Dr Oliver Mathematics
Advance Level Mathematics
Core Mathematics 2: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. Figure 1 shows a sketch of part of the curve with equation

$$y = \frac{(x+2)^{\frac{3}{2}}}{4}, \ x \geqslant -2.$$

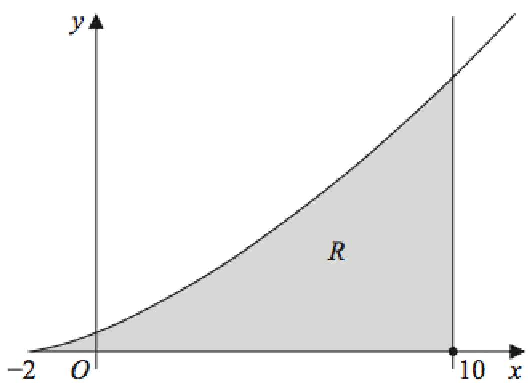

Figure 1: $y = \dfrac{(x+2)^{\frac{3}{2}}}{4}$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line with equation $x = 10$.

The table below shows corresponding values of x and y for $y = \dfrac{(x+2)^{\frac{3}{2}}}{4}$.

(a) Complete the table, giving values of y corresponding to $x = 2$ and $x = 6$. (1)

x	-2	2	6	10
y	0			$6\sqrt{3}$

Solution

x	-2	2	6	10
y	0	$\underline{\underline{2}}$	$\underline{\underline{4\sqrt{2}}}$	$6\sqrt{3}$

(b) Use the trapezium rule, with all the values of y from the completed table, to find an approximate value for the area of R, giving your answer to 3 decimal places. (4)

Solution

$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2} \times 4 \times [0 + 2(2 + 4\sqrt{2}) + 6\sqrt{3}] \\ &= 51.412\,026\,69 \text{ (FCD)} \\ &= \underline{\underline{51.412 \text{ (3 dp)}}}.\end{aligned}$$

2. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of (4)

$$(2 + kx)^7,$$

where k is a non-zero constant. Give each term in its simplest form.

Solution

$$\begin{aligned}&2^7 + \binom{7}{1}(2)^6(kx) + \binom{7}{2}(2)^5(kx)^2 + \binom{7}{3}(2)^4(kx)^3 + \ldots \\ =&\underline{\underline{128 + 448kx + 672k^2x^2 + 560k^3x^3 + \ldots}}.\end{aligned}$$

Given that the coefficient of x^3 in this expansion is 1 890,

(b) find the value of k. (3)

Solution

$$\begin{aligned}560k^3 = 1\,890 &\Rightarrow k^3 = 3\tfrac{3}{8} \\ &\Rightarrow \underline{\underline{k = 1\tfrac{1}{2}}}.\end{aligned}$$

3.

$$f(x) = 24x^3 + Ax^2 - 3x + B,$$

where A and B are constants.

When $f(x)$ is divided by $(2x - 1)$ the remainder is 30.

(a) Show that $A + 4B = 114$. (2)

Solution

$$\begin{aligned} f(\tfrac{1}{2}) = 30 &\Rightarrow 3 + \tfrac{1}{4}A - \tfrac{3}{2} + B = 30 \\ &\Rightarrow \tfrac{1}{4}A + B = 28\tfrac{1}{2} \\ &\Rightarrow \underline{\underline{A + 4B = 114}}, \end{aligned}$$

as required.

Given also that $(x + 1)$ is a factor of $f(x)$,

(b) find another equation in A and B. (2)

Solution

$$\begin{aligned} f(-1) = 0 &\Rightarrow -24 + A + 3 + B = 0 \\ &\Rightarrow \underline{\underline{A + B = 21}}. \end{aligned}$$

(c) Find the value of A and the value of B. (2)

Solution

Subtract:

$$\begin{aligned} 3B = 93 &\Rightarrow \underline{\underline{B = 31}} \\ &\Rightarrow \underline{\underline{A = -10}}. \end{aligned}$$

(d) Hence find a quadratic factor of $f(x)$. (2)

Solution

$$24x^3 - 10x^2 - 3x + 31 \equiv (x + 1)(24x^2 - 34x + 31)$$

and a quadratic factor of $\mathrm{f}(x)$ is

$$\underline{\underline{24x^2 - 34x + 31}}.$$

4. Figure 2 shows a flag $XYWZX$.

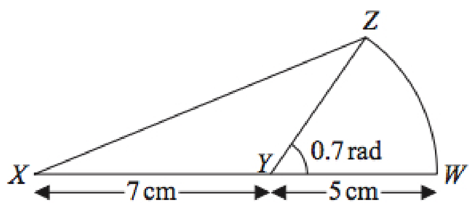

Figure 2: a flag

The flag consists of a triangle XYZ joined to a sector ZYW of a circle with radius 5 cm and centre Y.

