Dr Oliver Mathematics
Method of Exhaustion

In this note, we will examine the method of exhaustion.

Example 1
Every cube number is either a multiple of 9, or is 1 more or 1 less than a multiple of 9.

Solution
We will take $3n$, $3n+1$, and $3n+2$ as our examples for a^3.

$$(3n)^3 = 27n^3 = 3(9n^3)$$

which is a multiple of 9;

$$\begin{aligned}(3n+1)^3 &= 27n^3 + 27n^2 + 9n + 1 \\ &= 9(3n^3 + n^2 + n) + 1,\end{aligned}$$

which is 1 more than a multiple of 9;

$$\begin{aligned}(3n+2)^3 &= 27n^3 + 54n^2 + 36n + 8 \\ &= 9(3n^3 + 6n^2 + 4n) + 8;\end{aligned}$$

which is 1 less than a multiple of 9.

Here are some examples for you to try.

1. Suppose a and b are even integers. Prove that the sum and difference of a and b are divisible by 2.

Solution

Hence $a = 2m$ and $b = 2n$ for some constants m and n. Then

$$\begin{aligned}a + b &= 2m + 2n \\ &= 2(m+n)\end{aligned}$$

and

$$\begin{aligned}a - b &= 2m - 2n \\ &= 2(m-n)\end{aligned}$$

which are both even.

2. If n is a positive integer then $(n^7 - n)$ is divisible by 7.

Solution

$$\begin{aligned} n^7 - n &= n(n^6 - 1) \\ &= n(n^3 + 1)(n^3 - 1) \\ &= n(n + 1)(n^2 - n + 1)(n - 1)(n^2 + n + 1). \end{aligned}$$

There are seven cases to consider: $n = 7q+r$ where $q \in \mathbb{N}$ and $r = 0, 1, 2, 3, 4, 5,$ and 6.

Case 1: $n = 7q$: $n^7 - n$ has factor n which is divisible by 7.

Case 2: $n = 7q + 1$: $n^7 - n$ has factor $n - 1 = 7q$ which is divisible by 7.

Case 31: $n = 7q + 2$: $n^7 - n$ has factor $n^2 + n + 1$ which is divisible by 7:

$$\begin{aligned} (7q + 2)^2 + (7q + 2) + 1 &= (49q^2 + 28n + 4) + (7q + 2) + 1 \\ &= 49q^2 + 35n + 7 \\ &= 7(7q^2 + 5n + 1). \end{aligned}$$

Case 4: $n = 7q + 3$: $n^7 - n$ has factor $n^2 - n + 1$ which is divisible by 7:

$$\begin{aligned} (7q + 3)^2 - (7q + 3) + 1 &= (49q^2 + 42n + 9) - (7q + 3) + 1 \\ &= 49q^2 + 35n + 7 \\ &= 7(7q^2 + 5n + 1). \end{aligned}$$

Case 5: $n = 7q + 4$: $n^7 - n$ has factor $n^2 + n + 1$ which is divisible by 7:

$$\begin{aligned} (7q + 4)^2 + (7q + 4) + 1 &= (49q^2 + 56n + 16) + (7q + 4) + 1 \\ &= 49q^2 + 63n + 21 \\ &= 7(7q^2 + 9n + 3). \end{aligned}$$

Case 6: $n = 7q + 5$: $n^7 - n$ has factor $n^2 - n + 1$ which is divisible by 7:

$$\begin{aligned} (7q + 5)^2 - (7q + 5) + 1 &= (49q^2 + 70n + 25) - (7q + 5) + 1 \\ &= 49q^2 + 63n + 21 \\ &= 7(7q^2 + 9n + 3). \end{aligned}$$

Case 7: $n = 7q + 6$: $n^7 - n$ has factor $n + 1 = 7q + 7$ which is divisible by 7.