Dr Oliver Mathematics
Mathematics: Higher
2022 Paper 1: Non-Calculator
1 hour 15 minutes

The total number of marks available is 55.
You must write down all the stages in your working.

1. Determine the equation of the line perpendicular to (3)

$$5x + 2y = 7,$$

passing through $(-1, 6)$.

Solution

$$\begin{aligned} 5x + 2y = 7 &\Rightarrow 2y = -5x + 7 \\ &\Rightarrow y = -\tfrac{5}{2}x + \tfrac{7}{2} \end{aligned}$$

and this means the perpendicular has gradient

$$-\frac{1}{-\frac{5}{2}} = \tfrac{2}{5}.$$

The equation of this line is

$$y = \tfrac{2}{5}x + c$$

for some constant c. Now, since the line passes through $(-1, 6)$,

$$\begin{aligned} 6 = \tfrac{2}{5}(-1) + c &\Rightarrow 6 = -\tfrac{2}{5} + c \\ &\Rightarrow c = \tfrac{32}{5} \end{aligned}$$

and the equation is

$$\underline{\underline{y = \tfrac{2}{5}x + \tfrac{32}{5}}}.$$

2. Evaluate (3)

$$2\log_3 6 - \log_3 4.$$

Solution

$$\begin{aligned} 2\log_3 6 - \log_3 4 &= \log_3 6^2 - \log_3 4 \\ &= \log_3 36 - \log_3 4 \\ &= \log_3 \left(\frac{36}{4}\right) \\ &= \log_3 9 \\ &= \log_3 3^2 \\ &= 2\log_3 3 \\ &= \underline{\underline{2}}. \end{aligned}$$

3. A function, h, is defined by (3)

$$\mathrm{h}(x) = 4 + \tfrac{1}{3}x,$$

where $x \in \mathbb{R}$.

Find the inverse function, $\mathrm{h}^{-1}(x)$.

Solution

Let

$$y = 4 + \tfrac{1}{3}x.$$

Then

$$\begin{aligned} y = 4 + \tfrac{1}{3}x &\Rightarrow y - 4 = \tfrac{1}{3}x \\ &\Rightarrow 3(y-4) = x; \end{aligned}$$

hence,

$$\mathrm{h}^{-1}(x) = \underline{\underline{3(x-4),\ x \in \mathbb{R}}}.$$

4. Differentiate (3)

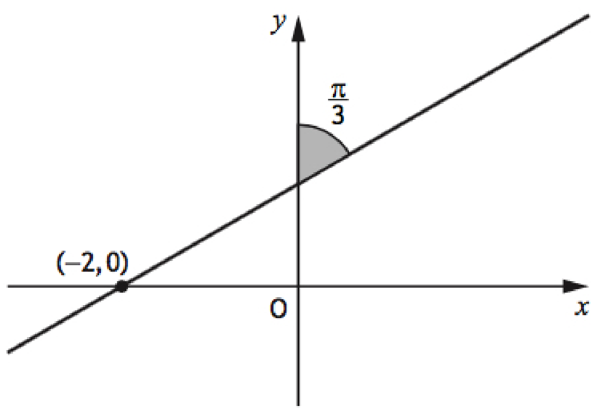
$$y = \sqrt{x^3} - 2x^{-1},$$

where $x > 0$.

Solution

$$y = \sqrt{x^3} - 2x^{-1} \Rightarrow y = x^{\frac{3}{2}} - 2x^{-1}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}x^{\frac{1}{2}} + 2x^{-2}.$$

5. A line makes an angle of $\frac{1}{3}\pi$ radians with the y-axis, and passes through the point $(-2, 0)$ as shown below. (3)



Determine the equation of the line.