The angle of the sector, angle ZYW, is 0.7 radians.

The points X, Y, and W lie on a straight line with $XY = 7$ cm and $YW = 5$ cm.

Find

(a) the area of the sector ZYW in cm^2, (2)

Solution

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}r^2\theta \\ &= \tfrac{1}{2} \times 5^2 \times 0.7 \\ &= \underline{\underline{8.75 \text{ cm}^2}}. \end{aligned}$$

(b) the area of the flag, in cm^2, to 2 decimal places, (3)

Solution

$$\begin{aligned}\text{Area} &= \text{area of triangle} + \text{area of sector}\\ &= (\tfrac{1}{2} \times 7 \times 5 \times \sin 0.7) + 8.75\\ &= 20.023\,809\,53 \text{ (FCD)}\\ &= \underline{\underline{20.02 \text{ cm}^2}}.\end{aligned}$$

(c) the length of the perimeter, $XYWZX$, of the flag, in cm to 2 decimal places. (4)

Solution

$$\begin{aligned}XZ &= \sqrt{5^2 + 7^2 - 2 \times 5 \times 7 \times \cos(\pi - 0.7)}\\ &= 11.293\,314\,53 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}\text{perimeter} &= 11.293\ldots + (5 \times 0.7) + (7 + 5)\\ &= 26.793\,314\,53 \text{ (FCD)}\\ &= \underline{\underline{26.79 \text{ cm (FCD)}}}.\end{aligned}$$

5. The circle C has equation

$$x^2 + y^2 - 2x + 14y = 0.$$

Find

(a) the coordinates of the centre of C, (2)

Solution

$$\begin{aligned}x^2 + y^2 - 2x + 14y = 0 &\Rightarrow (x^2 - 2x + 1) + (y^2 + 14y + 49) = 50\\ &\Rightarrow (x-1)^2 + (y+7)^2 = 50;\end{aligned}$$

hence, the coordinates of the centre of C is $\underline{\underline{(1, -7)}}$.

(b) the exact value of the radius of C, (2)

Solution

The exact value of the radius of C is $\sqrt{50} = \underline{\underline{5\sqrt{2}}}$.

(c) the y-coordinates of the points where the circle C crosses the y-axis. (2)

Solution

$$\begin{aligned} x = 0 &\Rightarrow y^2 + 14y = 0 \\ &\Rightarrow y(y+14) = 0 \\ &\Rightarrow \underline{\underline{y = -14 \text{ or } y = 0}}. \end{aligned}$$

(d) Find an equation of the tangent to C at the point $(2, 0)$, giving your answer in the form $ax + by + c = 0$, where a, b, and c are integers. (4)

Solution

Implicit differentiation:

$$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2 + 14\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow (2y+14)\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 2x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-2x}{2y+14}, \end{aligned}$$

and, at $(2, 0)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{1}{7}.$$

Finally, an equation of the tangent to C is

$$\begin{aligned} y - 0 = -\tfrac{1}{7}(x-2) &\Rightarrow 7y = -x + 2 \\ &\Rightarrow \underline{\underline{x + 7y - 2 = 0}}. \end{aligned}$$

6. A geometric series with common ratio $r = -0.9$ has sum to infinity $10\,000$.

For this series,

(a) find the first term, (2)

Solution

$$\begin{aligned} \frac{a}{1-(-0.9)} = 10\,000 &\Rightarrow \frac{a}{1.9} = 10\,000 \\ &\Rightarrow a = \underline{\underline{19\,000}}. \end{aligned}$$

(b) find the fifth term, (2)

Solution

$$\text{Fifth term} = 19\,000 \times (-0.9)^4 = \underline{\underline{12\,465.9}}.$$

(c) find the sum of the first twelve terms, giving this answer to the nearest integer. (3)

Solution

$$\begin{aligned}\text{Twelve terms} &= \frac{19\,000[1-(-0.9)^{12}]}{1-(-0.9)} \\ &= 7\,175.704\,635 \text{ (FCD)} \\ &= \underline{\underline{7\,176 \text{ (nearest integer)}}}.\end{aligned}$$

7. (a) Find the value of y for which (2)

$$1.01^{y-1} = 500.$$

Give your answer to 2 decimal places.

