Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2010 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Differentiate the following, simplifying your answers as appropriate.

   (a) $\mathrm{f}(x) = \mathrm{e}^{2x} \tan x$, $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. (3)

Solution

$$u = \mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^{2x}$$
$$v = \tan x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$$

$$\begin{aligned} \mathrm{f}(x) = \mathrm{e}^{2x} \tan x &\Rightarrow \mathrm{f}'(x) = (\mathrm{e}^{2x})(\sec^2 x) + (2\mathrm{e}^{2x})(\tan x) \\ &\Rightarrow \underline{\underline{\mathrm{f}'(x) = \mathrm{e}^{2x}(\sec^2 x + 2\tan x)}}. \end{aligned}$$

   (b) $\mathrm{g}(x) = \dfrac{\cos 2x}{x^3}$. (4)

Solution

$$u = \cos 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2\sin 2x$$
$$v = x^3 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 3x^2$$

$$\begin{aligned} \mathrm{g}(x) = \frac{\cos 2x}{x^3} &\Rightarrow \mathrm{g}'(x) = \frac{(x^3)(-2\sin 2x) - (\cos 2x)(3x^2)}{(x^3)^2} \\ &\Rightarrow \mathrm{g}'(x) = \frac{x^2(-3\cos 2x - 2x\sin 2x)}{x^6} \\ &\Rightarrow \underline{\underline{\mathrm{g}'(x) = \frac{-3\cos 2x - 2x\sin 2x}{x^4}}}. \end{aligned}$$

2. Find the term in a^6 in the binomial expansion of (4)

$$\left(\frac{1}{a} + 3a\right)^{10}.$$

Solution

The general term is

$$\begin{aligned}\binom{10}{r}\left(\frac{1}{a}\right)^r (3a)^{10-r} &= \binom{10}{r}(a^{-r})(3^{10-r})(a^{10-r}) \\ &= \binom{10}{r}(3^{10-r})(a^{10-2r}).\end{aligned}$$

Now,

$$\begin{aligned}10 - 2r = 6 &\Rightarrow 2r = 4 \\ &\Rightarrow r = 2\end{aligned}$$

and the term in a^6 is

$$\binom{10}{2}(3^8) = \underline{\underline{295\,245}}.$$

3. (a) Express (3)

$$\frac{3x}{(x+1)^2}$$

in partial fractions.

Solution

$$\begin{aligned}\frac{3x}{(x+1)^2} &\equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \\ &\equiv \frac{A(x+1) + B}{(x+1)^2}\end{aligned}$$

and so

$$3x \equiv A(x+1) + B.$$

$\underline{x = -1}$: $-3 = B$
$\underline{x = 0}$: $0 = A - 3 \Rightarrow A = 3$
Hence,

$$\frac{3x}{(x+1)^2} \equiv \underline{\underline{\frac{3}{(x+1)} - \frac{3}{(x+1)^2}}}.$$

(b) Hence obtain (2)

$$\int \frac{3x}{(x+1)^2}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int \frac{3x}{(x+1)^2}\,\mathrm{d}x &= \int \left(\frac{3}{(x+1)} - \frac{3}{(x+1)^2}\right)\mathrm{d}x \\ &= \underline{\underline{3\ln|x+1| + 3(x+1)^{-1} + c}}.\end{aligned}$$

4. An industrial process is modelled by the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{9t\mathrm{e}^{3t}}{y}$$

where $y > 0$ and $t \geqslant 0$.

Given that $y = 2$ when $t = 0$, find y explicitly in terms of t.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{9t\mathrm{e}^{3t}}{y} &\Rightarrow y\,\mathrm{d}y = 9t\mathrm{e}^{3t}\,\mathrm{d}t \\ &\Rightarrow \int y\,\mathrm{d}y = \int 9t\mathrm{e}^{3t}\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}u = 9t &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 9 \\ \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3t} &\Rightarrow v = \tfrac{1}{3}\mathrm{e}^{3t}\end{aligned}$$

$$\begin{aligned}&\Rightarrow \tfrac{1}{2}y^2 = 3t\mathrm{e}^{3t} - \int 3\mathrm{e}^{3t}\,\mathrm{d}t \\ &\Rightarrow \tfrac{1}{2}y^2 = 3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + c.\end{aligned}$$

Now,

$$t = 0, y = 2 \Rightarrow 2 = 0 - 1 + c \Rightarrow c = 3$$

and so

$$\tfrac{1}{2}y^2 = 3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + 3 \Rightarrow y^2 = 2(3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + 3)$$
$$\Rightarrow \underline{\underline{y = \sqrt{2(3t\mathrm{e}^{3t} - \mathrm{e}^{3t} + 3)}}}.$$

5. (a) Find the value(s) of m for which the matrix (3)

$$\begin{pmatrix} m & 1 & 1 \\ 0 & m & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

is singular.

Solution

$$\det \begin{pmatrix} m & 1 & 1 \\ 0 & m & -2 \\ 1 & 0 & 1 \end{pmatrix} = 0 \Rightarrow m(m-0) - 1(0+2) + 1(0-m) = 0$$
$$\Rightarrow m^2 - m - 2 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & -2 \end{array}\right\} -2, +1$$

$$\Rightarrow (m-2)(m+1) = 0$$
$$\Rightarrow m - 2 = 0 \text{ or } m + 1 = 0$$
$$\Rightarrow \underline{\underline{m = 2 \text{ or } m = -1}}.$$

The matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix}.$$

(b) Use elementary row operations to obtain $\mathbf{B}^{-1}$. (4)

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{array}\right)$$

Do $R_3 - R_1$:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right)$$

Do $R_3 + R_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

Do $R_1 + R_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

Do $R_1 - 2R_3$ and $R_2 + R_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

Do $-R_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right)$$

Hence,

$$\mathbf{B}^{-1} = \underline{\underline{\begin{pmatrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}}}.$$

(c) Hence, or otherwise, solve the system of equations (2)

$$\begin{aligned} x + y - z &= 3 \\ y + z &= -2 \\ x - 3z &= 7. \end{aligned}$$

Solution

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix};$$

hence,

$$\underline{\underline{x = 1,\ y = 0,\ z = -2}}.$$