Dr Oliver Mathematics
Further Mathematics
Complex Numbers: Loci
Past Examination Questions

This booklet consists of 17 questions across a variety of examination topics.
The total number of marks available is 188.

1. A transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}}, z \neq -\mathrm{i}.$$

(a) Show that T maps points on the half-line $\arg(z) = \frac{\pi}{4}$ in the z-plane into points on the circle $|w| = 1$ in the w-plane. (4)

Solution

$z = x + \mathrm{i}x$, $x > 0$. Now,

$$\begin{aligned} |w|^2 &= \left|\frac{(x+1)+x\mathrm{i}}{x+(x+1)\mathrm{i}}\right|^2 \\ &= \frac{(x+1)^2+x^2}{x^2+(x+1)^2} \\ &= 1, \end{aligned}$$

and so $\underline{\underline{|w| = 1}}$.

(b) Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane. (6)

Solution

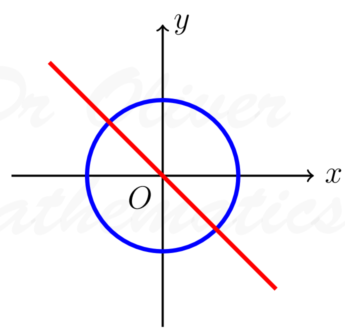
$$\begin{aligned} w = \frac{z+1}{z+\mathrm{i}} &\Rightarrow w(z+\mathrm{i}) = z+1 \\ &\Rightarrow wz + \mathrm{i}w = z+1 \\ &\Rightarrow wz - z = 1 - \mathrm{i}w \\ &\Rightarrow z(w-1) = 1 - \mathrm{i}w \\ &\Rightarrow z = \frac{1-\mathrm{i}w}{w-1}. \end{aligned}$$

Now,

$$\begin{aligned}|z| = 1 &\Rightarrow \left|\frac{1 - \mathrm{i}w}{w - 1}\right| = 1\\ &\Rightarrow |1 - \mathrm{i}w| = |w - 1|\\ &\Rightarrow |-\mathrm{i}(w + \mathrm{i})| = |w - 1|\\ &\Rightarrow |w + \mathrm{i}| = |w - 1|\\ &\Rightarrow |w + \mathrm{i}|^2 = |w - 1|^2\\ &\Rightarrow u^2 + (v + 1)^2 = (u - 1)^2 + v^2\\ &\Rightarrow u^2 + v^2 + 2v + 1 = u^2 - 2u + 1 + v^2\\ &\Rightarrow 2v = -2u\\ &\Rightarrow \underline{\underline{v = -u}}.\end{aligned}$$

(c) Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane. (2)

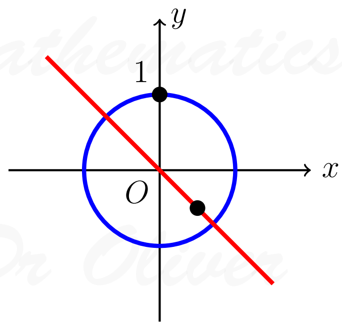
Solution



(d) Mark on your sketches the point P, where $z = \mathrm{i}$, and its image Q under T in the w-plane. (2)

Solution

$$w = \frac{\mathrm{i} + 1}{\mathrm{i} + \mathrm{i}} = \frac{\mathrm{i} + 1}{2\mathrm{i}} = -\frac{\mathrm{i}(\mathrm{i} + 1)}{2} = -\frac{-1 + \mathrm{i}}{2} = \frac{1 - \mathrm{i}}{2}.$$

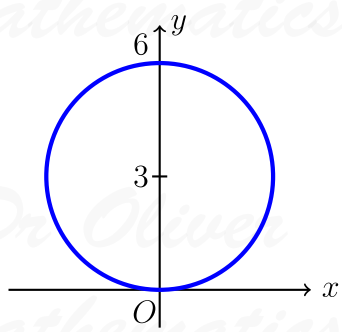


2. A complex number z is represented by a point P in the Argand diagram. Given that

$$|z - 3\mathrm{i}| = 3,$$

(a) sketch the locus of P. (2)

Solution



(b) Find the complex number z which satisfies both $|z - 3\mathrm{i}| = 3$ and $\arg(z - 3\mathrm{i}) = \frac{3}{4}\pi$. (4)

Solution

We have $x^2 + (y-3)^2 = 9$ and $y = 3 - x$:

$$x^2 + (-x)^2 = 9 \Rightarrow 2x^2 = 9 \Rightarrow x = \pm\tfrac{3\sqrt{2}}{2};$$

and, $x < 0$, hence the solution is

$$\underline{\underline{Z = -\tfrac{3\sqrt{2}}{2} + \mathrm{i}(3 + \tfrac{3\sqrt{2}}{2})}}.$$

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{2\mathrm{i}}{z}.$$

(c) Show that T maps $|z - 3\mathrm{i}| = 3$ to a line in the w-plane, and give the cartesian equation of this line. (5)

Solution

$$w = \frac{2\mathrm{i}}{z} \Rightarrow z = \frac{2\mathrm{i}}{w}$$

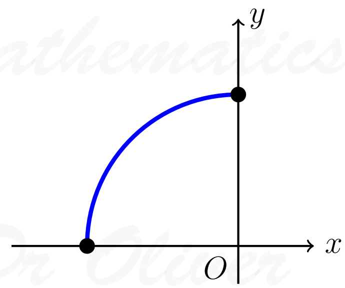
and

$$\begin{aligned}
|z - 3\mathrm{i}| = 3 &\Rightarrow \left|\frac{2\mathrm{i}}{w} - 3\mathrm{i}\right| = 3 \\
&\Rightarrow \left|\frac{2\mathrm{i} - 3\mathrm{i}w}{w}\right| = 3 \\
&\Rightarrow |\mathrm{i}(2 - 3w)| = 3|w| \\
&\Rightarrow |3w - 2| = 3|w| \\
&\Rightarrow |3w - 2|^2 = 9|w|^2 \\
&\Rightarrow (3u - 2)^2 + (3v)^2 = 9(u^2 + v^2) \\
&\Rightarrow 9u^2 - 12u + 4 + 9v^2 = 9u^2 + 9v^2 \\
&\Rightarrow -12u + 4 = 0 \\
&\Rightarrow \underline{\underline{u = \tfrac{1}{3}}}.
\end{aligned}$$