Solution

$$\begin{aligned}1.01^{y-1} = 500 &\Rightarrow (y-1)\log_{10} 1.01 = \log_{10} 500 \\ &\Rightarrow y - 1 = \frac{\log_{10} 500}{\log_{10} 1.01} \\ &\Rightarrow y = 1 + \frac{\log_{10} 500}{\log_{10} 1.01} \\ &\Rightarrow y = 625.562\,990\,8 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{y = 625.56 \text{ (2 dp)}}}.\end{aligned}$$

(b) Given that

$$2\log_4(3x+5) = \log_4(3x+8) + 1,\ x > -\tfrac{5}{3},$$

(i) show that (4)

$$9x^2 + 18x - 7 = 0.$$

Solution

$$
\begin{aligned}
& 2\log_4(3x+5) = \log_4(3x+8)+1 \\
\Rightarrow\ & \log_4(3x+5)^2 - \log_4(3x+8) = 1 \\
\Rightarrow\ & \log_4 \frac{(3x+5)^2}{3x+8} = 1 \\
\Rightarrow\ & \frac{(3x+5)^2}{3x+8} = 4 \\
\Rightarrow\ & (3x+5)^2 = 4(3x+8) \\
\Rightarrow\ & 9x^2+30x+25 = 12x+32 \\
\Rightarrow\ & \underline{\underline{9x^2+18x-7=0}},
\end{aligned}
$$

as required.

(ii) Hence solve the equation (2)

$$2\log_4(3x+5) = \log_4(3x+8)+1,\ x > -\tfrac{5}{3}.$$

Solution

$$
\left.\begin{array}{ll} \text{add to:} & +18 \\ \text{multiply to:} & (+9)\times(-7) = -63 \end{array}\right\} -3,\ +21
$$

$$
\begin{aligned}
9x^2+18x-7=0 &\Rightarrow 9x^2+21x-3x-7=0 \\
&\Rightarrow 3x(3x+7)-1(3x+7)=0 \\
&\Rightarrow (3x-1)(3x+7)=0 \\
&\Rightarrow x=-\tfrac{7}{3} \text{ or } x=\tfrac{1}{3};
\end{aligned}
$$

hence, $x > -\frac{5}{3}$ implies $\underline{\underline{x=\frac{1}{3}}}$.

8. *In this question solutions based entirely on graphical or numerical methods are not acceptable.*

(a) Solve for $0^\circ \leqslant x < 360^\circ$, (4)

$$4\cos(x+70)^\circ = 3,$$

giving your answers in degrees to one decimal place.

Solution

$$
\begin{aligned}
4\cos(x+70)^\circ = 3 \Rightarrow \cos(x+70)^\circ &= \tfrac{3}{4} \\
\Rightarrow (x+70)^\circ &= 318.590\,377\,9, 401.409\,622\,1 \text{ (FCD)} \\
\Rightarrow x &= 248.590\,377\,9, 331.409\,622\,1 \text{ (FCD)} \\
\Rightarrow \underline{\underline{x = 248.6, 331.4 \text{ (1 dp)}}}.
\end{aligned}
$$

(b) Find, for $0 \leqslant \theta < 2\pi$, all the solutions of (5)

$$6\cos^2\theta - 5 = 6\sin^2\theta + \sin\theta,$$

giving your answers in radians to 3 significant figures.

Solution

$$
\begin{aligned}
& 6\cos^2\theta - 5 = 6\sin^2\theta + \sin\theta \\
\Rightarrow\quad & 6(1-\sin^2\theta) - 5 = 6\sin^2\theta + \sin\theta \\
\Rightarrow\quad & 6 - 6\sin^2\theta - 5 = 6\sin^2\theta + \sin\theta \\
\Rightarrow\quad & 12\sin^2\theta + \sin\theta - 1 = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll}
\text{add to:} & \qquad\qquad +1 \\
\text{multiply to:} & (+12)\times(-1) = -12
\end{array}\right\} -3,\ +4
$$

$$
\begin{aligned}
\Rightarrow\quad & 12\sin^2\theta + 4\sin\theta - 3\sin\theta - 1 = 0 \\
\Rightarrow\quad & 4\sin\theta(3\sin\theta+1) - 1(3\sin\theta+1) = 0 \\
\Rightarrow\quad & (4\sin\theta - 1)(3\sin\theta + 1) = 0 \\
\Rightarrow\quad & \sin\theta = -\tfrac{1}{3} \text{ or } \sin\theta = \tfrac{1}{4}.
\end{aligned}
$$

