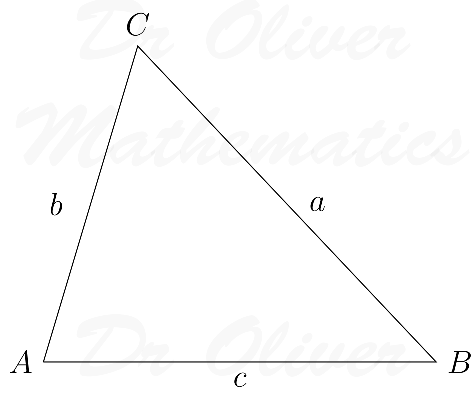
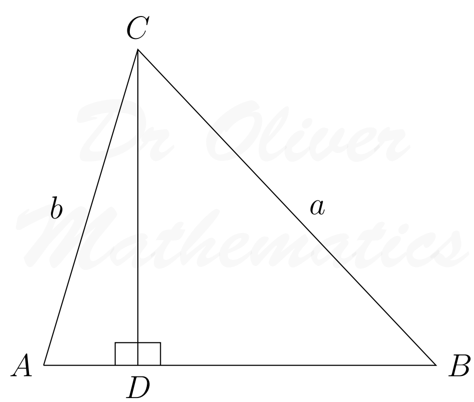
Dr Oliver Mathematics
The Sine Rule

In this note, we will investigate the sine rule.

Suppose we have the following triangle.



Split the triangle in two:



Then

$$\sin \angle CAD = \frac{CD}{b} \Rightarrow CD = b \sin \angle CAD$$

and

$$\sin \angle CBD = \frac{CD}{a} \Rightarrow CD = a \sin \angle CBD.$$

Hence,

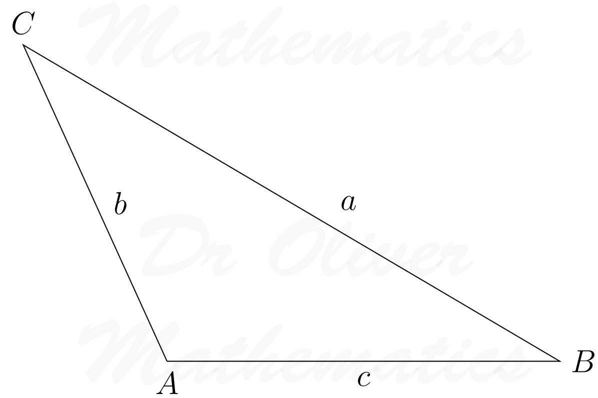
$$b \sin \angle CAD = a \sin \angle CBD \Rightarrow \frac{a}{\sin \angle CAD} = \frac{b}{\sin \angle CBD}.$$

"Now if were to divide $\triangle ABC$ into two right-angled triangles by drawing the perpendicular from A to BC the similar rest would be

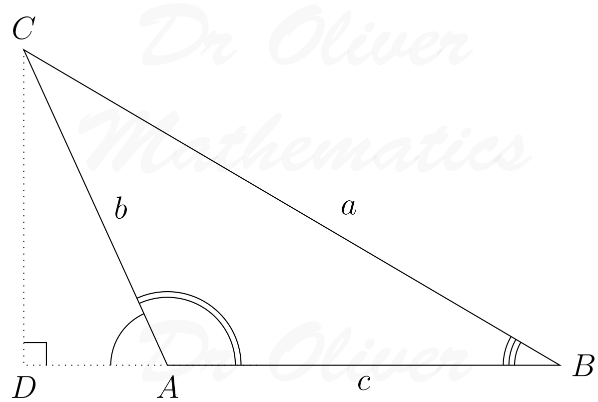
$$\frac{b}{\sin \widehat{B}} = \frac{c}{\sin \widehat{C}}."$$

I got that bit that from *National Curriculum Mathematics Higher GCSE 10A* by Bostock, Chandler, Shepherd, and Smith. It's a fine book. (No, really.)

Except ...



