Dr Oliver Mathematics
Further Mathematics
Series
Past Examination Questions

This booklet consists of 19 questions across a variety of examination topics.
The total number of marks available is 130.

$$\sum_{r=1}^{n} 1 = n$$
$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$
$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$
$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

1. Calculate (3)

$$\sum_{r=1}^{40} (3r-1)^2.$$

Solution

$$\begin{aligned}\sum_{r=1}^{40} (3r-1)^2 &= 9\sum_{r=1}^{40} r^2 - 6\sum_{r=1}^{40} r + \sum_{r=1}^{40} 1 \\ &= 9 \times \tfrac{1}{6} \times 40 \times 41 \times 81 - 6 \times \tfrac{1}{2} \times 40 \times 41 + 40 \\ &= 199\,260 - 4\,920 + 40 \\ &= \underline{\underline{194\,380}}.\end{aligned}$$

2. Show that (4)

$$\sum_{r=n}^{2n} r^2 = \tfrac{1}{6}n(n+1)(an+b),$$

where a and b are constants to be found.

Solution

$$\begin{aligned}\sum_{r=n}^{2n} r^2 &= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 \\ &= \tfrac{1}{6}(2n)(2n+1)(4n+1) - \tfrac{1}{6}(n-1)(n)(2n-1) \\ &= \tfrac{1}{6}n[2(2n+1)(4n+1) - (n-1)(2n-1)] \\ &= \tfrac{1}{6}n[(16n^2+12n+2) - (2n^2-3n+1)] \\ &= \tfrac{1}{6}n(14n^2+15n+1) \\ &= \underline{\underline{\tfrac{1}{6}n(n+1)(14n+1)}};\end{aligned}$$

so, $\underline{\underline{a = 14}}$ and $\underline{\underline{b = 1}}$.

3. (a) Show that (4)

$$\sum_{r=1}^{n}(r^2 - r - 1) = \tfrac{1}{3}n(n^2-4).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r^2 - r - 1) &= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\ &= \tfrac{1}{6}n(n+1)(2n+1) - \tfrac{1}{2}n(n+1) - n \\ &= \tfrac{1}{6}n[(n+1)(2n+1) - 3(n+1) - 6] \\ &= \tfrac{1}{6}n[(2n^2+3n+1) - (3n+3) - 6] \\ &= \tfrac{1}{6}n(2n^2-8) \\ &= \underline{\underline{\tfrac{1}{3}n(n^2-4)}}.\end{aligned}$$

(b) Hence, or otherwise, find the value of (2)

$$\sum_{r=10}^{20}(r^2 - r - 1).$$

Solution

$$\begin{aligned}\sum_{r=10}^{20}(r^2-r-1) &= \sum_{r=1}^{20}(r^2-r-1) - \sum_{r=1}^{9}(r^2-r-1) \\ &= \tfrac{1}{3}\times 20\times(20^2-4) - \tfrac{1}{3}\times 9\times(9^2-4) \\ &= 2\,640 - 231 \\ &= \underline{\underline{2\,409}}.\end{aligned}$$

4. (a) Show, using the formulae for $\sum r$ and $\sum r^2$, that (5)

$$\sum_{r=1}^{n}(6r^2+4r-1) = n(n+2)(2n+1).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(6r^2+4r-1) &= 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\ &= n(n+1)(2n+1) + 2n(n+1) - n \\ &= n[(n+1)(2n+1) + 2(n+1) - 1] \\ &= n[(n+1)(2n+1) + (2n+2) - 1] \\ &= n[(n+1)(2n+1) + (2n+1)] \\ &= \underline{\underline{n(n+2)(2n+1)}}.\end{aligned}$$

(b) Hence, or otherwise, find the value of (2)

$$\sum_{r=11}^{20}(6r^2+4r-1).$$

Solution

$$\begin{aligned}\sum_{r=11}^{20}(6r^2+4r-1) &= \sum_{r=1}^{20}(6r^2+4r-1) - \sum_{r=1}^{10}(6r^2+4r-1) \\ &= 20\times 22\times 41 - 10\times 12\times 21 \\ &= 18\,040 - 2\,520 \\ &= \underline{\underline{15\,520}}.\end{aligned}$$

