Dr Oliver Mathematics
Completing the Square

There are a number of reasons for writing a quadratic in completed square form:

(a) it will give us another way of solving a quadratic equation,

(b) it will allow us to find the the maximum or minimum value (as appropriate) of a quadratic and where this value occurs, and

(c) it will help us to sketch the graph of quadratic functions.

The idea is that we will take a quadratic expression and work out how much we need to add or subtract to that expression in order to write the expression using the square of a bracket. The technique has its origins in a time when numbers were, in street language, associated with the geometrical concepts of lengths, areas, and volumes.

1. (a) Express $x^2 + 6x$ in completed square form.

Solution

Mathematicians would initially have conceived of this problem as involving adding the areas of two shapes, one a square of side x and the other a rectangle of dimensions $x \times 6$, as shown in Figure 1.

Figure 1: Classical approach to completing the square, stage one

We now divide the rectangle in half to make two rectangles of dimension $x \times 3$, as shown in Figure 2.

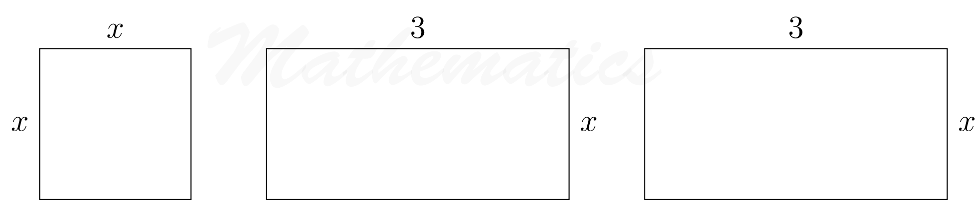


Figure 2: Classical approach to completing the square, stage two

We now put the three pieces together, as shown in Figure 3.

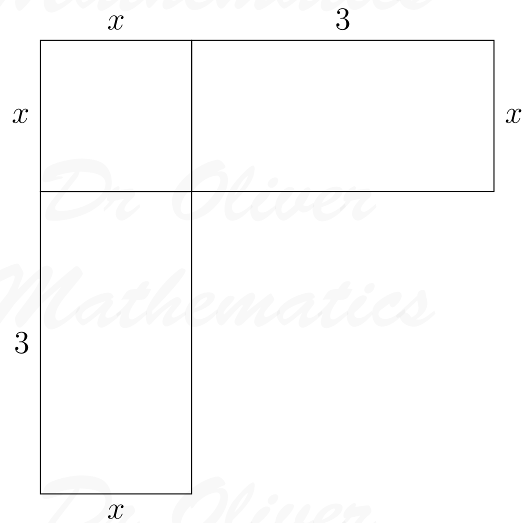


Figure 3: Classical approach to completing the square, stage three

Clearly, there is something that can be added to the bottom right-hand corner of this shape to make it a square; in fact, if we add a square of side 3 then we are able to make a big square of side $(x + 3)$, as shown in Figure 4

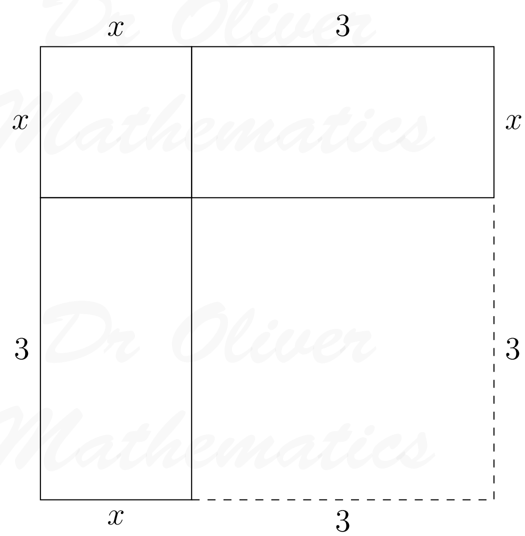


Figure 4: Classical approach to completing the square, stage four

So, we began with a square of side x and a rectangle of dimensions $x \times 6$, added on a square of side 3, and the result is a square of side $(x+3)$. Algebraically,

$$x^2 + 6x + 9 \equiv (x+3)^2$$

and hence

$$x^2 + 6x \equiv \underline{\underline{(x+3)^2 - 9}}.$$

(b) Does the graph have an absolute maximum or an absolute minimum?

