Dr Oliver Mathematics
Further Mathematics
Eigenvalues, Eigenvectors, and 3×3 Determinants
Past Examination Questions

This booklet consists of 24 questions across a variety of examination topics.
The total number of marks available is 236.

1. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix},$$

where p, a, b, and c are constants and $a > 0$. Given that $\mathbf{M}\mathbf{M}^{\mathrm{T}} = k\mathbf{I}$ for some constant k, find

(a) the value of p, (2)

(b) the value of k, (2)

(c) the values of a, b, and c, (6)

(d) $|\det \mathbf{M}|$. (2)

2. The transformation R is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(a) Find the eigenvectors of $\mathbf{A}$. (5)

(b) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that (5)

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

(c) Hence describe the transformation R as a combination of geometrical transformations, stating clearly their order. (4)

3.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & k \end{pmatrix}.$$

(a) Show that $\det \mathbf{A} = 20 - 4k$. (4)

(b) Find $\mathbf{A}^{-1}$. (6)

Given that $k = 3$ and that

$$\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

is an eigenvector of $\mathbf{A}$,

(c) find the corresponding eigenvalue. (2)

Given that the only other distinct eigenvalue of $\mathbf{A}$ is 8,

(d) find a corresponding eigenvector. (4)

4. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} k & 2 \\ 2 & -1 \end{pmatrix},$$

where k is a constant. For the case $k = -4$,

(a) find the image under T of the line with equation $y = 2x + 1$. (2)

For the case $k = 2$, find

(b) the two eigenvalues of $\mathbf{A}$, (4)

(c) a cartesian equation for each of the two lines passing through the origin which are invariant under T. (3)

5.

$$\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find values of k for which $\mathbf{A}$ is singular. (4)

Given that $\mathbf{A}$ is non-singular,

(b) find, in terms of k, $\mathbf{A}^{-1}$. (5)

6. (5)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove by induction, that for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

7. The eigenvalues of the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix},$$

and λ_1 and λ_2, where $\lambda_1 < \lambda_2$.

(a) Find the value of λ_1 and the value of λ_2. (3)

(b) Find $\mathbf{M}^{-1}$. (2)

(c) Verify that the eigenvalues of $\mathbf{M}^{-1}$ and λ_1^{-1} and λ_2^{-1}. (3)

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{M}$. There are two lines, passing through the origin, each of which is mapped onto itself under the transformation T.

(d) Find cartesian equations for each of these lines. (4)

8. Given that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix},$$

(a) find the eigenvalue of $\mathbf{A}$ corresponding to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, (2)

(b) find the value of p and the value of q. (4)

The image of the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ when transformed by $\mathbf{A}$ is $\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$.

(c) Using the values of p and q from part (b), find the values of the constants l, m, and n. (4)

9.

$$\mathbf{M} = \begin{pmatrix} 1 & p & 2 \\ 0 & 3 & q \\ 2 & p & 1 \end{pmatrix},$$

where p and q are constants. Given that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$,

(a) show that $q = 4p$. (3)

Given also that $\lambda = 5$ is an eigenvalue of $\mathbf{M}$, and $p < 0$ and $q < 0$, find

(b) the values of p and q, (4)

(c) an eigenvector corresponding to the eigenvalue $\lambda = 5$. (3)

10.
$$\mathbf{A} = \begin{pmatrix} k & -2 \\ 1-k & k \end{pmatrix},$$

where k is constant. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{A}$.

(a) Find the value of k for which the line $y = 2x$ is mapped onto itself under T. (3)

(b) Show that $\mathbf{A}$ is non-singular for all values of k. (3)

(c) Find $\mathbf{A}^{-1}$ in terms of k. (2)

A point P is mapped onto a point Q under T. The point Q has position vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ relative to an origin O. Given that $k = 3$,

(d) find the position vector of P. (3)

11.
$$\mathbf{M} = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}.$$

(a) Show that 7 is an eigenvalue of the matrix $\mathbf{M}$ and find the other two eigenvalues of $\mathbf{M}$. (5)

(b) Find an eigenvector corresponding to the eigenvalue 7. (4)

12. For $n \in \mathbb{Z}^+$, show, using mathematical induction, that (5)

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ n(n+2) & 2n & 1 \end{pmatrix}.$$