5. (a) Show that (5)

$$\sum_{r=1}^{n} r(r+2)(r+4) = \tfrac{1}{4}n(n+1)(n+4)(n+5).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n} r(r+2)(r+4) &= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 8\sum_{r=1}^{n} r \\ &= \tfrac{1}{4}n^2(n+1)^2 + n(n+1)(2n+1) + 4n(n+1) \\ &= \tfrac{1}{4}n(n+1)[n(n+1) + 4(2n+1) + 16] \\ &= \tfrac{1}{4}n(n+1)[(n^2+n) + (8n+4) + 16] \\ &= \tfrac{1}{4}n(n+1)(n^2+9n+20) \\ &= \underline{\underline{\tfrac{1}{4}n(n+1)(n+4)(n+5)}}.\end{aligned}$$

(b) Hence evaluate (2)

$$\sum_{r=21}^{30} r(r+2)(r+4).$$

Solution

$$\begin{aligned}\sum_{r=21}^{30} r(r+2)(r+4) &= \sum_{r=1}^{30} r(r+2)(r+4) - \sum_{r=1}^{20} r(r+2)(r+4) \\ &= \tfrac{1}{4} \times 30 \times 31 \times 34 \times 35 - \tfrac{1}{4} \times 20 \times 21 \times 24 \times 25 \\ &= 276\,675 - 63\,000 \\ &= \underline{\underline{213\,675}}.\end{aligned}$$

6. (a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$, show that (5)

$$\sum_{r=1}^{n} (r^3 + 3r + 2) = \tfrac{1}{4}n(n+2)(n^2+7).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r^3+3r+2) &= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 2\\ &= \tfrac{1}{4}n^2(n+1)^2 + \tfrac{3}{2}n(n+1) + 2n\\ &= \tfrac{1}{4}n[n(n+1)^2 + 6(n+1) + 8]\\ &= \tfrac{1}{4}n[(n^3+2n^2+n) + (6n+6) + 8]\\ &= \tfrac{1}{4}n(n^3+2n^2+7n+14)\\ &= \underline{\underline{\tfrac{1}{4}n(n+2)(n^2+7)}}.\end{aligned}$$

(b) Hence evaluate (2)

$$\sum_{r=15}^{25}(r^3+3r+2).$$

Solution

$$\begin{aligned}\sum_{r=15}^{25}(r^3+3r+2) &= \sum_{r=1}^{25}(r^3+3r+2) - \sum_{r=1}^{14}(r^3+3r+2)\\ &= \tfrac{1}{4}(25)(27)(25^2+7) - \tfrac{1}{4}(14)(16)(14^2+7)\\ &= 106\,650 - 11\,368\\ &= \underline{\underline{95\,282}}.\end{aligned}$$

7. (a) Using the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that (5)

$$\sum_{r=1}^{n}(r+2)(r+3) = \tfrac{1}{3}n(n^2+an+b),$$

where a and b are integers to be found.

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r+2)(r+3) &= \sum_{r=1}^{n}(r^2+5r+6)\\ &= \sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 6\\ &= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{5}{2}n(n+1) + 6n\\ &= \tfrac{1}{6}n[(n+1)(2n+1) + 15(n+1) + 36]\\ &= \tfrac{1}{6}n[(2n^2+3n+1) + (15n+15) + 36]\\ &= \tfrac{1}{6}n(2n^2+18n+52)\\ &= \underline{\underline{\tfrac{1}{3}n(n^2+9n+26)}};\end{aligned}$$

so, $\underline{\underline{a=9}}$ and $\underline{\underline{b=26}}$.

(b) Hence show that (3)

$$\sum_{r=n+1}^{2n}(r+2)(r+3) = \tfrac{1}{3}n(7n^2+27n+26).$$

Solution

$$\begin{aligned}\sum_{r=n+1}^{2n}(r+2)(r+3) &= \sum_{r=1}^{2n}(r+2)(r+3) - \sum_{r=1}^{n}(r+2)(r+3)\\ &= \tfrac{1}{3}(2n)[(2n)^2+9(2n)+26] - \tfrac{1}{3}n(n^2+9n+26)\\ &= \tfrac{1}{3}n(8n^2+36n+52) - \tfrac{1}{3}n(n^2+9n+26)\\ &= \underline{\underline{\tfrac{1}{3}n(7n^2+27n+26)}}.\end{aligned}$$

