\int \tan^2 x \, \mathrm{d}x.$$

10. A body moves along a straight line with velocity

$$v = t^3 - 12t^2 + 32t$$

at time t.

(a) Obtain the value of its acceleration when $t = 0$. (1)

(b) At time $t = 0$, the body is at the origin O. (2)
Obtain a formula for the displacement of the body at time t.

(c) Show that the body returns to O, and obtain the time, T, when this happens. (2)

11. For each of the following statements, decide whether it is true or false and prove your conclusion.

(a) For all natural numbers m, if m^2 is divisible by 4, then m is divisible by 4. (2)

(b) The cube of any odd integer p plus the square of any even integer q is always odd. (3)

12. Throughout this question, it can be assumed that $-2 < x < 2$.

(a) Obtain the first three non-zero terms in the Maclaurin expansion of (3)

$$x \ln(2 + x).$$

(b) Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of (2)

$$x \ln(2 - x).$$

(c) Hence obtain the first **two** non-zero terms in the Maclaurin expansion of (2)

$$x \ln(4 - x^2).$$

13. (a) Obtain the general solution of the differential equation (7)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2x^2.$$

(b) Given that $y = \frac{1}{2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$, when $x = 0$, find the particular solution. (3)

14. (a) Find an equation of the plane π_1 through the points $A(1, 1, 1)$, $B(2, -1, 1)$, and $C(0, 3, 3)$. (3)

The plane π_2 has equation $x + 3y - z = 2$.

(b) Given that the point $(0, a, b)$ lies on both the planes π_1 and π_2, find the values of a and b. (3)

(c) Hence find an equation of the line of intersection of the planes π_1 and π_2. (1)

(d) Find the size of the acute angle between the planes π_1 and π_2. (3)

15. Let

$$f(x) = \frac{x}{\ln x}$$

for $x > 1$.

(a) Derive expressions for $f'(x)$ and $f''(x)$, simplifying your answers. (4)

(b) Obtain the coordinates and nature of the stationary point of the curve $y = f(x)$. (3)

(c) Obtain the coordinates of the point of inflexion. (2)

16. (a) Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to write down an expression for z^k in terms of θ, where k is a positive integer. (1)

(b) Hence show that (2)

$$\frac{1}{z^k} = \cos k\theta - i\sin k\theta.$$

(c) Deduce expressions for $\cos k\theta$ and $\sin k\theta$ in terms of z. (2)

(d) Show that (3)

$$\cos^2\theta\sin^2\theta = -\tfrac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2.$$

(e) Hence show that (2)

$$\cos^2\theta\sin^2\theta = a + b\cos 4\theta,$$

for suitable constants a and b.