Dr Oliver Mathematics
Mathematics
Moments
Past Examination Questions

This booklet consists of 25 questions across a variety of examination topics. The total number of marks available is 254.

1. A uniform rod AB has length 3 m and weight 120 N. The rod rests in equilibrium in a horizontal position, smoothly supported at points C and D, where $AC = 0.5$ m and $AD = 2$ m, as shown in Figure 1.

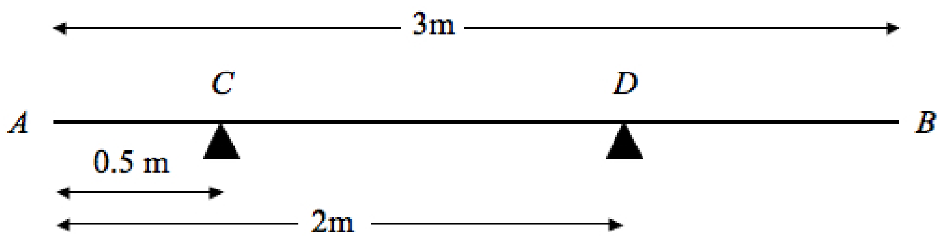


Figure 1: a uniform rod AB has length 3 m and weight 120 N

A particle of weight W newtons is attached to the rod at a point E where $AE = x$ metres. The rod remains in equilibrium and the magnitude of the reaction at C is now twice the magnitude of the reaction at D.

(a) Show that (8)

$$W = \frac{60}{1 - x}.$$

Solution

Let $2R$ and R be tensions at C and D respectively. Then

$$3R = W + 120.$$

Moments about A:

$$(0.5 \times 2R) + (2 \times R) = (x \times W) + (1.5 \times 120) \Rightarrow 3R = xW + 180.$$

The LHS of the two equations are equal:

$$\begin{aligned} W + 120 = xW + 180 &\Rightarrow W - xW = 60 \\ &\Rightarrow (1-x)W = 60 \\ &\Rightarrow \underline{\underline{W = \frac{60}{1-x}}}, \end{aligned}$$

as required.

(b) Hence deduce the range of possible values of x. (2)

Solution

$$\begin{aligned} W > 0 &\Rightarrow \frac{60}{1-x} > 0 \\ &\Rightarrow 1 - x > 0 \\ &\Rightarrow \underline{\underline{1 > x}}. \end{aligned}$$

2. A plank AB has mass 40 kg and length 3 m. A load of mass 20 kg is attached to the plank at B. The loaded plank is held in equilibrium, with AB horizontal, by two vertical ropes attached at A and C, as shown in Figure 2.

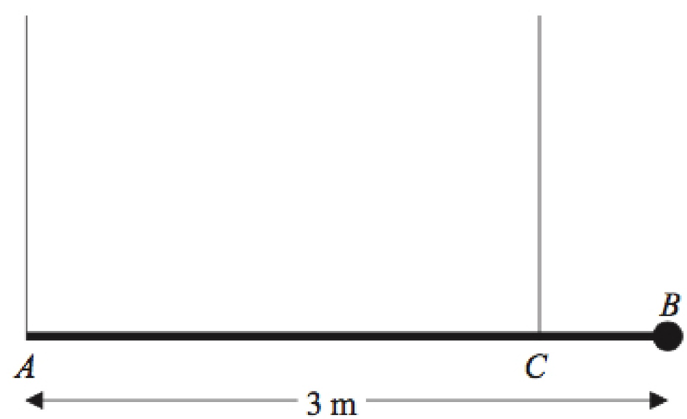


Figure 2: a plank AB has mass 40 kg and length 3 m

The plank is modelled as a uniform rod and the load as a particle. Given that the tension in the rope at C is three times the tension in the rope at A, calculate

(a) the tension in the rope at C, (2)

Solution

Let R and $3R$ be the tensions at A and C respectively. Then

$$\begin{aligned} R + 3R = 40g + 20g &\Rightarrow 4R = 60g \\ &\Rightarrow R = 15g \\ &\Rightarrow \underline{\underline{3R = 45g}}. \end{aligned}$$

(b) the distance CB. (5)

Solution

Let the distance CB Moments about B:

$$\begin{aligned} (3 \times 15g) + (CB \times 45g) = (1.5 \times 40g) &\Rightarrow CB \times 45g = 15g \\ &\Rightarrow \underline{\underline{CB = \tfrac{1}{3}\ m}}. \end{aligned}$$

3. A uniform beam AB has mass 12 kg and length 3 m. The beam rests in equilibrium in a horizontal position, resting on two smooth supports. One support is at end A, the other at a point C on the beam, where $BC = 1$ m, as shown in Figure 3.

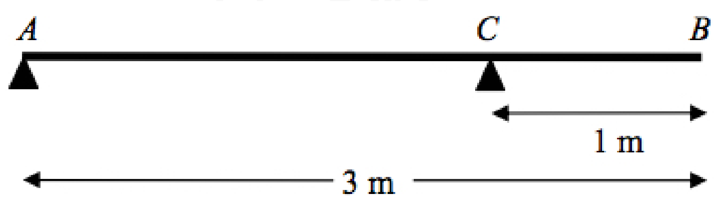


Figure 3: a uniform beam AB has mass 12 kg and length 3 m

The beam is modelled as a uniform rod.

(a) Find the reaction on the beam at C. (3)

Solution

Let S be the tension at C. Moments about A:

$$1.5 \times 12g = 2 \times S \Rightarrow \underline{\underline{S = 9g}}.$$

A woman of mass 48 kg stands on the beam at the point D. The beam remains in equilibrium. The reactions on the beam at A and C are now equal.

(b) Find the distance AD. (7)

Solution

Let R be the tension at A and C at respectively. Then

$$2R = 48g + 12g \Rightarrow R = 30g.$$

Then

$$\begin{aligned}(AD \times 48g) + (1.5 \times 12g) = (2 \times 30g) &\Rightarrow AD \times 48g = 42g \\ &\Rightarrow \underline{\underline{AD = \tfrac{7}{8}\text{ m}}}.\end{aligned}$$

4. A seesaw in a playground consists of a beam AB of length 4 m which is supported by a smooth pivot at its centre C. Jill has mass 25 kg and sits on the end A. David has mass 40 kg and sits at a distance x metres from C, as shown in Figure 4.

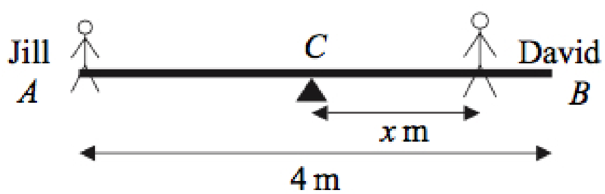


Figure 4: a seesaw in a playground consists of a beam AB

The beam is initially modelled as a uniform rod. Using this model,

(a) find the value of x for which the seesaw can rest in equilibrium in a horizontal position. (3)

Solution

Moments about C:

$$(2 \times 25g) = (x \times 40g) \Rightarrow \underline{\underline{x = 1.25\text{ m}}}.$$

(b) State what is implied by the modelling assumption that the beam is uniform. (1)

> **Solution**
> E.g., the mass acts at mid-point,
> mass is evenly distributed.