3. In the Argand diagram the point P represents the complex number z. Given that $\arg\left(\dfrac{z - 2\mathrm{i}}{z + 2}\right) = \dfrac{\pi}{2}$,

(a) sketch the locus of P, (2)

Solution



(b) deduce the value of $|z+1-\mathrm{i}|$. (2)

Solution
$|z+1-\mathrm{i}| = |z-(-1+\mathrm{i})| = \underline{\underline{\sqrt{2}}}.$

A transformation T from the z-plane to the w-plane is given by

$$w = \frac{2(1+\mathrm{i})}{z+2}, z \neq -2.$$

(c) Show that the locus of P in the z-plane is mapped to a straight line in the w-plane, and show this in the Argand diagram. (6)

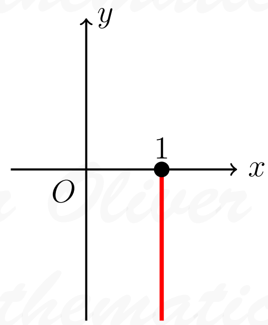
Solution

$$\begin{aligned} w = \frac{2(1+\mathrm{i})}{z+2} &\Rightarrow z+2 = \frac{2(1+\mathrm{i})}{w} \\ &\Rightarrow z = \frac{2(1+\mathrm{i})}{w} - 2 \\ &\Rightarrow z = \frac{2(1+\mathrm{i})-2w}{w}. \end{aligned}$$

Now,

$$\begin{aligned} \arg\left(\frac{z-2\mathrm{i}}{z+2}\right) &= \arg\left(\frac{\frac{2(1+\mathrm{i})}{w} - 2 - 2\mathrm{i}}{\frac{2(1+\mathrm{i})}{w}}\right) \\ &= \arg\left(\frac{\frac{2(1+\mathrm{i})-(2+2\mathrm{i})w}{w}}{\frac{2(1+\mathrm{i})}{w}}\right) \\ &= \arg\left(\frac{2(1+\mathrm{i})-2(1+\mathrm{i})w}{2(1+\mathrm{i})}\right) \\ &= \arg(1-w), \end{aligned}$$

and this is



4. The point P represents the complex number z on the Argand diagram, where

$$|z - 6 + 3\mathrm{i}| = 3|z + 2 - \mathrm{i}|.$$

(a) Show that the locus of P is a circle, giving the coordinates of the centre of the circle and the radius of this circle. (7)

Solution

$$\begin{aligned}
& |z - 6 + 3\mathrm{i}| = 3|z + 2 - \mathrm{i}| \\
\Rightarrow\ & |z - 6 + 3\mathrm{i}|^2 = 9|z + 2 - \mathrm{i}|^2 \\
\Rightarrow\ & |(x-6) + (y+3)\mathrm{i}|^2 = 9|(x+2) - (y-1)\mathrm{i}|^2 \\
\Rightarrow\ & (x-6)^2 + (y+3)^2 = 9(x+2)^2 + 9(y-1)^2 \\
\Rightarrow\ & (x^2 - 12x + 36) + (y^2 + 6y + 9) = 9(x^2 + 4x + 4) + 9(y^2 - 2y + 1) \\
\Rightarrow\ & 8x^2 + 48x + 8y^2 - 24y = 0 \\
\Rightarrow\ & x^2 + 6x + y^2 - 3y = 0 \\
\Rightarrow\ & x^2 + 6x + 9 + y^2 - 3y + \tfrac{9}{4} = \tfrac{45}{4} \\
\Rightarrow\ & (x+3)^2 + (y - \tfrac{3}{2})^2 = \tfrac{45}{4};
\end{aligned}$$

it is a circle, centre $(-3, \frac{3}{2})$, and radius $\frac{3\sqrt{5}}{2}$.

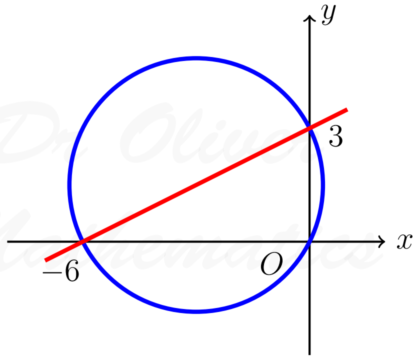
The point Q represents the complex number z on the Argand diagram, where

$$\tan|\arg(z + 6)| = \tfrac{1}{2}.$$

(b) On the Argand diagram, sketch the locus of P and the locus of Q. (5)

Solution

$\tan|\arg(z+6)| = \frac{1}{2}$ is a straight line through $(-6, 0)$ and $(0, 3)$ (why?).

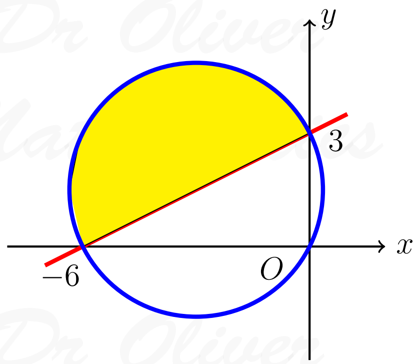


(c) On your diagram, shade the region which satisfies both (2)

$$|z-6+3\mathrm{i}| = 3|z+2-\mathrm{i}| \text{ and } \tan|\arg(z+6)| = \tfrac{1}{2}.$$

Solution

We want the 'top half' (why?).



5. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{z+\mathrm{i}}{z}, z \neq 0.$$

(a) The transformation T maps the points on the line with equation $y = x$ in the z-plane, other than $(0, 0)$, to points on the line l in the w-plane. Find a cartesian equation of l. (5)

Solution

We have $z = x + \mathrm{i}x$, $x \neq 0$, and

$$\begin{aligned} w &= \frac{x + (x+1)\mathrm{i}}{x + \mathrm{i}x} \\ &= \frac{x + (x+1)\mathrm{i}}{x + \mathrm{i}x} \times \frac{1 - \mathrm{i}}{1 - \mathrm{i}} \\ &= \frac{[x + (x+1)] + [(x+1) - x]\mathrm{i}}{x + x} \\ &= \frac{(2x+1) + \mathrm{i}}{2x} \\ &= 1 + \frac{1 + \mathrm{i}}{2x}, \end{aligned}$$

which means that

$$u = 1 + \frac{1}{2x}, v = \frac{1}{2x}$$

and

$$\underline{\underline{v = u - 1}}.$$

(b) Show that the image, under T, of the line with equation $x + y + 1 = 0$ in the z-plane is a circle C in the w-plane, where C has cartesian equation (7)

$$u^2 + v^2 - u + v = 0.$$

Solution

We have $z = x + \mathrm{i}(-x - 1)$, $x \neq 0$, and

$$\begin{aligned} w &= \frac{x + (-x)\mathrm{i}}{x + \mathrm{i}(-x-1)} \\ &= \frac{x - x\mathrm{i}}{x - \mathrm{i}(x+1)} \times \frac{x + \mathrm{i}(x+1)}{x + \mathrm{i}(x+1)} \\ &= \frac{[x^2 + x(x+1)] + [-x^2 + x(x+1)]\mathrm{i}}{x^2 + (x+1)^2} \\ &= \frac{(2x^2 + x) + x\mathrm{i}}{2x^2 + 2x + 1}, \end{aligned}$$

which means

$$u = \frac{2x^2 + x}{2x^2 + 2x + 1}, v = \frac{x}{2x^2 + 2x + 1}.$$

So

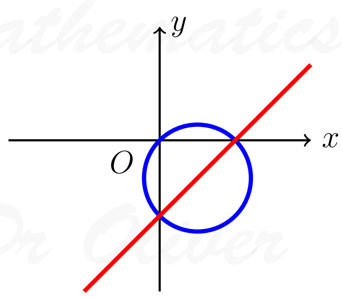
$$\begin{aligned} u^2 + v^2 - u + v &= \left(\frac{2x^2 + x}{2x^2 + 2x + 1}\right)^2 + \left(\frac{x}{2x^2 + 2x + 1}\right)^2 \\ &\qquad - \frac{2x^2 + x}{2x^2 + 2x + 1} + \frac{x}{2x^2 + 2x + 1} \\ &= \frac{(2x^2 + x)^2 + x^2 - (2x^2 + x)(2x^2 + 2x + 1) + x(2x^2 + 2x + 1)}{(2x^2 + 2x + 1)^2} \\ &= \frac{(4x^4 + 4x^3 + x^2) + x^2 - (4x^4 + 6x^3 + 4x^2 + x)}{(2x^2 + 2x + 1)^2} \\ &\qquad + \frac{(2x^3 + 2x^2 + x)}{(2x^2 + 2x + 1)^2} \\ &= \underline{\underline{0}}. \end{aligned}$$

(c) On the same Argand diagram, sketch l and C. (3)

Solution

$$\begin{aligned} u^2 + v^2 - u + v = 0 &\Rightarrow u^2 - u + \tfrac{1}{4} + v^2 + v + \tfrac{1}{4} = \tfrac{1}{2} \\ &\Rightarrow (u - \tfrac{1}{2})^2 + (v + \tfrac{1}{2})^2 = \tfrac{1}{2}; \end{aligned}$$

a circle, centre $(\frac{1}{2}, -\frac{1}{2})$, and radius $\frac{\sqrt{2}}{2}$.



6. The point P represents the complex number z on the Argand diagram such that

$$|z - 3| = 2|z|.$$

(a) Show that, as z varies, the locus of P is a circle, and give the coordinates of the centre and the radius of the circle. (5)

Solution

$$\begin{aligned}|z-3| = 2|z| &\Rightarrow |z-3|^2 = 4|z|^2 \\ &\Rightarrow (x-3)^2 + y^2 = 4x^2 + 4y^2 \\ &\Rightarrow x^2 - 6x + 9 + y^2 = 4x^2 + 4y^2 \\ &\Rightarrow 3x^2 + 6x + 3y^2 = 9 \\ &\Rightarrow x^2 + 2x + y^2 = 3 \\ &\Rightarrow x^2 + 2x + 1 + y^2 = 4 \\ &\Rightarrow (x+1)^2 + y^2 = 4;\end{aligned}$$

it is a circle, centre $(-1, 0)$, and radius 2.

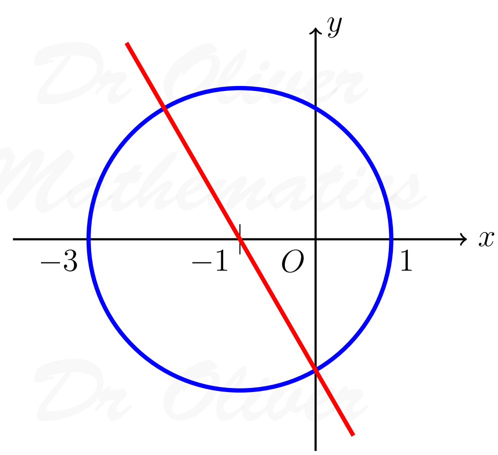
The point Q represents the complex number z on the Argand diagram such that

$$|z+3| = |z - \mathrm{i}\sqrt{3}|.$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies. (5)

Solution

$$\begin{aligned}|z+3| = |z - \mathrm{i}\sqrt{3}| &\Rightarrow |z+3|^2 = |z - \mathrm{i}\sqrt{3}|^2 \\ &\Rightarrow (x+3)^2 + y^2 = x^2 + (y - \sqrt{3})^2 \\ &\Rightarrow x^2 + 6x + 9 + y^2 = x^2 + y^2 - 2\sqrt{3}y + 3 \\ &\Rightarrow 2\sqrt{3}y = -6x - 6 \\ &\Rightarrow y = -\sqrt{3}x - \sqrt{3}.\end{aligned}$$