Solution

It has an absolute minimum.

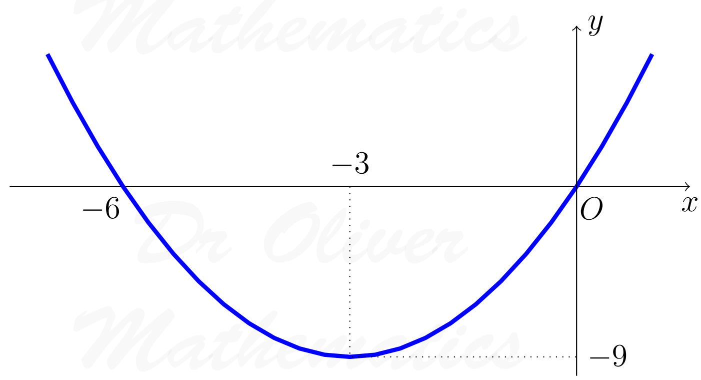
(c) Find the coordinates of the vertex.

Solution

$\underline{\underline{(-3, -9)}}$.

(d) Sketch the graph.

Solution



The idea of "halving and squaring" survives and although we no longer associate the quantities involved with geometrical shapes an echo of the approach shown above can be found in the more modern approach that uses a grid to present the working.

2. (a) Express $x^2 - 8x$ in completed square form.

Solution

The coefficient of x is -8 and so we divide that by two and we get the following grid.

	x	-4
x	x^2	$-4x$
-4	$-4x$	$+16$

This grid then tells us that

$$x^2 - 8x + 16 \equiv (x - 4)^2.$$

The original problem, however, involved just $x^2 - 8x$ and so, by subtracting 16 from each side, we get

$$x^2 - 8x \equiv \underline{\underline{(x - 4)^2 - 16}}.$$

(b) Does the graph have an absolute maximum or an absolute minimum?

Solution

It has an $\underline{\underline{\text{absolute minimum}}}$.

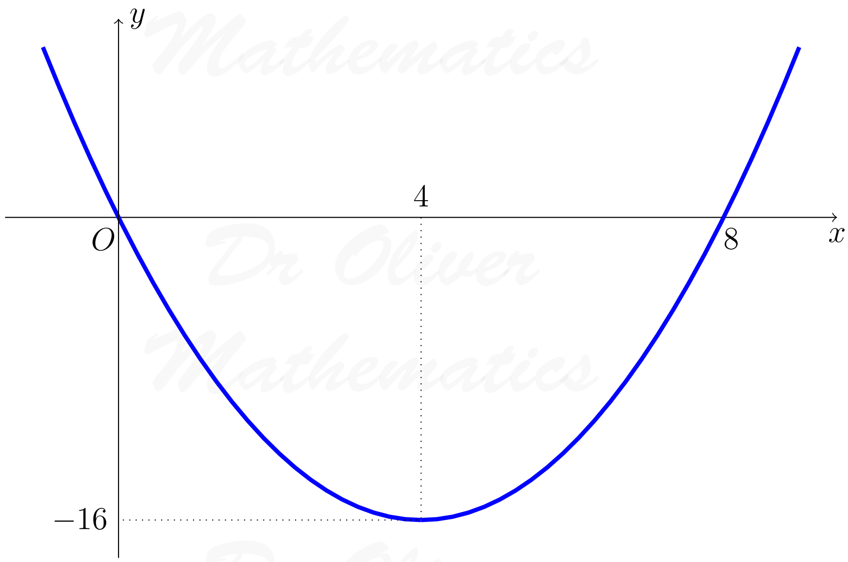
(c) Find the coordinates of the vertex.

Solution

$\underline{\underline{(4, -16)}}$.

(d) Sketch the graph.

Solution



3. (a) Express $x^2 + 10x + 17$ in completed square form.

Solution

The coefficient of x is $+10$ and so we divide that by two and we get the following grid.

	x	$+5$
x	x^2	$+5x$
$+5$	$+5x$	$+25$

This grid then tells us that

$$x^2 + 10x + 25 \equiv (x + 5)^2.$$

The original problem, however, involved $x^2 + 10x + 17$ and so, by subtracting 8 from each side, we get

$$x^2 + 10x + 17 \equiv \underline{\underline{(x + 5)^2 - 8}}.$$

(b) Does the graph have an absolute maximum or an absolute minimum?

Solution

It has an absolute minimum.