David realises that the beam is not uniform as he finds that he must sit at a distance 1.4 m from C for the seesaw to rest horizontally in equilibrium. The beam is now modelled as a non-uniform rod of mass 15 kg. Using this model,

(c) find the distance of the centre of mass of the beam from C. (4)

> **Solution**
> The seesaw's centre of mass must be to the left of C (why?).
> Moments about C:
>
> $$\begin{aligned}(2 \times 25g) + (y \times 15g) = (1.4 \times 40g) &\Rightarrow 50 + 15y = 56 \\ &\Rightarrow 15y = 6 \\ &\Rightarrow \underline{\underline{y = 0.4 \text{ m}}}.\end{aligned}$$

5. A steel girder AB has weight 210 N. It is held in equilibrium in a horizontal position by two vertical cables. One cable is attached to the end A. The other cable is attached to the point C on the girder, where $AC = 90$ cm, as shown in Figure 5.

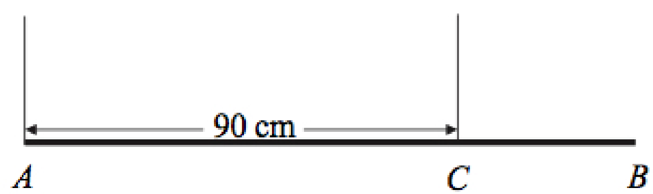


Figure 5: a steel girder AB has weight 210 N

The girder is modelled as a uniform rod, and the cables as light inextensible strings. Given that the tension in the cable at C is twice the tension in the cable at A, find

(a) the tension in the cable at A, (2)

> **Solution**
> Let $2R$ and R be the tensions at A and C respectively. Then
>
> $$3R = 210 \Rightarrow \underline{\underline{R = 70 \text{ N}}}.$$

(b) show that $AB = 120$ cm. (4)

Solution
Let x cm be the centre of mass of the steel bar.
Moments about A:

$$x \times 210 = 90 \times 140 \Rightarrow x = 60$$

and

$$AB = 2 \times 60 = \underline{\underline{120 \text{ cm}}}.$$

A small load of weight W newtons is attached to the girder at B. The load is modelled as a particle. The girder remains in equilibrium in a horizontal position. The tension in the cable at C is now three times the tension in the cable at A.

(c) Find the value of W. (7)

Solution
Let $3S$ and S be the tensions at A and C respectively. Then

$$4S = 210 + W.$$

Moments about A:

$$(60 \times 210) + (120 \times W) = (90 \times 3S).$$

Now,

$$\begin{aligned} S = 52.5 + \tfrac{1}{4}W &\Rightarrow 12\,600 + 120W = 270(52.5 + \tfrac{1}{4}W) \\ &\Rightarrow 12\,600 + 120W = 14\,175 + 67.5W \\ &\Rightarrow 52.5W = 1\,575 \\ &\Rightarrow \underline{\underline{W = 30}}. \end{aligned}$$

6. A uniform plank AB has weight 120 N and length 3 m. The plank rests horizontally in equilibrium on two smooth supports C and D, where $AC = 1$ m and $CD = x$ m, as shown in Figure 6.

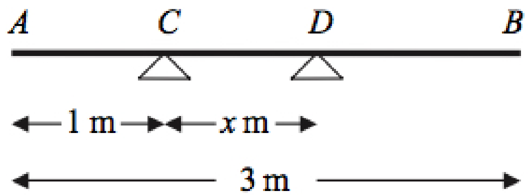


Figure 6: a uniform plank AB has weight 120 N and length 3 m

The reaction of the support on the plank at D has magnitude 80 N. Modelling the plank as a rod,

(a) show that $x = 0.75$. (3)

Solution

The centre of mass in the middle: 1.5 m from each end and 0.5 m to the right of C.
Moments about C:

$$x \times 80 = 0.5 \times 120 \Rightarrow \underline{\underline{x = 0.75}}.$$

A rock is now placed at B and the plank is on the point of tilting about D. Modelling the rock as a particle, find

(b) the weight of the rock, (4)

Solution

The reaction of the support on the plank at C has magnitude 0 N. This means the reaction at D equals 120 N and also means

$$1.75 - 1.5 = 0.25 \text{ m}$$

between C and D. Let W N be the weight of the rock.
Moments about D:

$$0.25 \times 120 = 1.25 \times W \Rightarrow \underline{\underline{W = 24}}.$$

(c) the magnitude of the reaction of the support on the plank at D. (2)

Solution

$$120 + 24 = \underline{\underline{144 \text{ N}}}.$$

(d) State how you have used the model of the rock as a particle. (1)

Solution
The weight of the rock $\underline{\underline{\text{acts exactly at } B}}$.

7. A uniform rod AB has length 1.5 m and mass 8 kg. A particle of mass m kg is attached to the rod at B. The rod is supported at the point C, where $AC = 0.9$ m, and the system is in equilibrium with AB horizontal, as shown in Figure 7.

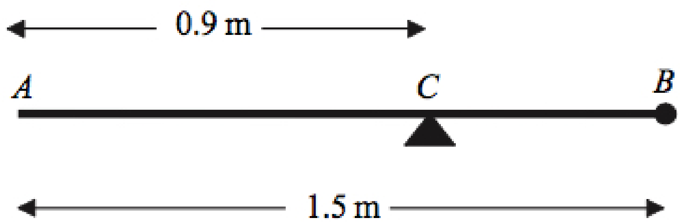


Figure 7: a uniform rod AB has length 1.5 m and mass 8 kg

(a) Show that $m = 2$. (4)

Solution
The distance between A and the centre of mass is

$$0.9 - 0.75 = 0.15 \text{ m}$$

and the distance between B and the centre of mass is

$$1.5 - 0.9 = 0.6 \text{ m}.$$

Moments about C:

$$0.15 \times 8g = 0.6 \times mg \Rightarrow \underline{\underline{m = 2}},$$

as required.

A particle of mass 5 kg is now attached to the rod at A and the support is moved from C to a point D of the rod. The system, including both particles, is again in equilibrium with AB horizontal.

(b) Find the distance AD. (5)

Solution

Moments about D:

$$\begin{aligned} & AD \times 5g = (0.75 - AD) \times 8g + (1.5 - AD) \times 2g \\ \Rightarrow \quad & 5AD = (6 - 8AD) + (3 - 2AD) \\ \Rightarrow \quad & 15AD = 9 \\ \Rightarrow \quad & \underline{\underline{AD = 0.6 \text{ m}}}. \end{aligned}$$

8. A beam AB has mass 12 kg and length 5 m. It is held in equilibrium in a horizontal position by two vertical ropes attached to the beam. One rope is attached to A, the other to the point C on the beam, where $BC = 1$ m, as shown in Figure 8.