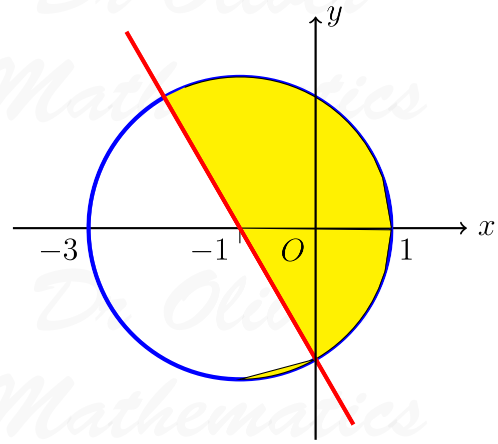


(c) On your diagram shade the region which satisfies (2)

$$|z - 3| \geqslant 2|z| \text{ and } |z + 3| \geqslant |z - \mathrm{i}\sqrt{3}|.$$

Solution

We want the right side of the circle (why?).



There is $(-1, -2)$ to $(0, -1.732)$ that should not be there!

7. The point P represents the complex number z on the Argand diagram. The locus of P is the curve C given by the equation

$$|z - 3| = 2|z - 4\mathrm{i}|.$$

(a) Show that C is a circle and give the coordinates of its centre and the value of its radius. (6)

Solution

$$\begin{aligned}
|z - 3| = 2|z - 4\mathrm{i}| &\Rightarrow |z - 3|^2 = 4|z - 4\mathrm{i}|^2 \\
&\Rightarrow (x - 3)^2 + y^2 = 4x^2 + 4(y - 4)^2 \\
&\Rightarrow x^2 - 6x + 9 + y^2 = 4x^2 + 4y^2 - 32y + 64 \\
&\Rightarrow 3x^2 + 6x + 3y^2 - 32y = -55 \\
&\Rightarrow x^2 + 2x + y^2 - \tfrac{32}{3}y = -\tfrac{55}{3} \\
&\Rightarrow x^2 + 2x + 1 + y^2 - \tfrac{32}{3}y + \tfrac{256}{9} = \tfrac{100}{9} \\
&\Rightarrow (x + 1)^2 + (y - \tfrac{16}{3})^2 = \tfrac{100}{9};
\end{aligned}$$

it is a circle, centre $(-1, \frac{16}{3})$, and radius $\frac{10}{3}$.

The point Q represents the complex number w. The point Q is related to the point P by

$$w = \frac{12}{z}.$$

As P describes the curve C,

(b) show that the locus of Q is given by the equation (5)

$$|w - a| = k|w - \mathrm{i}b|,$$

where a, b, and $k \in \mathbb{R}$, stating the value of a, b, and k.

Solution

We have

$$z = \frac{12}{w}$$

and

$$\begin{aligned} |z - 3| = 2|z - 4\mathrm{i}| &\Rightarrow \left|\frac{12}{w} - 3\right| = 2\left|\frac{12}{w} - 4\mathrm{i}\right| \\ &\Rightarrow \left|\frac{12 - 3w}{w}\right| = 2\left|\frac{12 - 4w\mathrm{i}}{w}\right| \\ &\Rightarrow |12 - 3w| = 2\,|12 - 4w\mathrm{i}| \\ &\Rightarrow 3\,|4 - w| = 8\,|3 - w\mathrm{i}| \\ &\Rightarrow |w - 4| = \tfrac{8}{3}\,|-\mathrm{i}(w + 3\mathrm{i})| \\ &\Rightarrow \underline{\underline{|w - 4| = \tfrac{8}{3}\,|w + 3\mathrm{i}|}}. \end{aligned}$$

8. The point P represents the complex number z on the Argand diagram. Point P moves on the curve C given by the equation

$$|z - 4 + 4\mathrm{i}| = 2|z - 1 + \mathrm{i}|.$$

(a) Show that C is a circle whose equation may be written $|z| = k$, giving the exact value of k. (5)

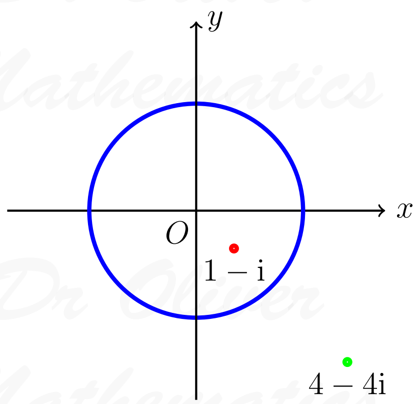
Solution

$$\begin{aligned} & |z-4+4\mathrm{i}| = 2|z-1+\mathrm{i}| \\ \Rightarrow\ & |z-4+4\mathrm{i}|^2 = 4|z-1+\mathrm{i}|^2 \\ \Rightarrow\ & (x-4)^2+(y+4)^2 = 4(x-1)^2+4(y+1)^2 \\ \Rightarrow\ & x^2-8x+16+y^2+8y+16 = 4x^2-8x+4+4y^2+8y+4 \\ \Rightarrow\ & 3x^2+3y^2 = 24 \\ \Rightarrow\ & x^2+y^2 = 8; \end{aligned}$$

it is a circle, centre $(0, 0)$, and radius $2\sqrt{2}$.

(b) Draw an Argand diagram showing the circle C and the points representing the complex numbers $1-\mathrm{i}$ and $4-4\mathrm{i}$. (3)

Solution



(c) For the points on the circle C, find the maximum and minimum values of $|z-4+4\mathrm{i}|$. (3)

Solution

$|4-4\mathrm{i}| = 4\sqrt{2}$ and the

$$\text{minimum value} = 4\sqrt{2} - 2\sqrt{2} = \underline{\underline{2\sqrt{2}}}$$

whilst

$$\text{minimum value} = 4\sqrt{2} + 2\sqrt{2} = \underline{\underline{6\sqrt{2}}}$$

The transformation T from the z-plane to the w-plane is given by

$$w = z + \frac{8}{z}.$$

(d) Show that T maps the curve C onto a line segment in the w-plane and define this line segment by giving its equation and the coordinates of its end points. (5)

Solution

$$\begin{aligned} w &= z + \frac{8}{z} \\ &= 2\sqrt{2}e^{i\theta} + 2\sqrt{2}e^{-i\theta} \\ &= 2\sqrt{2}(e^{i\theta} + e^{-i\theta}) \\ &= 4\sqrt{2}\cos\theta, \end{aligned}$$

so the locus is part of the real axis; as $-1 \leqslant \cos\theta \leqslant 1$, so the end points are $w = 4\sqrt{2}$ and $w = -4\sqrt{2}$.