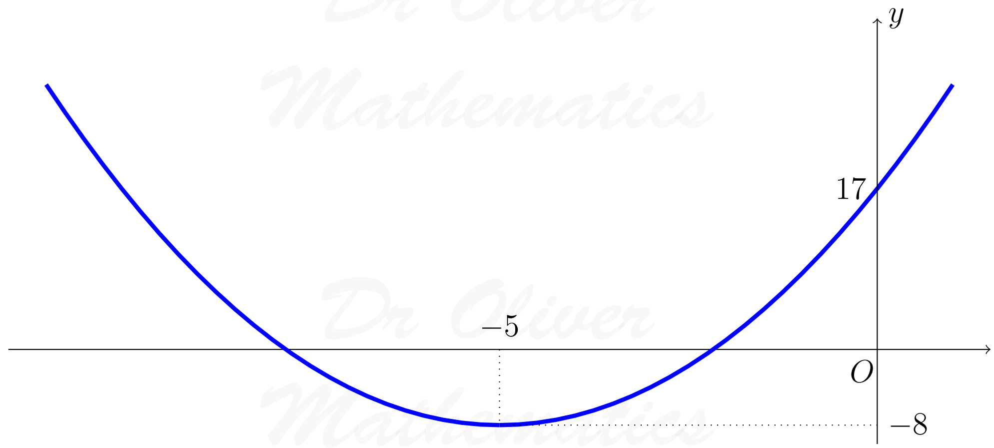
(c) Find the coordinates of the vertex.

Solution

$(-5, -8)$.

(d) Sketch the graph.

Solution



4. (a) Express $2x^2 + 7x - 3$ in completed square form.

Solution

Here, we begin by taking out the coefficient of x^2 as a factor of the terms involving x:

$$2x^2 + 7x - 3 \equiv 2\left[x^2 + \tfrac{7}{2}x\right] - 3.$$

We now proceed to complete the square on what is inside the square brackets, namely $x^2 + \frac{7}{2}x$. The coefficient of x is $+\frac{7}{2}$ and so we divide that by two and we get the following grid.

	x	$+\frac{7}{4}$
x	x^2	$+\frac{7}{4}x$
$+\frac{7}{4}$	$+\frac{7}{4}x$	$+\frac{49}{16}$

This grid then tells us that

$$x^2 + \tfrac{7}{2}x + \tfrac{49}{16} \equiv \left(x + \tfrac{7}{4}\right)^2$$

and hence

$$x^2 + \tfrac{7}{2}x \equiv \left(x + \tfrac{7}{4}\right)^2 - \tfrac{49}{16}.$$

Hence

$$\begin{aligned} 2x^2 + 7x - 3 &\equiv 2\left[x^2 + \tfrac{7}{2}x\right] - 3 \\ &\equiv 2\left[\left(x + \tfrac{7}{4}\right)^2 - \tfrac{49}{16}\right] - 3 \\ &\equiv 2\left(x + \tfrac{7}{4}\right)^2 - \tfrac{49}{8} - 3 \\ &\equiv \underline{\underline{2\left(x + \tfrac{7}{4}\right)^2 - \tfrac{73}{8}}}. \end{aligned}$$

You could, of course, work with mixed numbers rather than improper fractions but you may find it easier to square improper fractions than to square mixed numbers.

(b) Does the graph have an absolute maximum or an absolute minimum?

Solution

It has an absolute minimum.

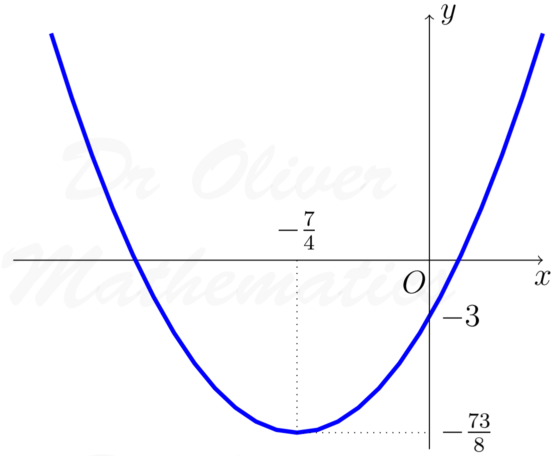
(c) Find the coordinates of the vertex.