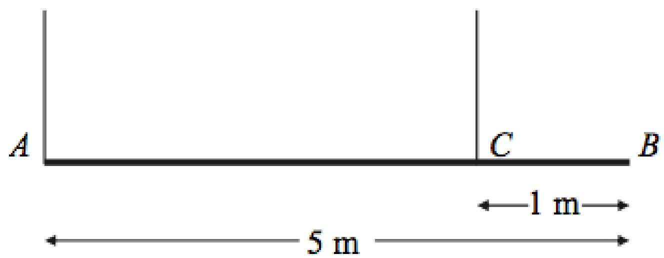


Figure 8: a beam AB has mass 12 kg and length 5 m

The beam is modelled as a uniform rod, and the ropes as light strings.

(a) Find (5)

(i) the tension in the rope at C,

Solution
Let R and S be the tensions at A and C respectively. Then

$$R + S = 12g.$$

Moments about A:

$$4 \times S = 2.5 \times 12g \Rightarrow \underline{\underline{S = 7.5g}}.$$

(ii) the tension in the rope at A.

Solution

$$R = 12g - 7.5g = \underline{\underline{4.5g}}.$$

A small load of mass 16 kg is attached to the beam at a point which is y metres from A. The load is modelled as a particle. Given that the beam remains in equilibrium in a horizontal position,

(b) find, in terms of y, an expression for the tension in the rope at C. (3)

Solution

Now, let T and U be the normal reactions at A and C respectively.
Moments about A:

$$4 \times U = 2.5 \times 12g + y \times 16g \Rightarrow \underline{\underline{U = (7.5 + 4y)g}}.$$

The rope at C will break if its tension exceeds 98 N. The rope at A cannot break.

(c) Find the range of possible positions on the beam where the load can be attached without the rope at C breaking. (3)

Solution

$$\begin{aligned} U \leqslant 98 &\Rightarrow (7.5 + 4y)g \leqslant 98 \\ &\Rightarrow 7.5 + 4y \leqslant 10 \\ &\Rightarrow 4y \leqslant 2.5 \\ &\Rightarrow y \leqslant \tfrac{5}{8}; \end{aligned}$$

hence, <u>the load must be no further than $\frac{5}{8}$ m from A</u>.

9. A plank AB has mass 12 kg and length 2.4 m. A load of mass 8 kg is attached to the plank at the point C, where $AC = 0.8$ m. The loaded plank is held in equilibrium, with AB horizontal, by two vertical ropes, one attached at A and the other attached at B, as shown in Figure 9.

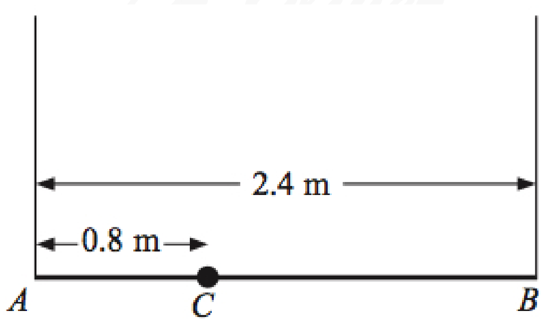


Figure 9: a plank AB has mass 12 kg and length 2.4 m

The plank is modelled as a uniform rod, the load as a particle and the ropes as light inextensible strings.

(a) Find the tension in the rope attached at B. (4)

Solution

Let R and S be the tensions at A and B respectively.
Moments about A:

$$0.8 \times 8g + 1.2 \times 12g = 2.4 \times S \Rightarrow \underline{\underline{S = \tfrac{26}{3}g}}.$$

The plank is now modelled as a non-uniform rod. With the new model, the tension in the rope attached at A is 10 N greater than the tension in the rope attached at B.

(b) Find the distance of the centre of mass of the plank from A. (6)

Solution

Let T and $(T-10)$ be the new tension at A and B respectively.
Then

$$T + (T-10) = 12g + 8g \Rightarrow T = 10g + 5.$$

Let x m be the distance from A to the centre of mass.
Moments about A:

$$\begin{aligned} & 0.8 \times 8g + x \times 12g = 2.4 \times (10g - 5) \\ \Rightarrow\quad & 12gx = 2.4(10g-5) - 6.4g \\ \Rightarrow\quad & x = 1\tfrac{268}{735} \\ \Rightarrow\quad & \underline{\underline{x = 1.4 \text{ m (2 sf)}}}. \end{aligned}$$

10. A bench consists of a plank which is resting in a horizontal position on two thin vertical legs. The plank is modelled as a uniform rod PS of length 2.4 m and mass 20 kg. The legs at Q and R are 0.4 m from each end of the plank, as shown in Figure 10.

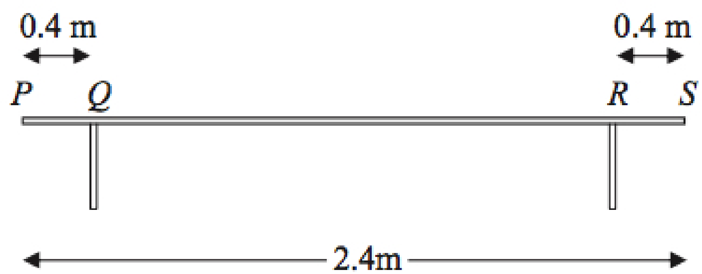


Figure 10: a plank is modelled as a uniform rod PS of length 2.4 m and mass 20 kg

Two pupils, Arthur and Beatrice, sit on the plank. Arthur has mass 60 kg and sits at the middle of the plank and Beatrice has mass 40 kg and sits at the end P. The plank remains horizontal and in equilibrium. By modelling the pupils as particles, find

(a) the magnitude of the normal reaction between the plank and the leg at Q and the magnitude of the normal reaction between the plank and the leg at R. (7)

Solution

Let R and S be the tensions at Q and R respectively. Then

$$R + S = 40g + 60g + 20g = 120g.$$

The distance between Arthur and one of the tables legs is

$$1.2 - 0.4 = 0.8 \text{ m}$$

and the distance between the two table legs is

$$2.4 - 2 \times 0.4 = 1.6 \text{ m}.$$

Moments about Q:

$$\begin{aligned} & 0.4 \times 40g + 1.6 \times S = 0.8 \times 80g \\ \Rightarrow \quad & 1.6S = 48g \\ \Rightarrow \quad & \underline{\underline{S = 30g \text{ N}}} \\ \Rightarrow \quad & \underline{\underline{R = 90g \text{ N}}}. \end{aligned}$$

Beatrice stays sitting at P but Arthur now moves and sits on the plank at the point X. Given that the plank remains horizontal and in equilibrium, and that the magnitude of the normal reaction between the plank and the leg at Q is now twice the magnitude of the normal reaction between the plank and the leg at R,

(b) find the distance QX. (6)

Solution

Let $2T$ and T be the normal reactions at Q and R respectively. Then

$$2T + T = 120g \Rightarrow T = 40g.$$

Moments about Q:

$$\begin{aligned} & 0.4 \times 40g + 1.6 \times 40g = QX \times 60g + 0.8 \times 20g \\ \Rightarrow\ & 60gQX = 16g + 64g - 16g \\ \Rightarrow\ & 60gQX = 64g \\ \Rightarrow\ & QX = 1\tfrac{1}{15} \\ \Rightarrow\ & \underline{\underline{QX = 1.1 \text{ m (2 sf)}}}. \end{aligned}$$

11. A beam AB is supported by two vertical ropes, which are attached to the beam at points P and Q, where $AP = 0.3$ m and $BQ = 0.3$ m. The beam is modelled as a uniform rod, of length 2 m and mass 20 kg. The ropes are modelled as light inextensible strings. A gymnast of mass 50 kg hangs on the beam between P and Q. The gymnast is modelled as a particle attached to the beam at the point X, where $PX = x$ m, $0 < x < 1.4$, as shown in Figure 11.