Solution

The x-axis makes an angle of

$$\tfrac{1}{2}\pi - \tfrac{1}{3}\pi = \tfrac{1}{6}\pi$$

with that line and

$$\tan(\tfrac{1}{6}\pi) = \tfrac{\sqrt{3}}{3}.$$

So, the equation of the line is

$$y - 0 = \tfrac{\sqrt{3}}{3}(x + 2) \Rightarrow y = \tfrac{\sqrt{3}}{3}(x + 2).$$

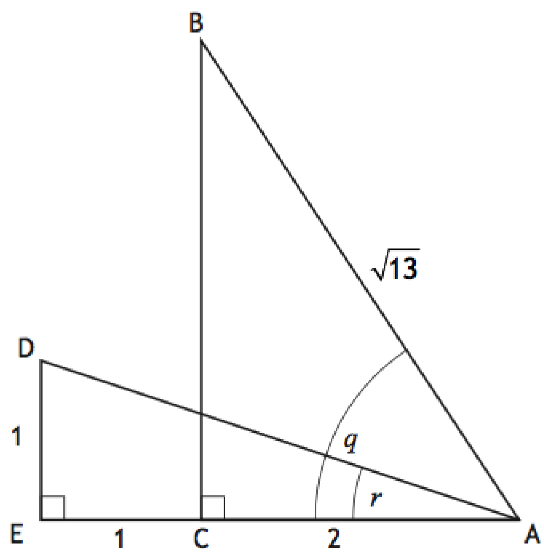
6. Evaluate (4)

$$\int_{-5}^{2} (10 - 3x)^{-\frac{1}{2}}\,\mathrm{d}x.$$

Solution

$$\int_{-5}^{2} (10-3x)^{-\frac{1}{2}}\,\mathrm{d}x = \left[-\tfrac{2}{3}(10-3x)^{\frac{1}{2}}\right]_{x=-5}^{2}$$
$$= -\tfrac{4}{3} - (-\tfrac{10}{3})$$
$$= \underline{\underline{2}}.$$

7. Triangles ABC and ADE are both right angled.
Angle $BAC = q$ and angle $DAE = r$,as shown in the diagram.



(a) Determine the value of:

(i) $\sin r$, (1)

Solution

$$AD^2 = DE^2 + AE^2 \Rightarrow AD^2 = 1^2 + 3^2$$
$$\Rightarrow AD^2 = 10$$
$$\Rightarrow AD = \sqrt{10}$$

and

$$\sin r = \underline{\underline{\frac{1}{\sqrt{10}}}}.$$

(ii) $\sin q$. (1)

Solution

$$\begin{aligned} AB^2 = BC^2 + AC^2 &\Rightarrow (\sqrt{13})^2 = BC^2 + 2^2 \\ &\Rightarrow 13 = BC^2 + 4 \\ &\Rightarrow BC^2 = 9 \\ &\Rightarrow BC = 3 \end{aligned}$$

and

$$\sin q = \frac{3}{\sqrt{13}}.$$

(b) Hence determine the value of $\sin(q - r)$. (3)

Solution

$$\begin{aligned} \sin(q-r) &= \sin q \cos r - \sin r \cos q \\ &= \left(\frac{3}{\sqrt{13}} \times \frac{3}{\sqrt{10}}\right) - \left(\frac{1}{\sqrt{10}} \times \frac{2}{\sqrt{13}}\right) \\ &= \frac{7}{\sqrt{130}}. \end{aligned}$$

8. Solve (4)

$$\log_6 x + \log_6(x+5) = 2,$$

where $x > 0$.

Solution

$$\begin{aligned} \log_6 x + \log_6(x+5) = 2 &\Rightarrow \log_6 x(x+5) = 2\log_6 6 \\ &\Rightarrow \log_6 x(x+5) = \log_6 6^2 \\ &\Rightarrow x(x+5) = 6^2 \\ &\Rightarrow x^2 + 5x - 36 = 0 \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +5 \\ \text{multiply to:} & -36\end{array}\right\} -4,\ +9$$

$$\begin{aligned} &\Rightarrow (x+9)(x-4) = 0 \\ &\Rightarrow x = -9 \text{ or } x = 4; \end{aligned}$$

hence, $\underline{\underline{x = 4}}$.

9. Solve the equation (5)

$$\cos 2x^\circ = 5\cos x^\circ - 3,$$

for $0 \leqslant x < 360$.

