Dr Oliver Mathematics
Mathematics
Integration Part 3
Past Examination Questions

This booklet consists of 67 questions across a variety of examination topics.
The total number of marks available is 451.

1. (a) Express (3)

$$\frac{5x+3}{(2x-3)(x+2)}$$

in partial fractions.

Solution

$$\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{A}{2x-3} + \frac{B}{x+2}$$
$$\equiv \frac{A(x+2)+B(2x-3)}{(2x-3)(x+2)}$$

and so

$$5x+3 \equiv A(x+2)+B(2x-3).$$

$\underline{x=-2}$: $-7=-7B \Rightarrow B=1$.
$\underline{x=\frac{3}{2}}$: $\frac{21}{2}=\frac{7}{2}A \Rightarrow A=3$.
Thus

$$\frac{5x+3}{(2x-3)(x+2)} \equiv \underline{\underline{\frac{3}{2x-3}+\frac{1}{x+2}}}.$$

(b) Hence find the exact value of (5)

$$\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,\mathrm{d}x,$$

giving your answer as a single logarithm.

Solution

$$
\begin{aligned}
\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,\mathrm{d}x &= \int_2^6 \left(\frac{3}{2x-3} + \frac{1}{x+2}\right)\,\mathrm{d}x \\
&= \left[\tfrac{3}{2}\ln|2x-3| + \ln|x+2|\right]_{x=2}^{6} \\
&= \left(\tfrac{3}{2}\ln 9 + \ln 8\right) - \left(\tfrac{3}{2}\ln 1 + \ln 4\right) \\
&= \ln 27 + \ln 8 - \ln 4 \\
&= \ln\left(\frac{27\times 8}{4}\right) \\
&= \underline{\underline{\ln 54}}.
\end{aligned}
$$

2. Use the substitution $x = \sin\theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x.$$

Solution

$$x = \sin\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta \Rightarrow \mathrm{d}x = \cos\theta\,\mathrm{d}\theta$$

and

$$x = 0 \Rightarrow \theta = 0 \text{ and } x = \tfrac{1}{2} \Rightarrow \theta = \tfrac{\pi}{6}.$$

Now,

$$
\begin{aligned}
\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x &= \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{(1-\sin^2\theta)^{\frac{3}{2}}}\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{\cos^3\theta}\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} \sec^2\theta\,\mathrm{d}\theta \\
&= \left[\tan\theta\right]_{\theta=0}^{\frac{\pi}{6}} \\
&= \frac{1}{\sqrt{3}} - 0 \\
&= \underline{\underline{\frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}}}.
\end{aligned}
$$

3. Using the substitution $u^2 = 2x - 1$, or otherwise, to find the exact value of

$$\int_1^5 \frac{3x}{\sqrt{2x-1}}\,\mathrm{d}x.$$

Solution

$$u^2 = 2x - 1 \Rightarrow 2u\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \Rightarrow u\,\mathrm{d}u = \mathrm{d}x$$

and

$$u^2 = 1 \Rightarrow u = 1 \text{ and } u^2 = 9 \Rightarrow u = 3.$$

Now,

$$\begin{aligned}\int_1^5 \frac{3x}{\sqrt{2x-1}}\,\mathrm{d}x &= \int_1^3 \frac{3(u^2+1)}{2u}u\,\mathrm{d}u \\ &= \int_1^3 \frac{3(u^2+1)}{2}\,\mathrm{d}u \\ &= \int_1^3 \frac{3u^2+3}{2}\,\mathrm{d}u \\ &= \int_1^3 \left(\frac{3u^2}{2}+\frac{3}{2}\right)\mathrm{d}u \\ &= \left[\frac{1}{2}u^3+\frac{3}{2}u\right]_{u=1}^3 \\ &= \left(\frac{27}{2}+\frac{9}{2}\right)-\left(\frac{1}{2}+\frac{3}{2}\right) \\ &= \underline{\underline{16}}.\end{aligned}$$

4. The finite region R, shown shaded in the figure, is bounded by the curve $y = xe^x$, the x-axis, and the lines $x = 1$ and $x = 3$.