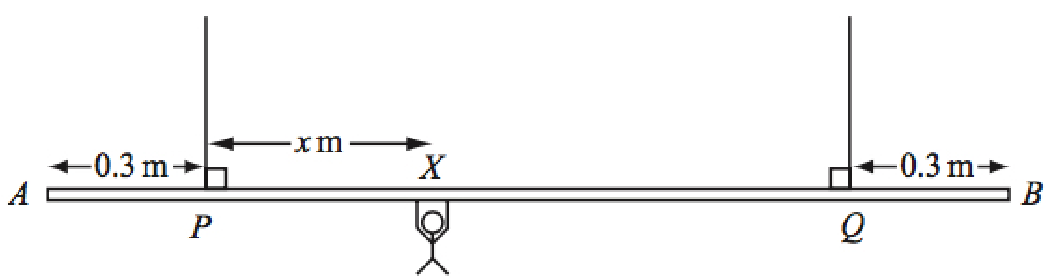


Figure 11: a beam AB is supported by two vertical ropes

The beam rests in equilibrium in a horizontal position.

(a) Show that the tension in the rope attached to the beam at P is $(588 - 350x)$ N. (3)

Solution

Let R and S be the tensions at P and Q respectively. Then

$$R + S = 20g + 50g = 70g.$$

The centre of mass is 1 m and the distance between the centre of mass and the rope is

$$1 - 0.3 = 0.7 \text{ m}.$$

Moments about Q:

$$\begin{aligned} & 1.4 \times R = (1.4 - x) \times 50g + 0.7 \times 20g \\ \Rightarrow\ & 1.4R = 84g - 50gx \\ \Rightarrow\ & R = 60g - \tfrac{250}{7}gx \\ \Rightarrow\ & \underline{\underline{R = (588 - 350x) \text{ N}}}. \end{aligned}$$

(b) Find, in terms of x, the tension in the rope attached to the beam at Q. (3)

Solution

$$S = 686 - (588 - 350x) = \underline{\underline{(98 + 350x) \text{ N}}}.$$

(c) Hence find, justifying your answer carefully, the range of values of the tension which could occur in each rope. (3)

Solution

Since $0 < x < 1.4$,

$$\underline{\underline{98 < R, S < 588 \text{ where } R + S = 686}}.$$

Given that the tension in the rope attached at Q is three times the tension in the rope attached at P,

(d) find the value of x. (3)

Solution

The tension at Q is

$$\tfrac{3}{4} \times 70g = 514.5 \text{ N}$$

and

$$\begin{aligned} 514.5 = 98 + 350x &\Rightarrow 350x = 416.5 \\ &\Rightarrow \underline{\underline{x = 1.19 \text{ m}}}. \end{aligned}$$

12. A pole AB has length 3 m and weight W newtons. The pole is held in a horizontal position in equilibrium by two vertical ropes attached to the pole at the points A and C where $AC = 1.8$ m, as shown in Figure 12.

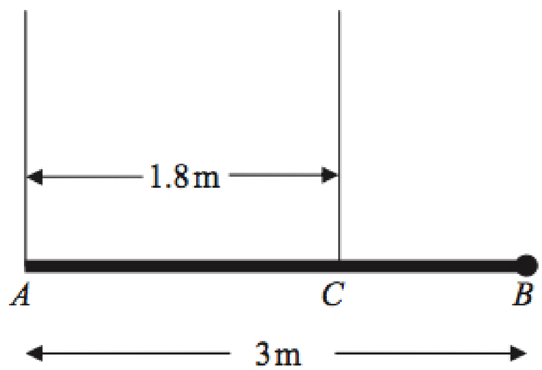


Figure 12: a pole AB has length 3 m and weight W newtons

A load of weight 20 N is attached to the rod at B. The pole is modelled as a uniform rod, the ropes as light inextensible strings and the load as a particle.

(a) Show that the tension in the rope attached to the pole at C is (4)

$$(\tfrac{5}{6}W + \tfrac{100}{3}) \text{ N}.$$

Solution

Let R and S be the tensions at A and C respectively.
Moments about A:

$$1.8 \times S = 1.5 \times W + 3 \times 20 \Rightarrow S = \underline{\underline{(\tfrac{5}{6}W + \tfrac{100}{3}) \text{ N}}}.$$

(b) Find, in terms of W, the tension in the rope attached to the pole at A. (3)

Solution

$$\begin{aligned} R + S = W + 20 &\Rightarrow R = W + 20 - (\tfrac{5}{6}W + \tfrac{100}{3}) \\ &\Rightarrow R = \underline{\underline{(\tfrac{1}{6}W - \tfrac{40}{3}) \text{ N}}}. \end{aligned}$$

Given that the tension in the rope attached to the pole at C is eight times the tension in the rope attached to the pole at A,

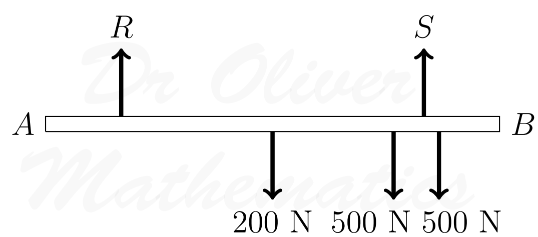
(c) find the value of W. (3)

Solution

$$\begin{aligned} & 8(\tfrac{1}{6}W - \tfrac{40}{3}) = (\tfrac{5}{6}W + \tfrac{100}{3}) \\ \Rightarrow\ & \tfrac{4}{3}W - \tfrac{320}{3} = \tfrac{5}{6}W + \tfrac{100}{3} \\ \Rightarrow\ & \tfrac{1}{2}W = 140 \\ \Rightarrow\ & \underline{\underline{W = 280 \text{ N}}}. \end{aligned}$$

13. A beam AB has length 6 m and weight 200 N. The beam rests in a horizontal position on two supports at the points C and D, where $AC = 1$ m and $DB = 1$ m . Two children, Sophie and Tom, each of weight 500 N, stand on the beam with Sophie standing twice as far from the end B as Tom. The beam remains horizontal and in equilibrium and the magnitude of the reaction at D is three times the magnitude of the reaction at C. By modelling the beam as a uniform rod and the two children as particles, find how far Tom is standing from the end B. (7)

Solution



Let x be how far Tom is standing from the end B. Then

$$R + S = 1200$$

and

$$3R = S$$

which gives

$$R = 300 \text{ and } S = 900.$$

Now, moments about B:

$$\begin{aligned} & 5 \times 300 + 1 \times 900 = x \times 500 + 2x \times 500 + 3 \times 200 \\ \Rightarrow \quad & 2400 = 1500x + 600 \\ \Rightarrow \quad & 1500x = 1800 \\ \Rightarrow \quad & \underline{\underline{x = 1.2 \text{ m}}}. \end{aligned}$$

14. A uniform beam AB has mass 20 kg and length 6 m. The beam rests in equilibrium in a horizontal position on two smooth supports. One support is at C, where $AC = 1$ m, and the other is at the end B, as shown in Figure 13.