9. The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z}{z + i}, z \neq -i.$$

The circle with equation $|z| = 3$ is mapped by T onto the curve C.

(a) Show that C is a circle and find its centre and radius. (8)

Solution

$$\begin{aligned} w = \frac{z}{z + i} &\Rightarrow w(z + i) = z \\ &\Rightarrow wz + wi = z \\ &\Rightarrow wi = z - wz \\ &\Rightarrow wi = z(1 - w) \\ &\Rightarrow z = \frac{wi}{1 - w}. \end{aligned}$$

Now,

$$\begin{aligned}
|z| = 3 &\Rightarrow \left|\frac{w\mathrm{i}}{1-w}\right| = 3 \\
&\Rightarrow |w| = 3|1-w| \\
&\Rightarrow |w|^2 = 9|1-w|^2 \\
&\Rightarrow u^2 + v^2 = 9(1-u)^2 + 9(-v)^2 \\
&\Rightarrow u^2 + v^2 = 9 - 18u + 9u^2 + 9v^2 \\
&\Rightarrow 8u^2 - 18u + 8v^2 = -9 \\
&\Rightarrow u^2 - \tfrac{9}{4}u + v^2 = -\tfrac{9}{8} \\
&\Rightarrow u^2 - \tfrac{9}{4}u + \tfrac{81}{64} + v^2 = \tfrac{63}{16} \\
&\Rightarrow (u - \tfrac{9}{8})^2 + v^2 = \tfrac{9}{64};
\end{aligned}$$

it is a circle, centre $(\frac{9}{8}, 0)$, and radius $\frac{3}{8}$.

The region $|z| < 3$ in the z-plane is mapped by T onto the region R in the w-plane.

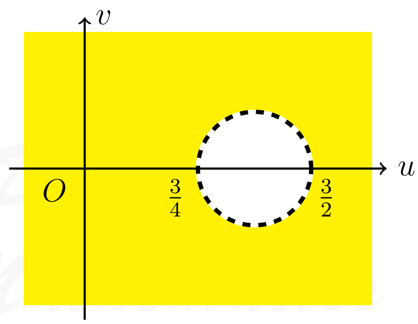
(b) Shade the region R on an Argand diagram. (2)

Solution

Where does $z = 4\mathrm{i}$ go?

$$w = \frac{4\mathrm{i}}{4\mathrm{i} + \mathrm{i}} = \tfrac{4}{5}$$

and it is inside the circle.



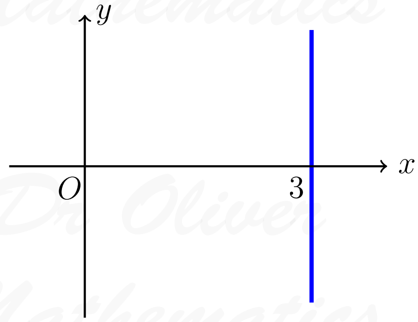
10. A complex number z is represented by the point P in the Argand diagram.

(a) Given that $|z - 6| = |z|$, sketch the locus of P. (2)

Solution

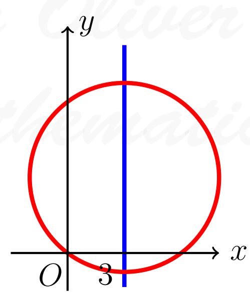
$|z-6| = |z|$ is a line through $x = 3$. In case you don't see it,

$$\begin{aligned}|z-6| = |z| &\Rightarrow |z-6|^2 = |z|^2 \\ &\Rightarrow (x-6)^2 + y^2 = x^2 + y^2 \\ &\Rightarrow x^2 - 12x + 36 + y^2 = x^2 + y^2 \\ &\Rightarrow 12x = 36 \\ &\Rightarrow x = 3.\end{aligned}$$



(b) Find the complex numbers z which satisfies both $|z-6| = |z|$ and $|z-3-4\text{i}| = 5$. (3)

Solution



$\underline{\underline{3+9\text{i}}}$ and $\underline{\underline{3-\text{i}}}$.

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{30}{z}.$$

(c) Show that T maps $|z-6| = |z|$ onto a circle in the w-plane and give the cartesian equation of this circle. (5)

Solution

$$\begin{aligned} w = \frac{30}{z} &\Rightarrow z = \frac{30}{w} \\ &\Rightarrow \left|\frac{30}{w} - 6\right| = \left|\frac{30}{w}\right| \\ &\Rightarrow \left|\frac{30 - 6w}{w}\right| = \left|\frac{30}{w}\right| \\ &\Rightarrow |6(5 - w)| = 30 \\ &\Rightarrow |w - 5| = 5 \\ &\Rightarrow \underline{\underline{(u-5)^2 + v^2 = 25}}. \end{aligned}$$

11. The point P represents the complex number z on an Argand diagram, where

$$|z - \mathrm{i}| = 2.$$

The locus of P as z varies is the curve C.