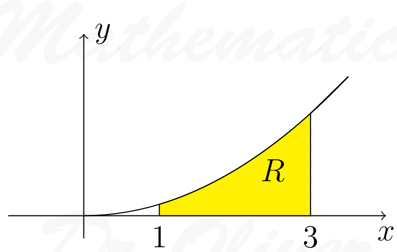

Figure 1: $y = xe^x$

The region R is rotated through 360 degrees about the x-axis. Use integration by parts to find an exact value for the **volume** of the solid generated.

Solution

We need to find $x^2 e^{2x}$:

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = e^{2x} \Rightarrow v = \tfrac{1}{2}e^{2x}$$

and

$$\begin{aligned}\int (xe^x)^2 \, dx &= \int x^2 e^{2x} \, dx \\ &= \tfrac{1}{2}x^2 e^{2x} - \int xe^{2x} \, dx \\ &= \tfrac{1}{2}x^2 e^{2x} - \left[\tfrac{1}{2}xe^{2x} - \int \tfrac{1}{2}e^{2x} \, dx\right] \\ &= \tfrac{1}{2}x^2 e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + c.\end{aligned}$$

Now,

$$\begin{aligned}\text{Volume} &= \pi\left[\tfrac{1}{2}x^2 e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x}\right]_{x=1}^{3} \\ &= \pi\left\{\left(\tfrac{9}{2}e^6 - \tfrac{3}{2}e^6 + \tfrac{1}{4}e^6\right) - \left(\tfrac{1}{2}e^2 - \tfrac{1}{2}e^2 + \tfrac{1}{4}e^2\right)\right\} \\ &= \underline{\underline{\pi\left(\tfrac{13}{4}e^6 - \tfrac{1}{4}e^2\right)}} \text{ or } \underline{\underline{\tfrac{1}{4}\pi e^2\left(13e^4 - 1\right)}}.\end{aligned}$$

5. The curve shown in the Figure 2 has parametric equations

$$x = t - 2\sin t, \; y = 1 - 2\cos t.$$

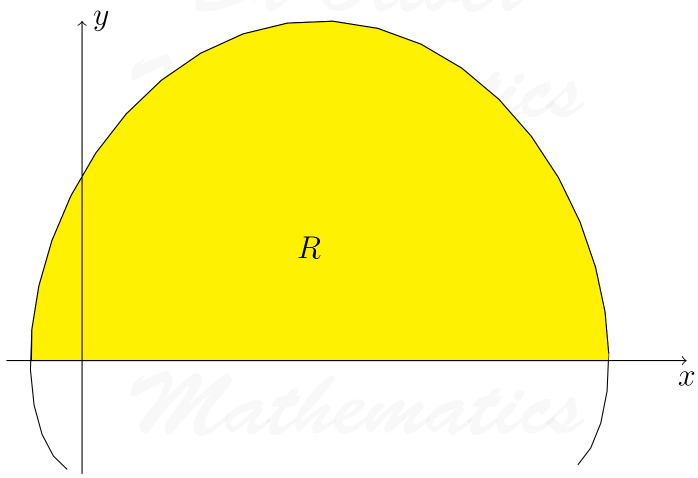

Figure 2: $x = t - 2\sin t$, $y = 1 - 2\cos t$

(a) Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. (2)

Solution

$$y = 0 \Rightarrow 1 - 2\cos t = 0 \Rightarrow \cos t = \tfrac{1}{2} \Rightarrow t = \underline{\underline{\tfrac{\pi}{3} \text{ or } \tfrac{5\pi}{3}}}.$$

The finite region R in enclosed by the curve and the x-axis.