Solution

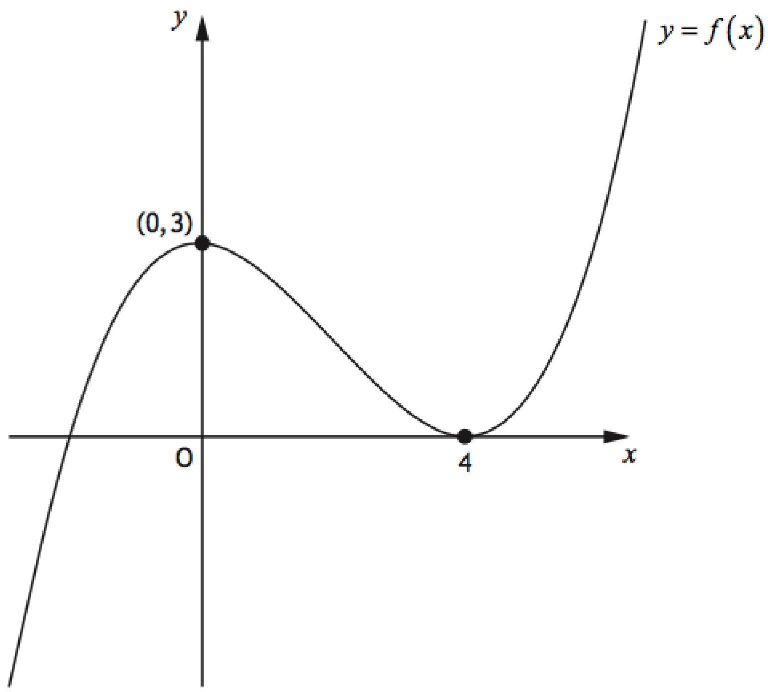
$$\begin{aligned} \cos 2x^\circ = 5\cos x^\circ - 3 &\Rightarrow 2\cos^2 x^\circ - 1 = 5\cos x^\circ - 3 \\ &\Rightarrow 2\cos^2 x^\circ - 5\cos x^\circ + 2 = 0 \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -5 \\ \text{multiply to:} & (+2)\times(+2) = +4\end{array}\right\} -4,\ -1$$

$$\begin{aligned} &\Rightarrow 2\cos^2 x^\circ - 4\cos x^\circ - \cos x^\circ + 2 = 0 \\ &\Rightarrow 2\cos x^\circ(\cos x^\circ - 2) - 1(\cos x^\circ - 2) = 0 \\ &\Rightarrow (2\cos x^\circ - 1)(\cos x^\circ - 2) = 0 \\ &\Rightarrow \cos x^\circ = \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{x^\circ = 60 \text{ or } x^\circ = 300}}, \end{aligned}$$

since $\cos x^\circ = 2$ does not have any solutions in $\mathbb{R}$.

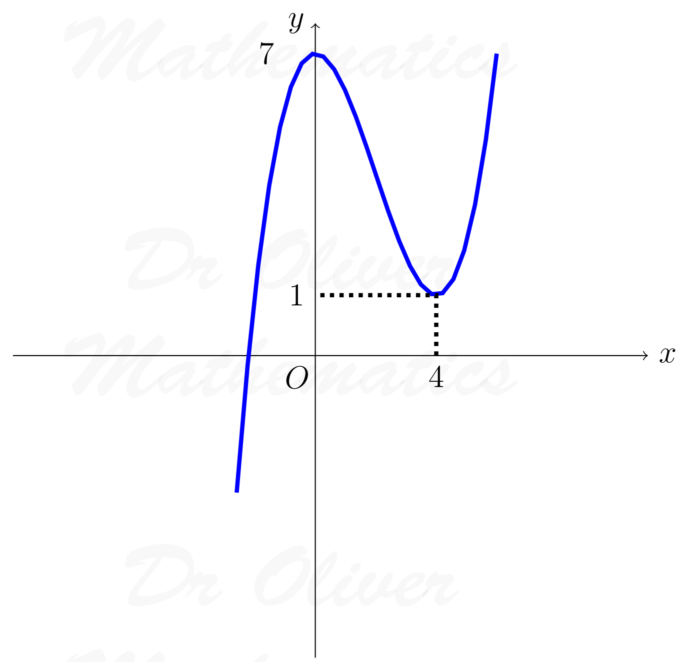
10. The diagram shows the graph of a cubic function with equation $y = \mathrm{f}(x)$.
The curve has stationary points at $(0, 3)$ and $(4, 0)$.