8. (a) Use the results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$, and $\sum_{r=1}^{n} r^3$, to prove that (5)

$$\sum_{r=1}^{n} r(r+1)(r+5) = \tfrac{1}{4}n(n+1)(n+2)(n+7).$$

Solution

$$\begin{aligned}
\sum_{r=1}^{n} r(r+1)(r+5) &= \sum_{r=1}^{n} (r^3 + 6r^2 + 5r) \\
&= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r \\
&= \tfrac{1}{4}n^2(n+1)^2 + n(n+1)(2n+1) + \tfrac{5}{2}n(n+1) \\
&= \tfrac{1}{4}n(n+1)[n(n+1) + 4(2n+1) + 10] \\
&= \tfrac{1}{4}n(n+1)[(n^2+n) + (8n+4) + 10] \\
&= \tfrac{1}{4}n(n+1)(n^2 + 9n + 14) \\
&= \underline{\underline{\tfrac{1}{4}n(n+1)(n+2)(n+7)}}.
\end{aligned}$$

(b) Hence, or otherwise, find the value of (2)

$$\sum_{r=20}^{50} r(r+1)(r+5).$$

Solution

$$\begin{aligned}
\sum_{r=20}^{50} r(r+1)(r+5) &= \sum_{r=1}^{50} r(r+1)(r+5) - \sum_{r=1}^{19} r(r+1)(r+5) \\
&= \tfrac{1}{4} \times 50 \times 51 \times 52 \times 57 - \tfrac{1}{4} \times 19 \times 20 \times 21 \times 26 \\
&= 1\,889\,550 - 51\,870 \\
&= \underline{\underline{1\,837\,680}}.
\end{aligned}$$

9. (a) Use the results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that (6)

$$\sum_{r=1}^{n} (2r-1)^2 = \tfrac{1}{3}n(2n-1)(2n+1),$$

for all positive integers n.

Solution

$$\begin{aligned}\sum_{r=1}^{n}(2r-1)^2 &= \sum_{r=1}^{n}(4r^2-4r+1)\\ &= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + \sum_{r=1}^{n}1\\ &= \tfrac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n\\ &= \tfrac{1}{3}n[2(n+1)(2n+1) - 6(n+1) + 3]\\ &= \tfrac{1}{3}n[(4n^2+6n+2) - (6n+6) + 3]\\ &= \tfrac{1}{3}n(4n^2-1)\\ &= \underline{\underline{\tfrac{1}{3}n(2n-1)(2n+1)}}.\end{aligned}$$

(b) Hence show that (4)

$$\sum_{r=n+1}^{3n}(2r-1)^2 = \tfrac{2}{3}n(an^2+b),$$

where a and b are integers to be found.

Solution

$$\begin{aligned}\sum_{r=n+1}^{3n}(2r-1)^2 &= \sum_{r=1}^{3n}(2r-1)^2 - \sum_{r=1}^{n}(2r-1)^2\\ &= \tfrac{1}{3}(3n)[2(3n)-1][2(3n)+1] - \tfrac{1}{3}n(2n-1)(2n+1)\\ &= n(6n-1)(6n+1) - \tfrac{1}{3}n(2n-1)(2n+1)\\ &= \tfrac{1}{3}n[3(36n^2-1) - (4n^2-1)]\\ &= \tfrac{1}{3}n(104n^2-2)\\ &= \underline{\underline{\tfrac{2}{3}n(52n^2-1)}};\end{aligned}$$

so, $\underline{\underline{a = 52}}$ and $\underline{\underline{b = -1}}$.

10. (a) Using the result (3)

$$\sum_{r=1}^{n}r^3 = \tfrac{1}{4}n^2(n+1)^2,$$

show that

$$\sum_{r=1}^{n}(r^3-2) = \tfrac{1}{4}n(n^3+2n^2+n-8).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r^3-2) &= \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} 2\\ &= \tfrac{1}{4}n^2(n+1)^2 - 2n\\ &= \tfrac{1}{4}n[n(n+1)^2-8]\\ &= \tfrac{1}{4}n[n(n^2+2n+1)-8]\\ &= \underline{\underline{\tfrac{1}{4}n(n^3+2n^2+n-8)}}.\end{aligned}$$

(b) Calculate the exact value of (3)

$$\sum_{r=20}^{50}(r^3-2).$$

Solution

$$\begin{aligned}\sum_{r=20}^{50}(r^3-2) &= \sum_{r=1}^{50}(r^3-2) - \sum_{r=1}^{19}(r^3-2)\\ &= \tfrac{1}{4}\times 50\times 130\,042 - \tfrac{1}{4}\times 19\times 7\,592\\ &= 1\,625\,525 - 36\,062\\ &= \underline{\underline{1\,589\,463}}.\end{aligned}$$

11. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that (5)

$$\sum_{r=1}^{n}(r^3+6r-3) = \tfrac{1}{4}n^2(n^2+2n+13),$$

for all positive integers n.

