Dr Oliver Mathematics
Further Mathematics
Complex Numbers: de Moivre's Theorem
Past Examination Questions

This booklet consists of 12 questions across a variety of examination topics.
The total number of marks available is 130.

Let $s = \sin\theta$, $c = \cos\theta$, and $t = \tan\theta$. Then

$$\begin{aligned}(1 + \mathrm{i}t)^n &\equiv \left[\tfrac{1}{c}(c + \mathrm{i}s)\right]^n \\ &\equiv \frac{1}{c^n}(c + \mathrm{i}s)^n \\ &\equiv \frac{1}{c^n}\left[(\text{real part}) + \mathrm{i}(\text{imaginary part})\right]\end{aligned}$$

and we have

$$\tan n\theta \equiv \frac{\frac{1}{c^n}(\text{imaginary part})}{\frac{1}{c^n}(\text{real part})} \equiv \frac{\text{imaginary part}}{\text{real part}}.$$

For example, use de Moivre's theorem to show that

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

Now,

$$\begin{aligned}(1 + \mathrm{i}t)^4 &\equiv 1 + 4(\mathrm{i}t) + 6(\mathrm{i}t)^2 + 4(\mathrm{i}t)^3 + (\mathrm{i}t)^4 \\ &\equiv 1 - 6t^2 + t^4 + \mathrm{i}(4t - 4t^3)\end{aligned}$$

and we have

$$\tan 4\theta \equiv \frac{\text{imaginary part}}{\text{real part}} \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

1. (a) Given that $z = \mathrm{e}^{\mathrm{i}\theta}$, show that (2)

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

where n is a positive integer.

(b) Show that (5)

$$\sin^5\theta = \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

(c) Hence solve, in the interval $0 \leqslant \theta < 2\pi$, (5)

$$\sin 5\theta - 5\sin 3\theta + 6\sin\theta = 0.$$

2. (a) Use de Moivre's theorem to show that (5)

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

(b) Hence, or otherwise, solve, for $0 \leqslant \theta < \pi$, (6)

$$\sin 5\theta + \cos\theta\sin 2\theta = 0.$$

3. (a) Use de Moivre's theorem to show that (2)

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

where n is a positive integer.

(b) Express $32\cos^6\theta$ in the form $p\cos 6\theta + q\cos 4\theta + r\cos 2\theta + s$, where p, q, r, s are integers. (5)

(c) Hence find the exact value of (4)

$$\int_0^{\frac{\pi}{3}} \cos^6\theta\,\mathrm{d}\theta.$$

4. De Moivre's theorem states that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta,\ \text{for } n \in \mathbb{R}.$$

(a) Use induction to prove de Moivre's theorem for $n \in \mathbb{Z}^+$. (5)

(b) Show that (5)

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(c) Hence show that $2\cos\frac{\pi}{10}$ is a root of the equation (3)

$$x^4 - 5x^2 + 5 = 0.$$

5. (a) Use de Moivre's theorem to show that (5)

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^6\theta - 1.$$

(b) Hence, or otherwise, solve the equation $\cos 6\theta = \cos 2\theta$, $0 \leqslant \theta \leqslant \pi$. (6)

6. (a) Use de Moivre's theorem to show that (5)

$$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(b) Hence find the two positive solutions of (6)

$$32x^5 - 40x^3 + 10x + 1 = 0,$$

giving your answers to 3 decimal places.

7. (a) Use de Moivre's theorem to show that (5)

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Hence, given that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$,

(b) find all the solutions of (6)

$$\sin 5\theta = 5\sin 3\theta,$$

in the interval $0 \leqslant \theta < 2\pi$. Give your answers to 3 decimal places.

8. Given than (5)

$$z = r(\cos\theta + \mathrm{i}\sin\theta), r \in \mathbb{R},$$

prove, by induction, that

$$z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta), n \in \mathbb{Z}^+.$$

9. The complex number $z = e^{i\theta}$, where θ is real.

(a) Use de Moivre's theorem to show that (2)

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

where n is a positive integer.

(b) Show that (5)

$$\cos^5\theta \equiv \tfrac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right).$$

(c) Hence find all of the solutions of (4)

$$\cos 5\theta + 5\cos 3\theta + 12\cos\theta = 0$$

in the interval $0 \leqslant \theta < 2\pi$.

10. (a) Use de Moivre's theorem to show that (5)

$$\cos 5\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

(b) Hence solve (5)

$$64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 3 = 0, 0 \leqslant \theta \leqslant \tfrac{\pi}{2},$$

giving your answers as exact multiples of π.

11. (a) Use de Moivre's theorem to show that (5)

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

(b) Hence find the five distinct solutions of the equation (5)

$$16x^5 - 20x^3 + 5x + \tfrac{1}{2} = 0,$$

giving your answers to 3 decimal places where necessary.

(c) Use the identity given in (a) to find (4)

$$\int_0^{\frac{\pi}{4}} (4\sin^5\theta - 5\sin^3\theta)\,\mathrm{d}\theta,$$

expressing your answer in the form $a\sqrt{2} + b$, where a and b are rational numbers.

12. (a) Use de Moivre's theorem to show that (5)

$$\sin^5\theta \equiv a\sin 5\theta + b\sin 3\theta + c\sin\theta,$$

where a, b, and c are constants to be found.

(b) Hence show that (5)

$$\int_0^{\frac{\pi}{3}} \sin^5\theta\,\mathrm{d}\theta = \tfrac{53}{480}.$$