(a) Sketch the graph of (3)

$$y = 2\,\mathrm{f}(x) + 1.$$

Solution



(b) State the coordinates of the stationary points on the graph of $y = \mathrm{f}(\frac{1}{2}x)$. (1)

Solution

$\underline{\underline{(0, 3)}}$ and $\underline{\underline{(8, 0)}}$ (twice).

11. Express (3)

$$2x^2 + 12x + 23$$

in the form

$$p(x + q)^2 + r.$$

Solution

$$\begin{aligned} 2x^2 + 12x + 23 &= 2[x^2 + 6x] + 23 \\ &= 2[(x^2 + 6x + 9) - 9] + 23 \\ &= 2(x + 3)^2 - 18 + 23 \\ &= \underline{\underline{2(x + 3)^2 + 5}}. \end{aligned}$$

12. Given that (3)

$$f(x) = 4\sin(3x - \tfrac{1}{3}\pi),$$

evaluate $f'(\frac{1}{6}\pi)$.

Solution

$$f(x) = 4\sin(3x - \tfrac{1}{3}\pi) \Rightarrow f'(x) = 12\cos(3x - \tfrac{1}{3}\pi)$$

and

$$\begin{aligned}(f'(\tfrac{1}{6}\pi) &= 12\cos\left[3(\tfrac{1}{6}\pi) - \tfrac{1}{3}\pi\right] \\ &= 12\cos(\tfrac{1}{6}\pi) \\ &= \underline{\underline{6\sqrt{3}}}.\end{aligned}$$

13. (a) (i) Show that $(x + 2)$ is a factor of (2)

$$f(x) = x^3 - 2x^2 - 20x - 24.$$

Solution

$$f(-2) = -8 - 8 + 40 - 24 = 0;$$

the remainder is zero which means $\underline{\underline{(x + 2)}}$ is a factor.

(ii) Hence, or otherwise, solve $f(x) = 0$. (3)

Solution

We use synthetic division:

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -20 & -24 \\ & \downarrow & -2 & 8 & 24 \\ \hline & 1 & -4 & -12 & 0 \end{array}$$

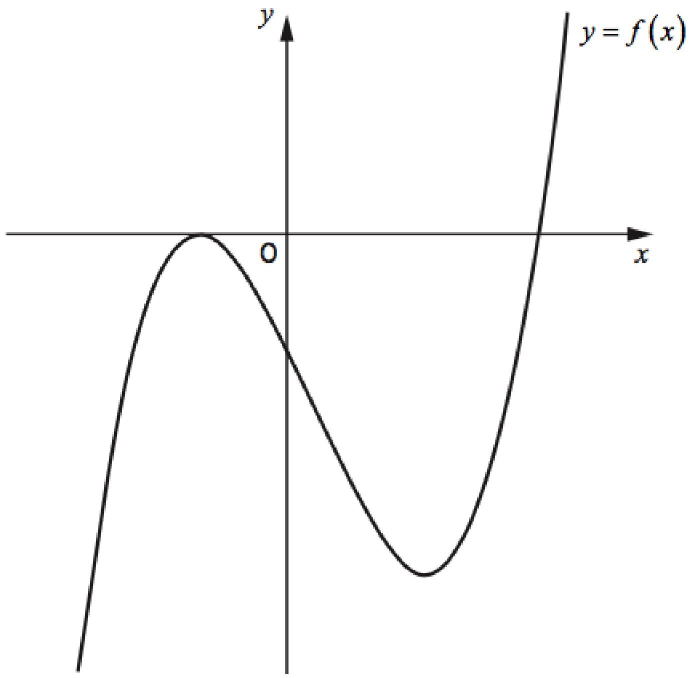
Now,

$$\begin{aligned}f(x) = 0 &\Rightarrow x^3 - 2x^2 - 20x - 24 = 0 \\ &\Rightarrow (x + 2)(x^2 - 4x - 12) = 0\end{aligned}$$