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r^3+6r-3) &= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3 \\ &= \tfrac{1}{4}n^2(n+1)^2 + 3n(n+1) - 3n \\ &= \tfrac{1}{4}n[n(n+1)^2 + 12(n+1) - 12] \\ &= \tfrac{1}{4}n[(n^3+2n^2+n) + (12n+12) - 12] \\ &= \tfrac{1}{4}n(n^3+2n^2+13n) \\ &= \underline{\underline{\tfrac{1}{4}n^2(n^2+2n+13)}}.\end{aligned}$$

(b) Hence find the exact value of (2)

$$\sum_{r=16}^{30}(r^3+6r-3).$$

Solution

$$\begin{aligned}\sum_{r=16}^{30}(r^3+6r-3) &= \sum_{r=1}^{30}(r^3+6r-3) - \sum_{r=1}^{15}(r^3+6r-3) \\ &= \tfrac{1}{4}\times 30^2 \times 973 - \tfrac{1}{4}\times 15^2 \times 268 \\ &= 218\,925 - 15\,075 \\ &= \underline{\underline{203\,850}}.\end{aligned}$$

12. Show, using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, that (5)

$$\sum_{r=1}^{n} 3(2r-1)^2 = n(2n-1)(2n+1),$$

for all positive integers n.

Solution

$$
\begin{aligned}
\sum_{r=1}^{n} 3(2r-1)^2 &= \sum_{r=1}^{n} 3(4r^2-4r+1) \\
&= 12\sum_{r=1}^{n} r^2 - 12\sum_{r=1}^{n} r + \sum_{r=1}^{n} 3 \\
&= 2n(n+1)(2n+1) - 6n(n+1) + 3n \\
&= n[2(n+1)(2n+1) - 6(n+1) + 3] \\
&= n[(4n^2+6n+2) - (6n+6) + 3] \\
&= n(4n^2-1) \\
&= \underline{\underline{n(2n-1)(2n+1)}}.
\end{aligned}
$$

13. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that (6)

$$\sum_{r=1}^{n}(r+2)(r+3) = \tfrac{1}{3}n(n^2+9n+26),$$

for all positive integers n.

Solution

$$
\begin{aligned}
\sum_{r=1}^{n}(r+2)(r+3) &= \sum_{r=1}^{n}(r^2+5r+6) \\
&= \sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 6 \\
&= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{5}{2}n(n+1) + 6n \\
&= \tfrac{1}{6}n[(n+1)(2n+1) + 15(n+1) + 36] \\
&= \tfrac{1}{6}n[(2n^2+3n+1) + (15n+15) + 36] \\
&= \tfrac{1}{6}n(2n^2+18n+52) \\
&= \underline{\underline{\tfrac{1}{3}n(n^2+9n+26)}}.
\end{aligned}
$$

(b) Hence show that (4)

$$\sum_{r=n+1}^{3n}(r+2)(r+3) = \tfrac{2}{3}n(an^2+bn+c),$$

where a, b, and c are integers to be found.

Solution

$$\begin{aligned}\sum_{r=n+1}^{3n}(r+2)(r+3) &= \sum_{r=1}^{3n}(r+2)(r+3) - \sum_{r=1}^{n}(r+2)(r+3)\\ &= \tfrac{1}{3}(3n)[(3n)^2+9(3n)+26] - \tfrac{1}{3}n(n^2+9n+26)\\ &= \tfrac{1}{3}n(27n^2+81n+78) - \tfrac{1}{3}n(n^2+9n+26)\\ &= \tfrac{1}{3}n(26n^2+72n+52)\\ &= \underline{\underline{\tfrac{2}{3}n(13n^2+36n+26)}};\end{aligned}$$

so, $\underline{\underline{a=13}}$, $\underline{\underline{b=36}}$, and $\underline{\underline{c=26}}$.

14. Given that (4)

$$\sum_{r=1}^{n} r(2r-1) = \tfrac{1}{6}n(n+1)(4n-1),$$

show that

$$\sum_{r=n+1}^{3n} r(2r-1) = \tfrac{1}{3}n(an^2+bn+c),$$

where a, b, and c are integers to be found.