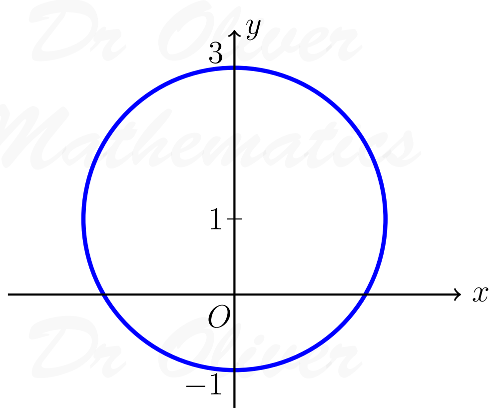
(a) Find a cartesian equation of C. (2)

Solution

$$\begin{aligned} |z - \mathrm{i}| = 2 &\Rightarrow |z - \mathrm{i}|^2 = 4 \\ &\Rightarrow \underline{\underline{x^2 + (y-1)^2 = 4}}. \end{aligned}$$

(b) Sketch the curve C. (2)

Solution



The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z + \mathrm{i}}{3 + \mathrm{i}z}, z \neq 3\mathrm{i}.$$

The point Q is mapped by T onto the point R. Given that R lies on the real axis,

(c) show that Q lies on C. (5)

Solution

$$\begin{aligned} w &= \frac{z + \mathrm{i}}{3 + \mathrm{i}z} \\ &= \frac{x + (y+1)\mathrm{i}}{3 + \mathrm{i}(x + \mathrm{i}y)} \\ &= \frac{x + (y+1)\mathrm{i}}{(3 - y) + \mathrm{i}x} \\ &= \frac{x + (y+1)\mathrm{i}}{(3 - y) + \mathrm{i}x} \times \frac{(3 - y) - \mathrm{i}x}{(3 - y) - \mathrm{i}x} \\ &= \frac{x(3 - y) + x(y + 1) + \mathrm{i}[(y+1)(3-y) - x^2]}{(3-y)^2 + x^2}. \end{aligned}$$

Now, $\mathrm{Im}(w) = 0$ and so

$$\begin{aligned} (y+1)(3-y) - x^2 = 0 &\Rightarrow 3 + 2y - y^2 - x^2 = 0 \\ &\Rightarrow x^2 + y^2 - 2y = 3 \\ &\Rightarrow x^2 + y^2 - 2y + 1 = 4 \\ &\Rightarrow x^2 + (y-1)^2 = 4, \end{aligned}$$

and so $\underline{\underline{Q \text{ is on } C}}$.

12. The point P represents a complex number z on an Argand diagram such that

$$|z - 6\mathrm{i}| = 2|z - 3|.$$

(a) Show that, as z varies, the locus of P is a circle, stating the radius and the coordinates of the centre of this circle. (6)

Solution

$$\begin{aligned}|z-6\mathrm{i}| = 2|z-3| &\Rightarrow |z-6\mathrm{i}|^2 = 4|z-3|^2\\ &\Rightarrow x^2+(y-6)^2 = 4(x-3)^2+4y^2\\ &\Rightarrow x^2+y^2-12y+36 = 4x^2-24x+36+4y^2\\ &\Rightarrow 3x^2-24x+3y^2+12y = 0\\ &\Rightarrow x^2-8x+y^2+4y = 0\\ &\Rightarrow x^2-8x+16+y^2+4y+4 = 20\\ &\Rightarrow (x-4)^2+(y+2)^2 = 20;\end{aligned}$$

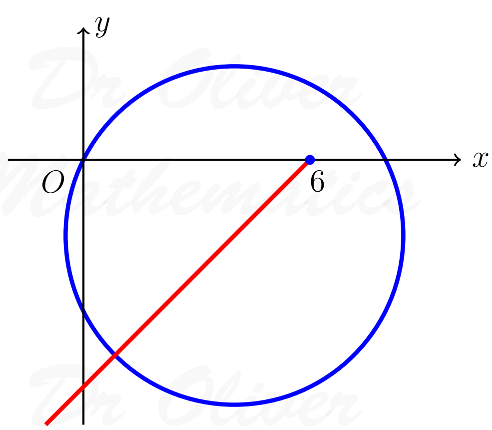
it is a circle, centre $(4, -2)$, and radius $2\sqrt{5}$.

The point Q represents the complex number $z$on an Argand diagram such that

$$\arg(z-6) = -\tfrac{3\pi}{4}.$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies. (4)

Solution



(c) Find the complex number for which both $|z-6\mathrm{i}| = 2|z-3|$ and $\arg(z-6) = -\frac{3\pi}{4}$. (4)

Solution

The straight line is $y = x - 6$:

$$\begin{aligned}(x-4)^2 + (y+2)^2 = 20 &\Rightarrow (x-4)^2 + (x-4)^2 = 20 \\ &\Rightarrow 2(x-4)^2 = 20 \\ &\Rightarrow (x-4)^2 = 10 \\ &\Rightarrow x - 4 = \sqrt{10} \text{ or } x - 4 = -\sqrt{10} \\ &\Rightarrow x = 4 + \sqrt{10} \text{ or } x = 4 - \sqrt{10},\end{aligned}$$

and hence the answer is $\underline{\underline{4 - \sqrt{10} + \mathrm{i}(-2 - \sqrt{10})}}$.

13. The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z + 2\mathrm{i}}{\mathrm{i}z}, z \neq 0.$$

The transformation maps points on the real line in the z-plane onto a line in the w-plane. Find an equation of this line.

Solution

In this case, $z = x + 0\mathrm{i}$ and so we have

$$\begin{aligned}w &= \frac{x + 2\mathrm{i}}{\mathrm{i}x} \\ &= \frac{x + 2\mathrm{i}}{\mathrm{i}x} \times \frac{-\mathrm{i}}{-\mathrm{i}} \\ &= \frac{2 - \mathrm{i}x}{x} \\ &= \frac{2}{x} - \mathrm{i}\end{aligned}$$

and we have $\underline{\underline{v = -1}}$.

14. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{4(1 - \mathrm{i})z - 8\mathrm{i}}{2(-1 + \mathrm{i})z - \mathrm{i}}, z \neq \tfrac{1}{4} - \tfrac{1}{4}\mathrm{i}.$$

The transformation T maps the points on the line l with equation $y = x$ in the z-plane to a circle C in the w-plane.

(a) Show that (6)

$$w = \frac{ax^2 + bx\mathrm{i} + c}{16x^2 + 1},$$

where a, b, and c are real constants to be found.