Solution

$\underline{\underline{\left(-\frac{7}{4}, -\frac{73}{8}\right)}}$.

(d) Sketch the graph.

Solution



And, of course, the approach still works even if the coefficient of x^2 is negative.

5. (a) Express $4 + 8x - 3x^2$ in completed square form.

Solution

Here, we begin by taking out the coefficient of x^2 as a factor of the terms involving x:

$$4 + 8x - 3x^2 \equiv 4 - 3\left[x^2 - \tfrac{8}{3}x\right].$$

We now proceed to complete the square on what is inside the square brackets, namely $x^2 - \frac{8}{3}x$. The coefficient of x is $-\frac{8}{3}$ and so we divide that by two and we get the following grid.

	x	$-\frac{4}{3}$
x	x^2	$-\frac{4}{3}x$
$-\frac{4}{3}$	$-\frac{4}{3}x$	$+\frac{16}{9}$

This grid then tells us that

$$x^2 - \tfrac{8}{3}x + \tfrac{16}{9} \equiv \left(x - \tfrac{4}{3}\right)^2$$

and hence

$$x^2 - \tfrac{8}{3}x \equiv \left(x - \tfrac{4}{3}\right)^2 - \tfrac{16}{9}.$$

Hence

$$\begin{aligned} 4 + 8x - 3x^2 &\equiv 4 - 3\left[x^2 - \tfrac{8}{3}x\right] \\ &\equiv 4 - 3\left[\left(x - \tfrac{4}{3}\right)^2 - \tfrac{16}{9}\right] \\ &\equiv 4 - 3\left(x - \tfrac{4}{3}\right)^2 + \tfrac{16}{3} \\ &\equiv \underline{\underline{\tfrac{28}{3} - 3\left(x - \tfrac{4}{3}\right)^2}}. \end{aligned}$$

(b) Does the graph have an absolute maximum or an absolute minimum?

Solution

It has an absolute maximum.

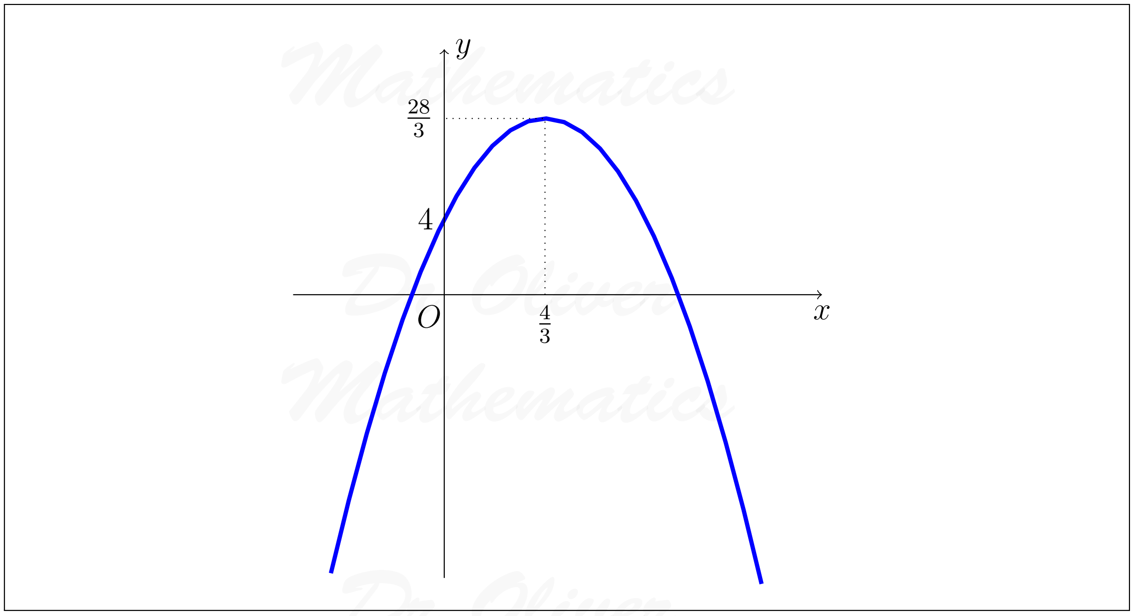
(c) Find the coordinates of the vertex.

Solution

$\underline{\underline{\left(\frac{4}{3}, \frac{28}{3}\right)}}$.

(d) Sketch the graph.

Solution


y
28/3
4
O
4/3
x