We have one obtuse angle: what about that? Do things still work? What do *you* think?



Well,

$$\sin \angle CAD = \frac{CD}{b} \Rightarrow CD = b \sin \angle CAD$$

and

$$\sin \angle ABC = \frac{CD}{a} \Rightarrow CD = a \sin \angle ABC.$$

In the first equation, we appear to have $b \sin \angle CAD$ rather than $b \sin \angle CAB$.

Can you spot what to do?

That's right:

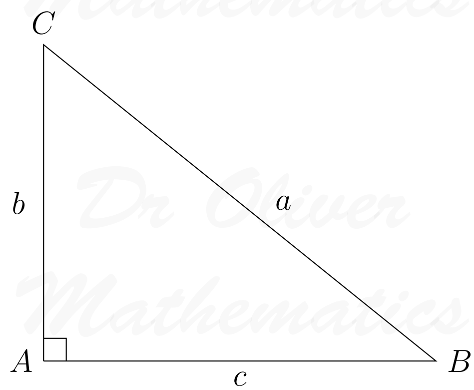
$$\angle CAD + \angle CAB = 180^\circ$$

and

$$CD = b \sin \angle CAD = b \sin \angle CAB$$

and the proof follows.

What about right-angled triangles?



Well,

$$\begin{aligned} b = a \sin \widehat{B} &\Rightarrow b \sin 90^\circ = a \sin \widehat{B} \\ &\Rightarrow \frac{b}{\sin \widehat{B}} = \frac{a}{\sin 90^\circ}, \end{aligned}$$

and the proof follows.

In summary,

$$\boxed{\frac{a}{\sin A^\circ} = \frac{b}{\sin B^\circ} = \frac{c}{\sin C^\circ}}$$

(assuming it is lengths that we want) or

$$\boxed{\frac{\sin A^\circ}{a} = \frac{\sin B^\circ}{b} = \frac{\sin C^\circ}{c}}$$

(assuming it is angles that we want).

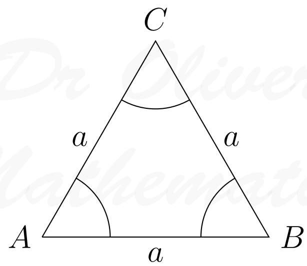
In fact,

(a) the smallest angle is opposite the smallest side,

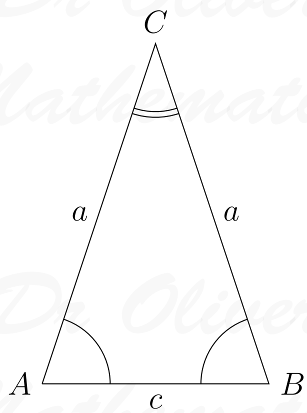
(b) the middle angle is opposite the middle side,

(c) the largest angle is opposite the largest side.

Assuming, that is, the three lengths are different. In a equilateral triangle, the three sides are all the same length and they are opposite 60°

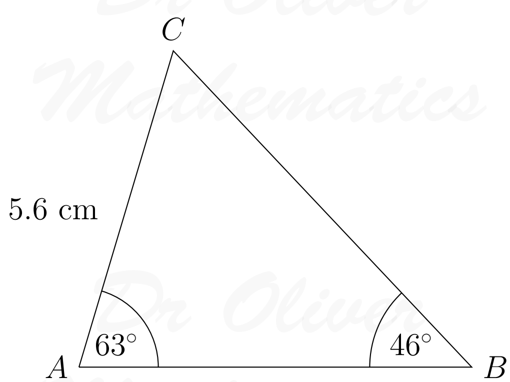


and, in an isosceles triangle, the base angles are the same size.