Solution

Let $z = x + \mathrm{i}x$. Then

$$\begin{aligned} w &= \frac{4(1-\mathrm{i})z - 8\mathrm{i}}{2(-1+\mathrm{i})z - \mathrm{i}} \\ &= \frac{4(1-\mathrm{i})(x+\mathrm{i}x) - 8\mathrm{i}}{2(-1+\mathrm{i})(x+\mathrm{i}x) - \mathrm{i}} \\ &= \frac{8x - 8\mathrm{i}}{-4x - \mathrm{i}} \\ &= \frac{8x - 8\mathrm{i}}{-4x - \mathrm{i}} \times \frac{-4x + \mathrm{i}}{-4x + \mathrm{i}} \\ &= \frac{-32x^2 + 8 + x\mathrm{i}(8 + 32)}{16x^2 + 1} \\ &= \underline{\underline{\frac{-32x^2 + 40x\mathrm{i} + 8}{16x^2 + 1}}}. \end{aligned}$$

(b) Hence show that the circle C has equation (4)

$$(u-3)^2 + v^2 = k^2,$$

where k is a constant to be found.

Solution

$$u = \frac{-32x^2 + 8}{16x^2 + 1} \text{ and } v = \frac{40x}{16x^2 + 1}.$$

Now,

$$
\begin{aligned}
(u-3)^2+v^2 &= \left(\frac{-32x^2+8}{16x^2+1}-3\right)^2+\left(\frac{40x}{16x^2+1}\right)^2 \\
&= \frac{1}{(16x^2+1)^2}\left\{\left[(-32x^2+8-3(16x^2+1)\right]^2+(40x)^2\right\} \\
&= \frac{(-80x^2+5)^2+(40x)^2}{(16x^2+1)^2} \\
&= \frac{6400x^4-800x^2+25+1600x^2}{(16x^2+1)^2} \\
&= \frac{6400x^4+800x^2+25}{(16x^2+1)^2} \\
&= \frac{(80x^2+5)^2}{(16x^2+1)^2} \\
&= \frac{25(16x^2+1)^2}{(16x^2+1)^2} \\
&= 25;
\end{aligned}
$$

it is a circle, centre $(3,0)$, and radius 5.

15. The transformation T maps from from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$. The transformation T is given by

$$w = \frac{z}{\mathrm{i}z+1}, z \neq \mathrm{i}.$$

The transformation T maps the line l in the z-plane onto the line with equation $v = -1$ in the w-plane.

(a) Find a cartesian equation of l in terms of x and y. (5)

Solution

$$\begin{aligned}
w = \frac{z}{\mathrm{i}z+1} \Rightarrow u - \mathrm{i} &= \frac{x+\mathrm{i}y}{\mathrm{i}(x+\mathrm{i}y)+1} \\
\Rightarrow u - \mathrm{i} &= \frac{x+\mathrm{i}y}{(1-y)+\mathrm{i}x} \\
\Rightarrow u - \mathrm{i} &= \frac{x+\mathrm{i}y}{(1-y)+\mathrm{i}x} \times \frac{(1-y)-\mathrm{i}x}{(1-y)-\mathrm{i}x} \\
\Rightarrow u - \mathrm{i} &= \frac{x(1-y)+xy+\mathrm{i}[y(1-y)-x^2]}{(1-y)^2+x^2} \\
\Rightarrow -1 &= \frac{y(1-y)-x^2}{(1-y)^2+x^2} \\
&\Rightarrow -(1-y)^2 - x^2 = y(1-y) - x^2 \\
&\Rightarrow -1 + 2y - y^2 = y - y^2 \\
&\Rightarrow \underline{\underline{y = 1}}.
\end{aligned}$$

The transformation T maps the line with equation $y = \frac{1}{2}$ in the z-plane onto the curve C in the w-plane.

(b) (i) Show that C is a circle with centre the origin. (4)

Solution
$z = x + \frac{1}{2}\mathrm{i}$:

$$\begin{aligned}
u + \mathrm{i}v &= \frac{x+\frac{1}{2}\mathrm{i}}{\mathrm{i}(x+\frac{1}{2}\mathrm{i})+1} \\
&= \frac{x+\frac{1}{2}\mathrm{i}}{\frac{1}{2}+x\mathrm{i}} \\
&= \frac{x+\frac{1}{2}\mathrm{i}}{\frac{1}{2}+x\mathrm{i}} \times \frac{\frac{1}{2}-x\mathrm{i}}{\frac{1}{2}-x\mathrm{i}} \\
&= \frac{x+\mathrm{i}(\frac{1}{4}-x^2)}{\frac{1}{4}+x^2}
\end{aligned}$$

and so

$$u = \frac{x}{\frac{1}{4}+x^2} \text{ and } v = \frac{\frac{1}{4}-x^2}{\frac{1}{4}+x^2}.$$

Now,

$$
\begin{aligned}
u^2 + v^2 &= \left(\frac{x}{\frac{1}{4} + x^2}\right)^2 + \left(\frac{\frac{1}{4} - x^2}{\frac{1}{4} + x^2}\right)^2 \\
&= \frac{x^2 + (\frac{1}{4} - x^2)^2}{(\frac{1}{4} + x^2)^2} \\
&= \frac{x^2 + (\frac{1}{16} - \frac{1}{2}x^2 + x^4)}{(\frac{1}{4} + x^2)^2} \\
&= \frac{\frac{1}{16} + \frac{1}{2}x^2 + x^4}{(\frac{1}{4} + x^2)^2} \\
&= \frac{(\frac{1}{4} + x^2)^2}{(\frac{1}{4} + x^2)^2} \\
&= 1,
\end{aligned}
$$

it is a circle, centre $(0, 0)$, and radius 1.

(ii) Write down a cartesian equation of C in terms of u and v. (2)

Solution
$u^2 + v^2 = 1$.

16. The transformation T maps from from the z-plane to the w-plane is given by

$$w = \frac{z}{z + 3\mathrm{i}}, z \neq -3\mathrm{i}.$$

The circle with equation $|z| = 2$ is mapped by T onto the curve C.