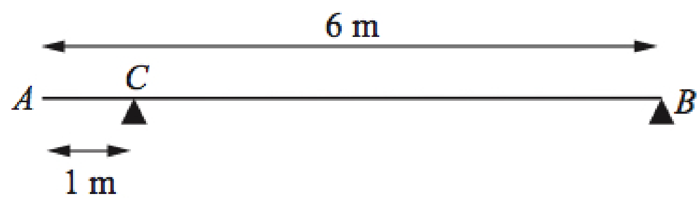


Figure 13: a uniform beam AB has mass 20 kg and length 6 m

The beam is modelled as a rod.

(a) Find the magnitudes of the reactions on the beam at B and at C. (5)

Solution

Let R and S be the tensions at B and C respectively. Then

$$R + S = 20g \Rightarrow R = 20g - S.$$

Moments about A:

$$\begin{aligned} (1 \times R) + (6 \times S) = 3 \times 20g & \Rightarrow (20g - S) + 6S = 60g \\ & \Rightarrow 5S = 40g \\ & \Rightarrow \underline{\underline{S = 8g \text{ N}}} \\ & \Rightarrow \underline{\underline{R = 12g \text{ N}}}. \end{aligned}$$

A boy of mass 30 kg stands on the beam at the point D. The beam remains in equilibrium. The magnitudes of the reactions on the beam at B and at C are now equal. The boy is modelled as a particle.

(b) Find the distance AD. (5)

Solution

Let T be the tensions at B and C respectively. Then

$$2T = 20g + 30g \Rightarrow T = 25g.$$

Moments about A:

$$\begin{aligned}(1 \times 25g) + (6 \times 25g) = (3 \times 20g) + (AD \times 30g) &\Rightarrow 175g = 60g + 30gAD \\ &\Rightarrow 30gAD = 115g \\ &\Rightarrow AD = \tfrac{23}{6} \\ &\Rightarrow \underline{\underline{AD = 3.8 \text{ m (2 sf)}}}.\end{aligned}$$

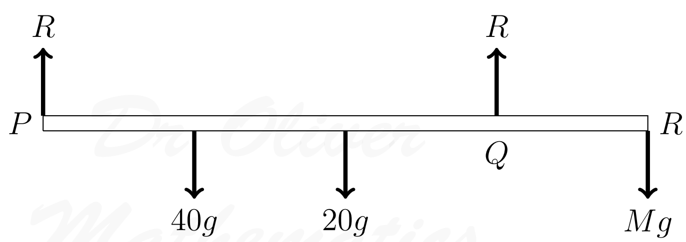
15. A plank PQR, of length 8 m and mass 20 kg, is in equilibrium in a horizontal position on two supports at P and Q, where $PQ = 6$ m.

A child of mass 40 kg stands on the plank at a distance of 2 m from P and a block of mass M kg is placed on the plank at the end R. The plank remains horizontal and in equilibrium. The force exerted on the plank by the support at P is equal to the force exerted on the plank by the support at Q.

By modelling the plank as a uniform rod, and the child and the block as particles,

(a) (i) find the magnitude of the force exerted on the plank by the support at P, (10)

Solution



Let R be the tensions at P and Q respectively.

Moments about R:

$$\begin{aligned} & 8 \times R + 2 \times R = 6 \times 40g + 4 \times 20g \\ \Rightarrow \quad & 10R = 320g \\ \Rightarrow \quad & \underline{\underline{R = 32g \text{ N}}}. \end{aligned}$$

(ii) find the value of M.

Solution

$$\begin{aligned} & 6 \times 32g = 2 \times 40g + 4 \times 20g + 8 \times Mg \\ \Rightarrow \quad & 192g = 160g + 8Mg \\ \Rightarrow \quad & 8Mg = 32g \\ \Rightarrow \quad & \underline{\underline{M = 4 \text{ kg}}} \end{aligned}$$

(b) State how, in your calculations, you have used the fact that the child and the block can be modelled as particles. (1)

Solution

The mass of the child and the block are concentrated at a point.

16. A non-uniform rod AB, of mass m and length $5d$, rests horizontally in equilibrium on two supports at C and D, where $AC = DB = d$, as shown in Figure 14.

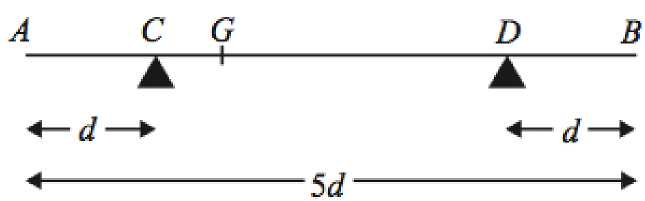


Figure 14: a non-uniform rod AB of mass m and length $5d$

The centre of mass of the rod is at the point G. A particle of mass $\frac{5}{2}m$ is placed on the rod at B and the rod is on the point of tipping about D.

(a) Show that $GD = \frac{5}{2}d$. (4)

Solution

Moments about D:

$$GD \times mg = d \times \tfrac{5}{2}mg \Rightarrow \underline{\underline{GD = \tfrac{5}{2}d}}.$$

The particle is moved from B to the mid-point of the rod and the rod remains in equilibrium.

(b) Find the magnitude of the normal reaction between the support at D and the rod. (5)

Solution

Let S be the tension at D. The distance from the centre of mass to C is

$$3d - \tfrac{5}{2}d = \tfrac{1}{2}d$$

and the distance from the mid-point to C is

$$\tfrac{5}{2}d - d = \tfrac{3}{2}d.$$

Moments about C:

$$\begin{aligned} & (\tfrac{1}{2}d \times mg) + (\tfrac{3}{2}d \times \tfrac{5}{2}mg) = 3d \times S \\ \Rightarrow \quad & 3dS = \tfrac{17}{4}dmg \\ \Rightarrow \quad & \underline{\underline{S = \tfrac{17}{12}mg}}. \end{aligned}$$

17. A non-uniform rod AB has length 3 m and mass 4.5 kg. The rod rests in equilibrium, in a horizontal position, on two smooth supports at P and at Q, where $AP = 0.8$ m and $QB = 0.6$ m, as shown in Figure 15.