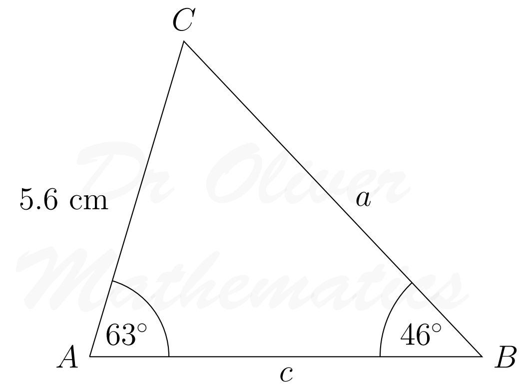


Okay: a few examples. We will give our answers to 3 significant figures. Oh, the diagrams are *not* accurately drawn...

1. In $\triangle ABC$, find BC.

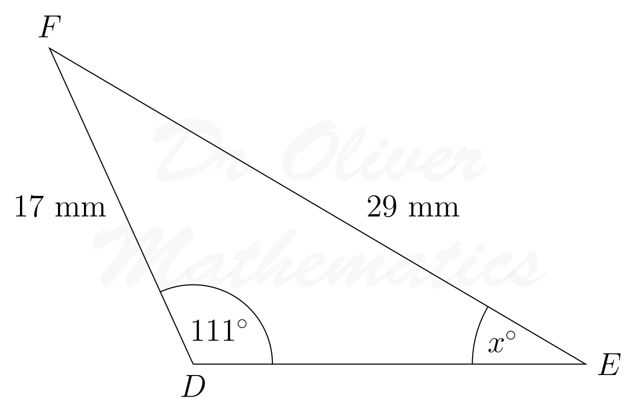


Solution



We know
$$\frac{a}{\sin A^\circ} = \frac{b}{\sin B^\circ} = \frac{c}{\sin C^\circ}$$
so we take only the first two parts of the equation:
$$\frac{a}{\sin 63^\circ} = \frac{5.6}{\sin 46^\circ} \Rightarrow a = \frac{5.6 \sin 63^\circ}{\sin 46^\circ}$$
$$\Rightarrow a = 6.936\,411\,044 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{a = 6.94 \text{ cm (3 sf)}}}.$$

2. In $\triangle DEF$, find x°.



Solution

We want angles:

$$\frac{\sin D^\circ}{d} = \frac{\sin E^\circ}{e} = \frac{\sin F^\circ}{f}$$

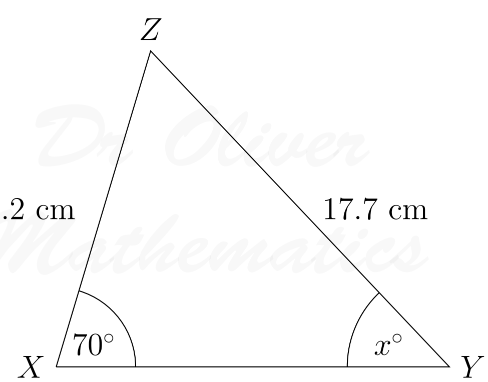
so we take only the first two parts of the equation:

$$\begin{aligned}\frac{\sin 111^\circ}{29} = \frac{\sin x^\circ}{17} &\Rightarrow \sin x^\circ = \frac{17 \sin 111^\circ}{29} \\ &\Rightarrow x = 33.180\,012\,34 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 33.2 \text{ (3 sf)}}}.\end{aligned}$$

3. In $\triangle XYZ$, $XZ = 18.2$ cm, $YZ = 17.7$ cm, and $\angle YXZ = 70^\circ$.

   Find $\angle XYZ$.

Solution



Hmm: we have a bigger angle in the pair $(x^\circ, 18.2 \text{ cm})$ than do in the pair $(70^\circ, 17.7 \text{ cm})$:

$$\begin{aligned}\frac{\sin 70^\circ}{17.7} = \frac{\sin x^\circ}{18.2} &\Rightarrow \sin x^\circ = \frac{18.2 \sin 70^\circ}{17.7} \\ &\Rightarrow x = 75.069\,176\,66 \text{ or } 104.930\,823\,3 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 75.1^\circ \text{ or } x = 105^\circ \text{ (3 sf)}}}.\end{aligned}$$