Solution

$$\begin{aligned}\sum_{r=n+1}^{3n} r(2r-1) &= \sum_{r=1}^{3n} r(2r-1) - \sum_{r=1}^{n} r(2r-1)\\ &= \tfrac{1}{6}(3n)(3n+1)(12n-1) - \tfrac{1}{6}n(n+1)(4n-1)\\ &= \tfrac{1}{6}n[3(3n+1)(12n-1)-(n+1)(4n-1)]\\ &= \tfrac{1}{6}n[(108n^2+27n-3)-(4n^2+3n-1)]\\ &= \tfrac{1}{6}n(104n^2+24n-2)\\ &= \underline{\underline{\tfrac{1}{3}n(52n^2+12n-1)}};\end{aligned}$$

so, $\underline{\underline{a=52}}$, $\underline{\underline{b=12}}$, and $\underline{\underline{c=-1}}$.

15. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that (6)

$$\sum_{r=1}^{n}(2r-1)^2 = \tfrac{1}{3}n(4n^2-1),$$

for all positive integers n.

Solution

$$\begin{aligned}\sum_{r=1}^{n}(2r-1)^2 &= \sum_{r=1}^{n}(4r^2-4r+1)\\ &= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + \sum_{r=1}^{n}1\\ &= \tfrac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n\\ &= \tfrac{1}{3}n[2(n+1)(2n+1) - 6(n+1) + 3]\\ &= \tfrac{1}{3}n[(4n^2+6n+2) - (6n+6) + 3]\\ &= \underline{\underline{\tfrac{1}{3}n(4n^2-1)}}.\end{aligned}$$

(b) Hence show that (3)

$$\sum_{r=2n+1}^{4n}(2r-1)^2 = an(bn^2-1),$$

where a and b are integers to be found.

Solution

$$\begin{aligned}\sum_{r=2n+1}^{4n}(2r-1)^2 &= \sum_{r=1}^{4n}(2r-1)^2 - \sum_{r=1}^{2n}(2r-1)^2\\ &= \tfrac{1}{3}(4n)[4(4n)^2-1] - \tfrac{1}{3}(2n)[4(2n)^2-1]\\ &= \tfrac{1}{3}(256n^2-4) - \tfrac{1}{3}n(32n^2-2)\\ &= \tfrac{1}{3}n(224n^2-2)\\ &= \underline{\underline{\tfrac{2}{3}n(112n^2-1)}};\end{aligned}$$

so, $\underline{\underline{a=\tfrac{2}{3}}}$ and $\underline{\underline{b=112}}$.

16. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that (5)

$$\sum_{r=1}^{n} r(r^2-3) = \tfrac{1}{4}n(n+1)(n+3)(n-2).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n} r(r^2-3) &= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r \\ &= \tfrac{1}{4}n^2(n+1)^2 - \tfrac{3}{2}n(n+1) \\ &= \tfrac{1}{4}n(n+1)[n(n+1)-6] \\ &= \tfrac{1}{4}n(n+1)(n^2+n-6) \\ &= \underline{\underline{\tfrac{1}{4}n(n+1)(n+3)(n-2)}}.\end{aligned}$$

(b) Calculate the value of (3)

$$\sum_{r=10}^{50} r(r^2-3).$$

Solution

$$\begin{aligned}\sum_{r=10}^{50} r(r^2-3) &= \sum_{r=1}^{50} r(r^2-3) - \sum_{r=1}^{9} r(r^2-3) \\ &= \tfrac{1}{4}\times 50\times 51\times 53\times 48 - \tfrac{1}{4}\times 9\times 10\times 12\times 7 \\ &= 1\,621\,800 - 1\,890 \\ &= \underline{\underline{1\,619\,910}}.\end{aligned}$$

17. (a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that (5)

$$\sum_{r=1}^{n} (r+1)(r+4) = \tfrac{1}{3}n(n+4)(n+5),$$

for all positive integers n.

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r+1)(r+4) &= \sum_{r=1}^{n}(r^2+5r+4)\\ &= \sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 4\\ &= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{5}{2}n(n+1) + 4n\\ &= \tfrac{1}{6}n[(n+1)(2n+1) + 15(n+1) + 24]\\ &= \tfrac{1}{6}n[(2n^2+3n+1) + (15n+15) + 24]\\ &= \tfrac{1}{6}n(2n^2+18n+40)\\ &= \tfrac{1}{3}n(n^2+9n+20)\\ &= \underline{\underline{\tfrac{1}{3}n(n+4)(n+5)}}.\end{aligned}$$

(b) Hence show that (3)

$$\sum_{r=n+1}^{2n}(r+1)(r+4) = \tfrac{1}{3}n(n+1)(an+b),$$

where a and b are integers to be found.