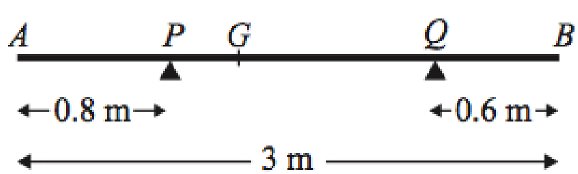


Figure 15: a non-uniform rod AB has length 3 m and mass 4.5 kg

The centre of mass of the rod is at G. Given that the magnitude of the reaction of the support at P on the rod is twice the magnitude of the reaction of the support at Q on the rod, find

(a) the magnitude of the reaction of the support at Q on the rod, (3)

Solution

Let $2R$ and R be the tensions at P and Q respectively.
Then
$$2R + R = 4.5g \Rightarrow \underline{\underline{R = 1.5g \text{ N}}}.$$

(b) the distance AG. (4)

Solution

Moments about G:
$$\begin{aligned} PG \times 3g = (1.6 - PG) \times 1.5g &\Rightarrow 3PG = 2.4 - 1.5PG \\ &\Rightarrow 4.5PG = 2.4 \\ &\Rightarrow PG = \tfrac{8}{15} \\ &\Rightarrow AG = 1\tfrac{1}{3} \\ &\Rightarrow \underline{\underline{AG = 1.3 \text{ m (2 sf)}}}. \end{aligned}$$

18. A steel girder AB, of mass 200 kg and length 12 m, rests horizontally in equilibrium on two smooth supports at C and at D, where $AC = 2$ m and $DB = 2$ m. A man of mass 80 kg stands on the girder at the point P, where $AP = 4$ m, as shown in Figure 16.

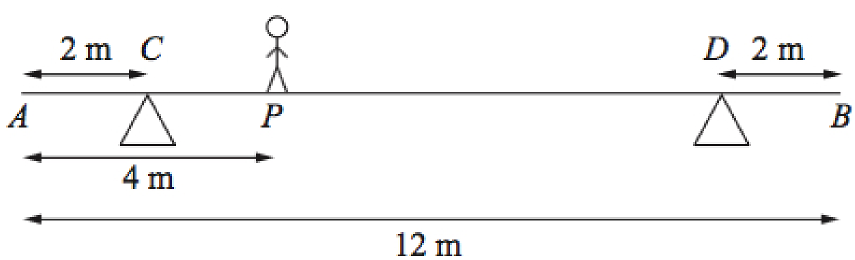


Figure 16: a steel girder AB, of mass 200 kg and length 12 m

The man is modelled as a particle and the girder is modelled as a uniform rod.

(a) Find the magnitude of the reaction on the girder at the support at C. (3)

Solution

Let R be the tension at C. Then the separation between the supports in

$$12 - 2 - 2 = 8 \text{ m},$$

the distance between the man and D is

$$12 - 2 - 4 = 6 \text{ m},$$

and the distance between the centre of mass and D is

$$6 - 2 = 4 \text{ m}.$$

Moments about D:

$$8 \times R = (4 \times 200g) + (6 \times 80g) \Rightarrow \underline{\underline{R = 160g}}.$$

The support at D is now moved to the point X on the girder, where $XB = x$ metres. The man remains on the girder at P, as shown in Figure 17.

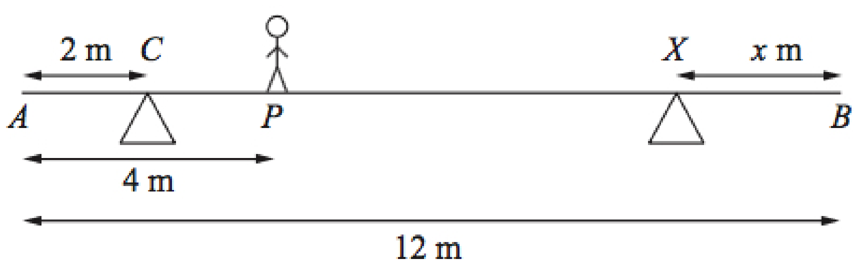


Figure 17: the support at D is now moved to the point X

Given that the magnitudes of the reactions at the two supports are now equal and that the girder again rests horizontally in equilibrium, find

(b) the magnitude of the reaction at the support at X, (2)

Solution

Let S be the tension at C and X. Then

$$S + S = 80g + 200g \Rightarrow \underline{\underline{S = 140g}}.$$

(c) the value of x. (4)

Solution

The separation between C and B is

$$12 - 2 = 10 \text{ m},$$

the distance between the man and B is

$$12 - 4 = 8 \text{ m}$$

and the distance between the centre of mass and B is 6 m.
Moments about B:

$$\begin{aligned} & (6 \times 200g) + (8 \times 80g) = (10 \times 140g) + (x \times 140g) \\ \Rightarrow\quad & 1840 = 1400 + 140x \\ \Rightarrow\quad & 140x = 440 \\ \Rightarrow\quad & x = 3\tfrac{1}{7} \\ \Rightarrow\quad & \underline{\underline{x = 3.1 \text{ m}}}. \end{aligned}$$

19. A beam AB has length 15 m. The beam rests horizontally in equilibrium on two smooth supports at the points P and Q, where $AP = 2$ m and $QB = 3$ m. When a child of mass 50 kg stands on the beam at A, the beam remains in equilibrium and is on the point of tilting about P. When the same child of mass 50 kg stands on the beam at B, the beam remains in equilibrium and is on the point of tilting about Q. The child is modelled as a particle and the beam is modelled as a non-uniform rod.

(a) (i) Find the mass of the beam. (8)

Solution

Let m kg be the mass of the beam and let x m be the distance between A and the centre of mass.
Moments about P:

$$2 \times 50g = (y - 2) \times mg.$$

Moments about Q:

$$3 \times 50g = (12 - y) \times mg.$$

Add:
$$250g = 10mg \Rightarrow \underline{\underline{m = 25}}.$$

(ii) Find the distance of the centre of mass of the beam from A.

Solution
$$100g = 25(y-2)g \Rightarrow y - 2 = 4 \Rightarrow \underline{\underline{y = 6}}.$$

When the child stands at the point X on the beam, it remains horizontal and in equilibrium. Given that the reactions at the two supports are equal in magnitude,

(b) find AX. (6)

Solution
Let R be the tension at A and B. Then
$$R + R = 25g + 50g \Rightarrow R = 37.5g.$$
Moments about A:
$$\begin{aligned} & (2 \times 37.5g) + (12 \times 37.5g) = (AX \times 50g) + (6 \times 25g) \\ \Rightarrow\ & 525 = 50AX + 150 \\ \Rightarrow\ & 50AX = 375 \\ \Rightarrow\ & \underline{\underline{AX = 7.5 \text{ m}}}. \end{aligned}$$

20. A uniform rod AB has length 2 m and mass 50 kg. The rod is in equilibrium in a horizontal position, resting on two smooth supports at C and D, where $AC = 0.2$ metres and $DB = x$ metres, as shown in Figure 18.