13.
$$\mathbf{M} = \begin{pmatrix} 11 & -5\sqrt{3} \\ -5\sqrt{3} & 1 \end{pmatrix}.$$

Given that λ_1 and λ_2 are the eigenvalues of $\mathbf{M}$ and $\lambda_1 > \lambda_2$,

(a) show that $\lambda_1 = 16$ and find the value of λ_2. (4)

(b) Find eigenvectors corresponding to the eigenvalues λ_1 and λ_2. (4)

Given that there is an orthogonal matrix $\mathbf{M}$ such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ is the diagonal matrix $\mathbf{D}$, where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

(c) find the matrix $\mathbf{P}$, (2)

(d) verify that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P} = \mathbf{D}$. (4)

14.
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix},$$

where k is a constant. Given that $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$,

(a) find the eigenvalue of $\mathbf{M}$ corresponding to $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$, (2)

(b) show that $k = 3$, (2)

(c) show that $\mathbf{M}$ has exactly two eigenvalues. (4)

15. The matrix $\mathbf{M}$ is given by
$$\mathbf{M} = \begin{pmatrix} k & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & -1 \end{pmatrix},$$

where $k \neq 1$.

(a) Show that $\det \mathbf{M} = 2 - 2k$. (2)

(b) Find $\mathbf{M}^{-1}$, in terms of k. (5)

16. The matrix $\mathbf{M}$ is given by
$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

(a) Show that 4 is an eigenvalue of $\mathbf{M}$, and find the other two eigenvalues. (5)

(b) For the eigenvalue 4, find a corresponding eigenvector. (3)

17. The matrix $\mathbf{M}$ is given by
$$\mathbf{M} = \begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix},$$

where a, b, and c are constants.

(a) Given that $\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{k}$ are two of the eigenvectors of $\mathbf{M}$, find (8)

  (i) the values of a, b, and c,

  (ii) the eigenvalues which correspond to the two given eigenvectors.

The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & d \\ -1 & 0 & 1 \end{pmatrix},$$

where d is constant and $d \neq 1$. Find

(b) (i) the determinant of $\mathbf{P}$ in terms of d, (5)

(ii) the matrix $\mathbf{P}^{-1}$ in terms of d.

18. It is given that $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 3 \\ 2 & b & 0 \\ a & 1 & 8 \end{pmatrix},$$

and a and b are constants.

(a) Find the eigenvalue of $\mathbf{A}$ corresponding to the eigenvector $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. (3)

(b) Find the values of a and b. (3)

(c) Find the other eigenvalues of $\mathbf{A}$. (5)

19.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 1 \end{pmatrix}.$$

(a) Show that matrix $\mathbf{M}$ is not orthogonal. (2)

(b) Using algebra, show that 1 is an eigenvalue of M and find the other two eigenvalues of $\mathbf{M}$. (5)

(c) Find an eigenvector of $\mathbf{M}$ which corresponds to the eigenvalue 1. (2)

20. The symmetric matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ with eigenvalues 5, 2, and -1 respectively.

(a) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that (4)

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}.$$

Given that $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$,

(b) show that (2)

$$\mathbf{M} = \mathbf{PDP}^{-1}.$$

(c) Hence find the matrix $\mathbf{M}$. (5)

21.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of $\mathbf{A}$. (5)

(b) Find a normalised eigenvector for each of the eigenvalues of $\mathbf{A}$. (5)

(c) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$. (2)

22. (4)

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix},$$

where k is a constant. Given that the matrix $\mathbf{A}$ is singular, find the possible values of k.

23.

$$\mathbf{M} = \begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix},$$

where p and q are constants. Given that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{M}$,

(a) find the eigenvalue corresponding to this eigenvector, (3)

(b) find the value of p and the value of q. (3)

Given that 6 is another eigenvalue of $\mathbf{M}$,

(c) find a corresponding eigenvector. (2)

Given that $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is a third eigenvector of $\mathbf{M}$ with eigenvalue 3,

(d) find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that (3)

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}.$$

24. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}, \; k \in \mathbb{R}, \; k \neq \tfrac{1}{2}.$$

(a) Show that $\det \mathbf{M} = 1 - 2k$. (2)

(b) Find $\mathbf{M}^{-1}$ in terms of k. (4)