Dr Oliver Mathematics
Applied Mathematics: Parametric Equations

The total number of marks available is 28.
You must write down all the stages in your working.

1. A curve is defined by the parametric equations

$$x = 5t^2 - 5 \text{ and } y = 3t^3.$$

   (a) Find the value of t corresponding to the point $(0, -3)$. (2)

Solution

Select y (why?):

$$\begin{aligned} 3t^3 = -3 &\Rightarrow t^3 = -1 \\ &\Rightarrow \underline{\underline{t = -1}}. \end{aligned}$$

   (b) Calculate the gradient of the curve at this point. (3)

Solution

$$\begin{aligned} x = 5t^2 - 5 &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 10t \\ y = 3t^3 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 9t^2. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{9t^2}{10t} \\ &= \tfrac{9}{10}t. \end{aligned}$$

Finally,

$$t = -1 \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{9}{10}}}.$$

2. A curve is defined parametrically by (5)

$$x = \frac{t}{t^2 + 1} \text{ and } y = \frac{t - 1}{t^2 + 1}.$$

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of t.

Solution

$$x = \frac{t}{t^2+1} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(t^2+1)\cdot 1 - t\cdot(2t)}{(t^2+1)^2}$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t^2+1-2t^2}{(t^2+1)^2}$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1-t^2}{(t^2+1)^2}$$

and

$$y = \frac{t-1}{t^2+1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(t^2+1)\cdot 1 - (t-1)\cdot(2t)}{(t^2+1)^2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t^2+1-2t^2+2t}{(t^2+1)^2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1+2t-t^2}{(t^2+1)^2}.$$

Finally,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{1+2t-t^2}{(t^2+1)^2}}{\frac{1-t^2}{(t^2+1)^2}} = \underline{\underline{\frac{1+2t-t^2}{1-t^2}}}.$$

3. A particle moves along a curve in the x-y plane. The curve is defined by the parametric equations

$$x = t^2+1,\ y = 1-3t^3,$$

where t is the time elapsed since the start.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. (3)

Solution

$$x = t^2 + 1 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 2t$$
$$y = 1 - 3t^3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -9t^2.$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{-9t^2}{2t} \\ &= \underline{\underline{-\tfrac{9}{2}t}}.\end{aligned}$$

(b) Hence obtain an equation of the tangent to the curve when $t = 2$. (2)

Solution

Well,

$$t = 2 \Rightarrow x = 5, y = -23, \frac{\mathrm{d}y}{\mathrm{d}x} = -9$$

and an equation of the tangent is

$$\begin{aligned}y + 23 = -9(x - 5) &\Rightarrow y + 23 = -9x + 45 \\ &\Rightarrow \underline{\underline{y = -9x + 22}}.\end{aligned}$$

4. A curve is defined by the equations (4)

$$x = 5\cos t \text{ and } y = 3\sin t,\ 0 \leqslant t < 2\pi.$$

Find the gradient of the curve when $t = \frac{1}{6}\pi$.

Solution

$$x = 5\cos t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = -5\sin t$$
$$y = 3\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 3\cos t.$$

Now,

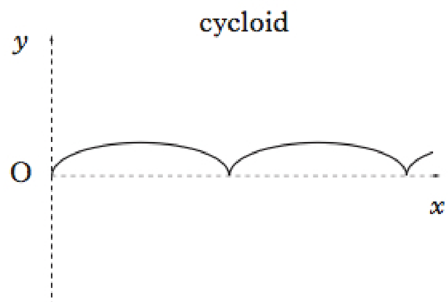
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{3\cos t}{-5\sin t}$$

and

$$t = \tfrac{1}{6}\pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\underline{-\tfrac{3}{5}\sqrt{3}}}.$$

5. The cycloid curve below is defined by the parametric equations

$$x = t - \sin t,\ y = 1 - \cos t.$$



(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. (2)

Solution

$$x = t - \sin t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \cos t$$
$$y = 1 - \cos t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \sin t$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \underline{\underline{\frac{\sin t}{1 - \cos t}}}.$$

(b) Show that the value of $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is always negative, in the case where $0 < t < 2\pi$. (5)

Solution

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin t}{1-\cos t}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sin t}{1-\cos t}\right) \times \frac{\mathrm{d}t}{\mathrm{d}x}\end{aligned}$$

$$\begin{aligned}u = \sin t &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = \cos t\\ v = 1-\cos t &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} = \sin t\end{aligned}$$

$$\begin{aligned}&= \frac{(1-\cos t)(\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2} \times \frac{1}{1-\cos t}\\ &= \frac{\cos t - \cos^2 - \sin^2 t}{(1-\cos t)^3}\\ &= \frac{\cos t - 1}{(1-\cos t)^3}\\ &= \frac{-(1-\cos t)}{(1-\cos t)^3}\\ &= -\frac{1}{(1-\cos t)^2}\end{aligned}$$

and this is negative except when

$$\cos t = 1 \Rightarrow t = 0,\ 2\pi.$$

A particle follows the path of the cycloid where t is the time elapsed since the particle's motion commenced.

(c) Calculate the speed of the particle when $t = \frac{1}{3}\pi$. (2)

Solution

$$
\begin{aligned}
\text{Speed} &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \\
&= \sqrt{\left(1 - \cos\tfrac{1}{3}\pi\right)^2 + \left(\sin\tfrac{1}{3}\pi\right)^2} \\
&= \sqrt{\left(1 - \cos\tfrac{1}{3}\pi\right)^2 + \left(\sin\tfrac{1}{3}\pi\right)^2} \\
&= \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} \\
&= \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} \\
&= \underline{\underline{1}}.
\end{aligned}
$$