(a) (i) Show that C is a circle. (6)

Solution

$$
\begin{aligned}
w = \frac{z}{z + 3\mathrm{i}} &\Rightarrow w(z + 3\mathrm{i}) = z \\
&\Rightarrow wz + 3w\mathrm{i} = z \\
&\Rightarrow 3w\mathrm{i} = z - wz \\
&\Rightarrow 3w\mathrm{i} = z(1 - w) \\
&\Rightarrow z = \frac{3w\mathrm{i}}{1 - w}
\end{aligned}
$$

and

$$\begin{aligned}|z| = 2 &\Rightarrow \left|\frac{3w\mathrm{i}}{1-w}\right| = 2\\ &\Rightarrow |3w\mathrm{i}| = 2|1-w|\\ &\Rightarrow 3\,|w| = 2|1-w|\\ &\Rightarrow 9\,|w|^2 = 4|1-w|^2\\ &\Rightarrow 9u^2 + 9v^2 = 4(1-u)^2 + 4(-v)^2\\ &\Rightarrow 9u^2 + 9v^2 = 4 - 8u + 4u^2 + 4v^2\\ &\Rightarrow 5u^2 + 8u + 5v^2 = 4\\ &\Rightarrow u^2 + \tfrac{8}{5}u + v^2 = \tfrac{4}{5}\\ &\Rightarrow u^2 + \tfrac{8}{5}u + \tfrac{16}{25} + v^2 = \tfrac{36}{25}\\ &\Rightarrow (u + \tfrac{4}{5})^2 + v^2 = \tfrac{36}{25},\end{aligned}$$

it is a circle.

(ii) Find the centre and radius of C. (2)

Solution
Centre $(-\frac{4}{5}, 0)$, and radius $\frac{6}{5}$.

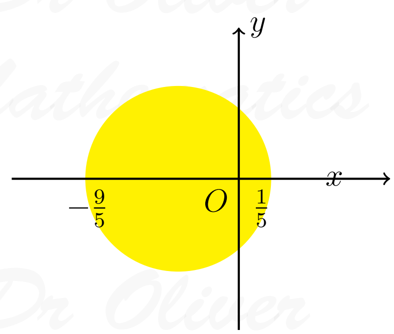
The region $|z| \leqslant 2$ in the z-plane is mapped by T onto the region R in to the w-plane.

(b) Shade the region R on an Argand diagram. (2)

Solution

$$z = \mathrm{i} \Rightarrow w = \frac{\mathrm{i}}{4\mathrm{i}} = \frac{1}{4}$$

which is inside the circle.



17. The transformation T maps from from the z-plane to the w-plane is given by

$$w = \frac{z + 3\mathrm{i}}{1 + \mathrm{i}z}, z \neq \mathrm{i}.$$

The transformation T maps the circle $|z| = 1$ in the z-plane onto the line l in the w-plane.

(a) Find a cartesian equation of the line l. (5)

Solution

$$\begin{aligned} w = \frac{z + 3\mathrm{i}}{1 + \mathrm{i}z} &\Rightarrow w(1 + \mathrm{i}z) = z + 3\mathrm{i} \\ &\Rightarrow w + \mathrm{i}zw = z + 3\mathrm{i} \\ &\Rightarrow w - 3\mathrm{i} = z - \mathrm{i}zw \\ &\Rightarrow w - 3\mathrm{i} = z(1 - \mathrm{i}w) \\ &\Rightarrow z = \frac{w - 3\mathrm{i}}{1 - \mathrm{i}w}. \end{aligned}$$

Now,

$$\begin{aligned} |z| = 1 \Rightarrow 1 &= \left| \frac{w - 3\mathrm{i}}{1 - \mathrm{i}w} \right| \\ \Rightarrow 1 &= \left| \frac{w - 3\mathrm{i}}{-\mathrm{i}(w + \mathrm{i})} \right| \\ &\Rightarrow |w + \mathrm{i}| = |w - 3\mathrm{i}| \\ &\Rightarrow |u + \mathrm{i}(v + 1)| = |u + \mathrm{i}(v - 3)| \\ &\Rightarrow u^2 + (v + 1)^2 = u^2 + (v - 3)^2 \\ &\Rightarrow v^2 + 2v + 1 = v^2 - 6v + 9 \\ &\Rightarrow 8v = 8 \\ &\Rightarrow \underline{\underline{v = 1}}. \end{aligned}$$

The circle $|z - a - b\mathrm{i}| = c$ in the z-plane is mapped by T onto the circle $|w| = 5$ in the w-plane.

(b) Find the exact values of the real constants a, b, and c. (6)

Solution

$$
\begin{aligned}
w = \frac{z+3\mathrm{i}}{1+\mathrm{i}z} &\Rightarrow |w| = \left|\frac{z+3\mathrm{i}}{1+\mathrm{i}z}\right| \\
&\Rightarrow 5 = \left|\frac{z+3\mathrm{i}}{1+\mathrm{i}z}\right| \\
&\Rightarrow 5|\mathrm{i}(z-\mathrm{i})| = |z+3\mathrm{i}| \\
&\Rightarrow 5|x+\mathrm{i}(y-1)| = |x+\mathrm{i}(y+3)| \\
&\Rightarrow 25x^2 + (y-1)^2 = x^2 + (y+3)^2 \\
&\Rightarrow 25x^2 + 25(y^2-2y+1) = x^2 + (y^2+6y+9) \\
&\Rightarrow 24x^2 + 24y^2 - 56y = -16 \\
&\Rightarrow x^2 + y^2 - \tfrac{7}{3}y = -\tfrac{2}{3} \\
&\Rightarrow x^2 + y^2 - \tfrac{7}{3}y + \tfrac{49}{36} = -\tfrac{2}{3} + \tfrac{49}{36} \\
&\Rightarrow x^2 + (y-\tfrac{7}{6})^2 = \tfrac{25}{36};
\end{aligned}
$$

so, $\underline{\underline{a=0}}$, $\underline{\underline{b=\frac{7}{6}}}$, and $\underline{\underline{c=\frac{5}{6}}}$.