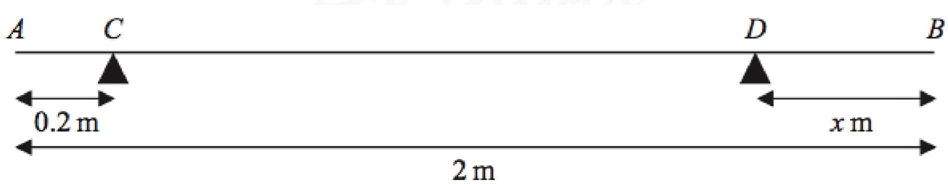


Figure 18: a uniform rod AB has length 2 m and mass 50 kg

Given that the magnitude of the reaction on the rod at D is twice the magnitude of the reaction on the rod at C,

(a) find the value of x. (6)

Solution
Let R and $2R$ be the tensions at C and D. Then

$$R + 2R = 50g \Rightarrow R = \tfrac{50g}{3}.$$

The distance between the centre of mass of C is

$$1 - 0.2 = 0.8 \text{ m}$$

and the distance between C and D is $(1.8 - x)$ m.
Moments about C:

$$\begin{aligned} 0.8 \times 50g = (1.8 - x) \times \tfrac{100g}{3} &\Rightarrow 2.4 = 2(1.8 - x) \\ &\Rightarrow 2.4 = 3.6 - 2x \\ &\Rightarrow 2x = 1.2 \\ &\Rightarrow \underline{\underline{x = 0.6 \text{ m}}}. \end{aligned}$$

The support at D is now moved to the point E on the rod, where $EB = 0.4$ metres. A particle of mass m kg is placed on the rod at B, and the rod remains in equilibrium in a horizontal position. Given that the magnitude of the reaction on the rod at E is four times the magnitude of the reaction on the rod at C,

(b) find the value of m. (7)

Solution
Let S and $4S$ be the tensions at C and E. Then

$$S + 4S = 50g + mg \Rightarrow S = \tfrac{(50+m)g}{5}.$$

The distance between C and E is

$$2 - 0.2 - 0.4 = 1.4 \text{ m}.$$

Moments about C:

$$\begin{aligned} & (0.8 \times 50g) + (1.8 \times mg) = 1.4 \times \tfrac{4(50+m)g}{5} \\ \Rightarrow \quad & 200 + 9m = 5.6(50 + m) \\ \Rightarrow \quad & 200 + 9m = 280 + 5.6m \\ \Rightarrow \quad & 3.4m = 80 \\ \Rightarrow \quad & m = 23\tfrac{9}{17} \\ \Rightarrow \quad & \underline{\underline{m = 24 \text{ kg (2 sf)}}}. \end{aligned}$$

21. A beam AB has weight W newtons and length 4 m. The beam is held in equilibrium in a horizontal position by two vertical ropes attached to the beam. One rope is attached to A and the other rope is attached to the point C on the beam, where $AC = d$ metres, as shown in Figure 19.

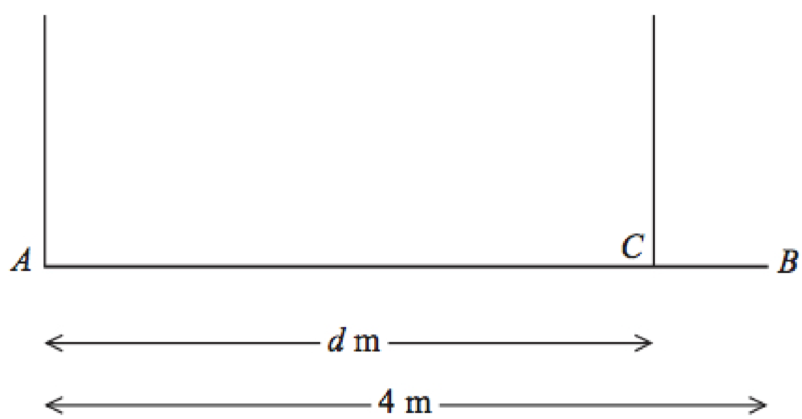


Figure 19: a beam AB has weight W newtons and length 4 m

The beam is modelled as a uniform rod and the ropes as light inextensible strings. The tension in the rope attached at C is double the tension in the rope attached at A.

(a) Find the value of d. (6)

Solution

Let R and $2R$ be the tensions at A and C. Then

$$R + 2R = W \Rightarrow R = \tfrac{W}{3}.$$

Moments about A:

$$2 \times W = d \times \tfrac{2W}{3} \Rightarrow 6 = 2d$$
$$\Rightarrow \underline{\underline{d = 3 \text{ m}}}.$$

A small load of weight kW newtons is attached to the beam at B. The beam remains in equilibrium in a horizontal position. The load is modelled as a particle. The tension in the rope attached at C is now four times the tension in the rope attached at A.

(b) Find the value of k. (6)

Solution

Let S and $4S$ be the tensions at A and C. Then

$$S + 4S = W + kW \Rightarrow S = \tfrac{W(k+1)}{5}.$$

Moments about A:

$$\begin{aligned}
& (2 \times W) + (4 \times kW) = 3 \times \tfrac{4W(k+1)}{5} \\
\Rightarrow\quad & 10 + 20k = 12(k+1) \\
\Rightarrow\quad & 10 + 20k = 12k + 12 \\
\Rightarrow\quad & 8k = 2 \\
\Rightarrow\quad & \underline{\underline{k = \tfrac{1}{4}}}.
\end{aligned}$$

22. A non-uniform beam AD has weight W newtons and length 4 m. It is held in equilibrium in a horizontal position by two vertical ropes attached to the beam. The ropes are attached to two points B and C on the beam, where $AB = 1$ m and $CD = 1$ m, as shown in Figure 20.

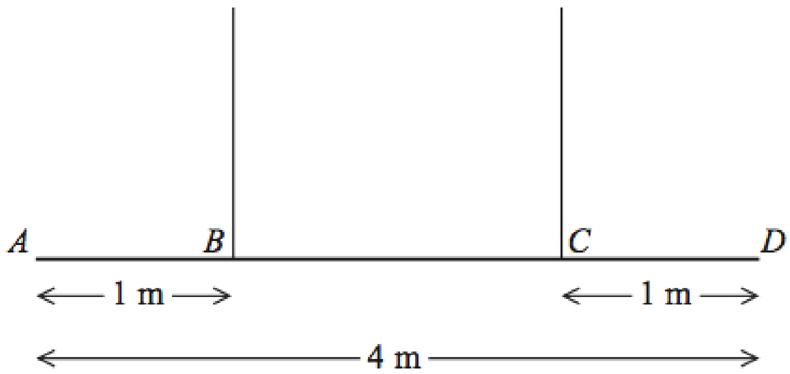


Figure 20: a non-uniform beam AD has weight W newtons and length 4 m

The tension in the rope attached to C is double the tension in the rope attached to B. The beam is modelled as a rod and the ropes are modelled as light inextensible strings.

(a) Find the distance of the centre of mass of the beam from A. (6)

Solution

Let R and $2R$ be the tensions at B and C. Then

$$R + 2R = W \Rightarrow R = \tfrac{W}{3}.$$

Let the distance between A and the centre of mass be d m.
Moments about B:

$$\begin{aligned} & (d-1) \times W = 2 \times \tfrac{2W}{3} \\ \Rightarrow\ & 3(d-1) = 4 \\ \Rightarrow\ & 3d - 3 = 4 \\ \Rightarrow\ & 3d = 7 \\ \Rightarrow\ & \underline{\underline{d = \tfrac{7}{3}\text{ m}}}. \end{aligned}$$

A small load of weight kW newtons is attached to the beam at D. The beam remains in equilibrium in a horizontal position. The load is modelled as a particle.