Solution

$$\begin{aligned}\sum_{r=n+1}^{2n}(r+1)(r+4) &= \sum_{r=1}^{2n}(r+1)(r+4) - \sum_{r=1}^{n}(r+1)(r+4)\\ &= \tfrac{1}{3}(2n)(2n+4)(2n+5) - \tfrac{1}{3}n(n+4)(n+5)\\ &= \tfrac{1}{3}n(8n^2+36n+40) - \tfrac{1}{3}n(n^2+9n+20)\\ &= \tfrac{1}{3}n(7n^2+27n+20)\\ &= \underline{\underline{\tfrac{1}{3}n(n+1)(7n+20)}};\end{aligned}$$

so, $\underline{\underline{a=7}}$ and $\underline{\underline{b=20}}$.

18. (a) Using the formula for $\sum_{r=1}^{n} r^2$ write down, in terms of n only, an expression for (1)

$$\sum_{r=1}^{3n} r^2.$$

Solution

$$\sum_{r=1}^{3n} r^2 = \tfrac{1}{6}(3n)(3n+1)(6n+1)$$
$$= \underline{\underline{\tfrac{1}{2}n(3n+1)(6n+1)}}.$$

(b) Show that, for all integers n, where $n > 0$, (4)

$$\sum_{r=2n+1}^{3n} r^2 = \tfrac{1}{6}n(an^2+bn+c),$$

where the values of the constants a, b, and c are to be found.

Solution

$$\begin{aligned}\sum_{r=2n+1}^{3n} r^2 &= \sum_{r=1}^{3n} r^2 - \sum_{r=1}^{2n} r^2 \\ &= \tfrac{1}{2}n(3n+1)(6n+1) - \tfrac{1}{6}(2n)(2n+1)(4n+1) \\ &= \tfrac{1}{2}n(3n+1)(6n+1) - \tfrac{1}{3}n(2n+1)(4n+1) \\ &= \tfrac{1}{6}n[3(3n+1)(6n+1) - 2(2n+1)(4n+1)] \\ &= \tfrac{1}{6}n[(54n^2+27n+3) - (16n^2+12n+2)] \\ &= \underline{\underline{\tfrac{1}{6}n(38n^2+15n+1)}};\end{aligned}$$

so, $\underline{\underline{a = 38}}$, $\underline{\underline{b = 15}}$, and $\underline{\underline{c = 1}}$.

19. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that (5)

$$\sum_{r=1}^{n} (3r^2+8r+3) = \tfrac{1}{2}n(2n+5)(n+3),$$

or all positive integers n.

Solution

$$
\begin{aligned}
\sum_{r=1}^{n}(3r^2+8r+3) &= 3\sum_{r=1}^{n} r^2 + 8\sum_{r=1}^{n} r + \sum_{r=1}^{n} 3 \\
&= \tfrac{1}{2}n(n+1)(2n+1) + 4n(n+1) + 3n \\
&= \tfrac{1}{2}n[(n+1)(2n+1) + 8(n+1) + 6] \\
&= \tfrac{1}{2}n[(2n^2+3n+1) + (8n+8) + 6] \\
&= \tfrac{1}{2}n(2n^2+11n+15) \\
&= \underline{\underline{\tfrac{1}{2}n(2n+5)(n+3)}}.
\end{aligned}
$$

Given that

$$\sum_{r=1}^{12}[3r^2+8r+3+k(2^{r-1})] = 3\,520,$$

(b) find the exact value of the constant k. (4)

Solution

$$
\begin{aligned}
& \sum_{r=1}^{12}[3r^2+8r+3+k(2^{r-1})] = 3\,520 \\
\Rightarrow\quad & \sum_{r=1}^{12}(3r^2+8r+3) + k\sum_{r=1}^{12}(2^{r-1})] = 3\,520 \\
\Rightarrow\quad & \tfrac{1}{2}(12)(29)(15) + k(2^{12}-1) = 3\,520 \\
\Rightarrow\quad & 2\,610 + 4\,095k = 3\,520 \\
\Rightarrow\quad & 4\,095k = 910 \\
\Rightarrow\quad & \underline{\underline{k = \tfrac{2}{9}}}.
\end{aligned}
$$