$\underline{\sin\theta = \frac{1}{4}}$:

$$
\begin{aligned}
\sin\theta = \tfrac{1}{4} \Rightarrow \theta &= 0.252\,680\,255\,1, 2.888\,912\,398 \text{ (FCD)} \\
\Rightarrow \underline{\underline{\theta = 0.253, 2.89 \text{ (3 sf)}}}.
\end{aligned}
$$

$\underline{\sin\theta = \frac{1}{4}}$:

$$
\begin{aligned}
\sin\theta = -\tfrac{1}{3} \Rightarrow \theta &= 3.481\,429\,563, 5.943\,348\,398 \text{ (FCD)} \\
\Rightarrow \underline{\underline{\theta = 3.48, 5.94 \text{ (3 sf)}}}.
\end{aligned}
$$

9. Figure 3 shows a sketch of part of the curve with equation

$$y = 7x^2(5 - 2\sqrt{x}),\ x \geqslant 0.$$

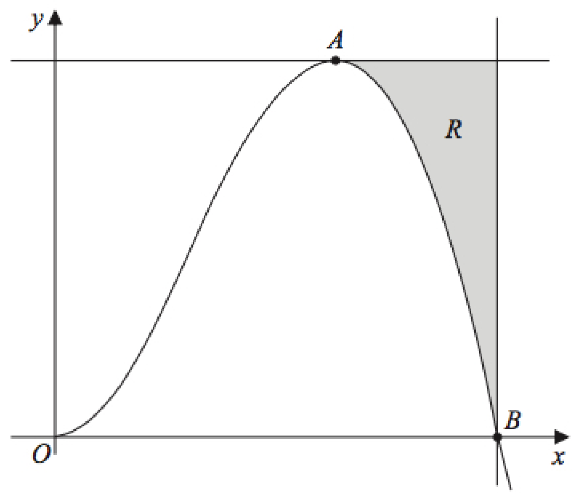

Figure 3: $y = 7x^2(5 - 2\sqrt{x})$

The curve has a turning point at the point A, where $x > 0$, as shown in Figure 3.

(a) Using calculus, find the coordinates of the point A. (5)

Solution

$$\begin{aligned} y = 7x^2(5 - 2\sqrt{x}) &\Rightarrow y = 35x^2 - 14x^{\frac{5}{2}} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 70x - 35x^{\frac{3}{2}}. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 70x - 35x^{\frac{3}{2}} = 0 \\ &\Rightarrow 35x(2 - x^{\frac{1}{2}}) = 0 \\ &\Rightarrow x = 0 \text{ or } x^{\frac{1}{2}} = 2 \\ &\Rightarrow x = 0 \text{ or } x = 4; \end{aligned}$$

hence, as $x > 0$, $x = 4$ and the point is $\underline{\underline{A(4, 112)}}$.

The curve crosses the x-axis at the point B, as shown in Figure 3.

(b) Use algebra to find the x-coordinate of the point B. (2)

Solution

$$7x^2(5 - 2\sqrt{x}) = 0 \Rightarrow \sqrt{x} = 2\tfrac{1}{2}$$
$$\Rightarrow \underline{\underline{x = 6\tfrac{1}{4}}}.$$

The finite region R, shown shaded in Figure 3, is bounded by the curve, the line through A parallel to the x-axis and the line through B parallel to the y-axis.

(c) Use integration to find the area of the region R, giving your answer to 2 decimal places. (5)

Solution

We have

$$\begin{aligned} \text{area of } R &= (112 \times 2\tfrac{1}{4}) - \int_4^{6\frac{1}{4}} (35x^2 - 14x^{\frac{5}{2}})\,\mathrm{d}x \\ &= 252 - \left[\tfrac{35}{3}x^3 - 4x^{\frac{7}{2}}\right]_{x=4}^{6\frac{1}{4}} \\ &= 252 - \left[\left(\tfrac{35}{3} \times 244\tfrac{9}{64} - 4 \times 610\tfrac{45}{128}\right) - \left(746\tfrac{2}{3} - 512\right)\right] \\ &= 79.765\,625 \text{ (exact)} \\ &= \underline{\underline{79.77 \text{ (2 dp)}}}. \end{aligned}$$