Find

(b) an expression for the tension in the rope attached to B, giving your answer in terms of k and W, (3)

Solution

Let S be the tension at B.

Moments about C:

$$\left(\tfrac{2}{3} \times W\right) = (2 \times S) + (1 \times kW)$$
$$\Rightarrow \quad 2S = \tfrac{2}{3}W - kW$$
$$\Rightarrow \quad \underline{\underline{S = \tfrac{1}{6}W(2 - 3k) \text{ N}}}.$$

(c) the set of possible values of k for which both ropes remain taut. (2)

Solution

$$S > 0 \Rightarrow 2 - 3k > 0 \Rightarrow \underline{\underline{0 < k < \tfrac{2}{3}}}.$$

23. A beam AB has length 5 m and mass 25 kg. The beam is suspended in equilibrium in a horizontal position by two vertical ropes. One rope is attached to the beam at A and the other rope is attached to the point C on the beam where $CB = 0.5$ m, as shown in Figure 21.

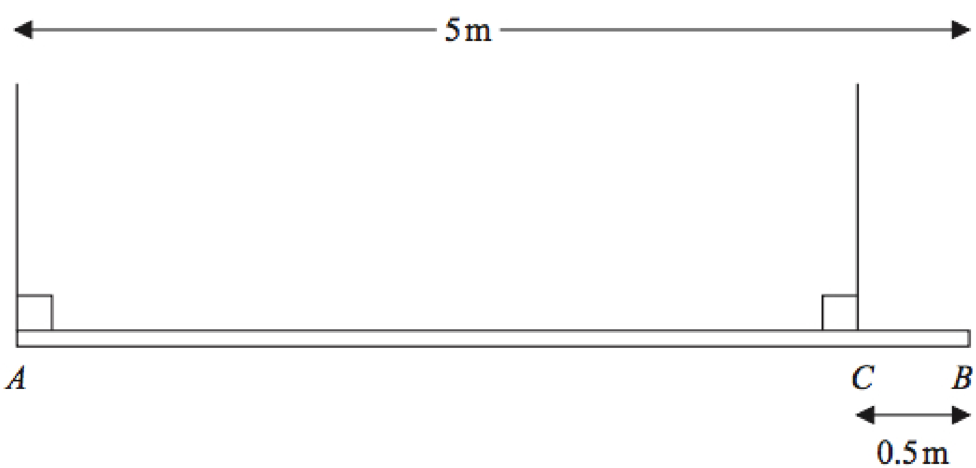


Figure 21: a beam AB has length 5 m and mass 25 kg

A particle P of mass 60 kg is attached to the beam at B and the beam remains in equilibrium in a horizontal position. The beam is modelled as a uniform rod and the ropes are modelled as light strings.

(a) Find (6)

(i) the tension in the rope attached to the beam at A,

Solution
Let R and S be the tensions at A and C. Then

$$R + S = 25g + 60g = 85g.$$

Moments about B:

$$\begin{aligned} 2.5 \times 25g = 5 \times R + 0.5 \times S &\Rightarrow 62.5g = 5R + 0.5(85g - R) \\ &\Rightarrow 62.5g = 5R + (42.5g - 0.5R) \\ &\Rightarrow 4.5R = 20g \\ &\Rightarrow R = \tfrac{40}{9}g \\ &\Rightarrow \underline{\underline{R = 44 \text{ N (2 sf)}}}. \end{aligned}$$

(ii) the tension in the rope attached to the beam at C.

Solution

$$\begin{aligned} S &= 85g - \tfrac{40}{9}g \\ &= \tfrac{725}{9}g \\ &= \underline{\underline{790 \text{ N (2 sf)}}}. \end{aligned}$$

Particle P is removed and replaced by a particle Q of mass M kg at B. Given that the beam remains in equilibrium in a horizontal position,

(b) find (6)

(i) the greatest possible value of M,

Solution
Moments about C:

$$2 \times 25g = 0.5 \times Mg \Rightarrow \underline{\underline{M = 100}}.$$

(ii) the greatest possible tension in the rope attached to the beam at C.

Solution
Let T be the tension at C.

Moments about A:

$$(2.5 \times 25g) + (5 \times 100g) = 4.5 \times T \Rightarrow 4.5T = 562.5$$
$$\Rightarrow \underline{\underline{T = 125g \text{ N}}}.$$

24. A non-uniform plank AB has length 6 m and mass 30 kg. The plank rests in equilibrium in a horizontal position on supports at the points S and T of the plank where $AS = 0.5$ m and $TB = 2$ m. (7)

When a block of mass M kg is placed on the plank at A, the plank remains horizontal and in equilibrium and the plank is on the point of tilting about S.

When the block is moved to B, the plank remains horizontal and in equilibrium and the plank is on the point of tilting about T.

The distance of the centre of mass of the plank from A is d metres. The block is modelled as a particle and the plank is modelled as a non-uniform rod. Find

(a) the value of d,

Solution

First, there is 0 N tension about T and so the moments about S:

$$0.5 \times Mg = (d - 0.5) \times 30g. \quad (1)$$

Second, there is 0 N tension about S and so the moments about T:

$$2 \times Mg = (4 - d) \times 30g. \quad (2)$$

Divide (2) by (1):

$$\frac{2Mg}{0.5Mg} = \frac{30g(4-d)}{30g(d-0.5)} \Rightarrow 4 = \frac{4-d}{d-0.5}$$
$$\Rightarrow 4(d - 0.5) = 4 - d$$
$$\Rightarrow 4d - 2 = 4 - d$$
$$\Rightarrow 5d = 6$$
$$\Rightarrow \underline{\underline{d = 1.2 \text{ m}}}.$$

(b) the value of M.

Solution

$$0.5 \times Mg = 0.7 \times 30g \Rightarrow \underline{\underline{M = 42 \text{ kg}}}.$$

25. A plank AB has length 6 m and mass 30 kg. The point C is on the plank with $CB = 2$ m. The plank rests in equilibrium in a horizontal position on supports at A and C. Two people, each of mass 75 kg, stand on the plank. One person stands at the point P of the plank, where $AP = x$ metres, and the other person stands at the point Q of the plank, where $AQ = 2x$ metres. The plank remains horizontal and in equilibrium with the magnitude of the reaction at C five times the magnitude of the reaction at A. The plank is modelled as a uniform rod and each person is modelled as a particle.

(a) Find the value of x. (7)

Solution

Let R and $5R$ be the tensions at P and Q. Then

$$R + 5R = 75g + 75g + 30g \Rightarrow R = 30g.$$

Moments about A:

$$\begin{aligned} & (x \times 75g) + (2x \times 75g) + (3 \times 30g) = 4 \times 150g \\ \Rightarrow \quad & 225gx + 90g = 600g \\ \Rightarrow \quad & 225x = 510 \\ \Rightarrow \quad & x = 2\tfrac{4}{15} \\ \Rightarrow \quad & \underline{\underline{x = 2.3 \text{ m (2 sf)}}}. \end{aligned}$$

(b) State two ways in which you have used the assumptions made in modelling the plank as a uniform rod. (2)

Solution

E.g., the <u>rod is rigid</u> and <u>centre of mass is at the middle of the plank</u>.