add to: -4 \
multiply to: -12 $\Big\} -6, +2$

$$\Rightarrow (x+2)(x-6)(x+2) = 0$$
$$\Rightarrow \underline{\underline{x = -2 \text{ (twice) or } x = 6}}.$$

The diagram shows the graph of $y = \mathrm{f}(x)$.



The graph of $y = \mathrm{f}(x - k)$, $k > 0$, has a stationary point at $(1, 0)$.

(b) State the value of k. (1)

Solution

$\underline{\underline{k = 3}}$.

14. C_1 is the circle with equation

$$(x-7)^2 + (y+5)^2 = 100.$$

(a) (i) State the centre and radius of C_1. (2)

Solution
The centre is $\underline{\underline{(7, -5)}}$ and the radius is $\sqrt{100} = \underline{\underline{10}}$.

(ii) Hence, or otherwise, show that the point $P(-2, 7)$ lies outside C_1. (2)

Solution

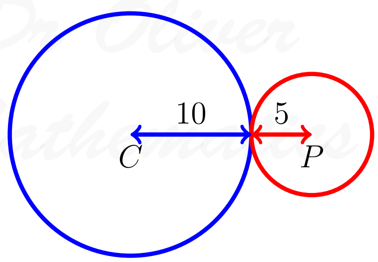
$$\begin{aligned}\sqrt{(-2-7)^2 + (7+5)^2} &= \sqrt{(-9)^2 + (12)^2} \\ &= \sqrt{225} \\ &= 15 \\ &> 10;\end{aligned}$$

hence, the point $P(-2, 7)$ lies $\underline{\underline{\text{outside}}}$ C_1.

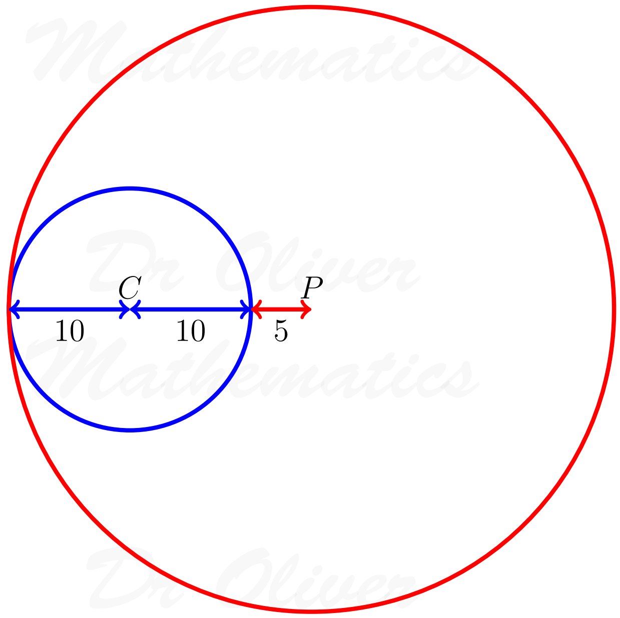
C_2 is a circle with centre P and radius r.

(b) Determine the value(s) of r for which circles C_1 and C_2 have exactly one point of intersection. (2)

Solution
Let r_1 and r_2 be the radii for C_1 and C_2 respectively.



or



Then either

$$r_2 + r_1 = 15 \Rightarrow r_2 + 10 = 15$$
$$\Rightarrow \underline{\underline{\underline{r_2 = 5}}}$$

or

$$r_2 - r_1 = 15 \Rightarrow r_2 - 10 = 15$$
$$\Rightarrow \underline{\underline{\underline{r_2 = 25}}}$$