(b) Show that the area of R is given by the integral (3)

$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, \mathrm{d}t.$$

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - 2\cos t$$

and

$$\begin{aligned}\int_{x_A}^{x_B} y \, \mathrm{d}x &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)(1 - 2\cos t)\mathrm{d}t \\ &= \underline{\underline{\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, \mathrm{d}t.}}\end{aligned}$$

(c) Use this integral to find the exact value of the shaded area. (7)

Solution

$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, \mathrm{d}t \\
&= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 4\cos t + 4\cos^2 t) \, \mathrm{d}t \\
&= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 4\cos t + 4(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t)) \, \mathrm{d}t \\
&= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (3 - 4\cos t + 2\cos 2t) \, \mathrm{d}t \\
&= \left[3t - 4\sin t + \sin 2t\right]_{t=\frac{\pi}{3}}^{\frac{5\pi}{3}} \\
&= \left(5\pi + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) - \left(\pi - 2\sqrt{3} + \tfrac{\sqrt{3}}{2}\right) \\
&= \underline{\underline{4\pi + 3\sqrt{3}}}.
\end{aligned}$$

6. The curve with equation $y = 3\sin\frac{x}{2}$, $0 \leqslant x \leqslant 2\pi$, is shown in Figure 3.

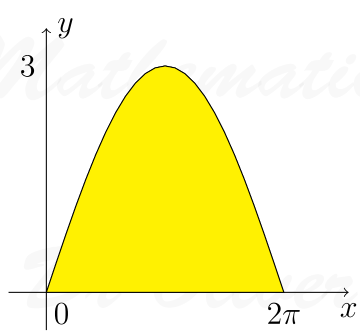

Figure 3: $y = 3\sin\frac{x}{2}$

The finite region enclosed by the curve and the x-axis is shaded.

(a) Find, by integration, the area of the shaded region. (3)

Solution

$$\int_0^{2\pi} 3\sin\tfrac{x}{2}\,\mathrm{d}x = [-6\cos\tfrac{x}{2}]_{x=0}^{2\pi}$$
$$= 6-(-6)$$
$$= \underline{\underline{12}}.$$

This region is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. (6)

Solution

$$\begin{aligned}\text{Volume} &= \pi\int_0^{2\pi}(3\sin\tfrac{x}{2})^2\,\mathrm{d}x\\ &= \pi\int_0^{2\pi} 9\sin^2\tfrac{x}{2}\,\mathrm{d}x\\ &= 9\pi\int_0^{2\pi}(\tfrac{1}{2}-\tfrac{1}{2}\cos x)\,\mathrm{d}x\\ &= 9\pi[\tfrac{1}{2}x-\tfrac{1}{2}\sin x]_{x=0}^{2\pi}\\ &= 9\pi\,\{(\pi-0)-(0-0)\}\\ &= \underline{\underline{9\pi^2}}.\end{aligned}$$

7. Show, by integration, that the exact value of $\displaystyle\int_1^3 (x-1)\ln x\,\mathrm{d}x$ is $\tfrac{3}{2}\ln 3$. (6)

Solution

$$u=\ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x}=x-1 \Rightarrow v=\tfrac{1}{2}x^2-x.$$

Now,

$$\begin{aligned}\int_1^3 (x-1)\ln x\,\mathrm{d}x &= \left[(\tfrac{1}{2}x^2-x)\ln x\right]_{x=1}^3 - \int_1^3 \frac{\tfrac{1}{2}x^2-x}{x}\,\mathrm{d}x\\ &= (\tfrac{3}{2}\ln 3-0)-\int_1^3(\tfrac{1}{2}x-1)\,\mathrm{d}x\\ &= \tfrac{3}{2}\ln 3-\left[\tfrac{1}{4}x^2-x\right]_{x=1}^3\\ &= \tfrac{3}{2}\ln 3-\left[(\tfrac{9}{4}-3)-(\tfrac{1}{4}-1)\right]\\ &= \underline{\underline{\tfrac{3}{2}\ln 3}}.\end{aligned}$$

8. The curve with equation $y = \dfrac{1}{3(1+2x)}$, $x > -\frac{1}{2}$, is shown in Figure 4.

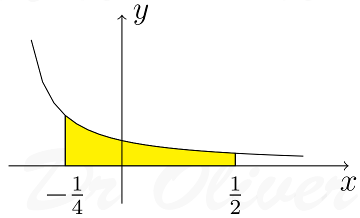

Figure 4: $y = \dfrac{1}{3(1+2x)}$

The region bounded by the lines $x = -\frac{1}{4}$, $x = \frac{1}{2}$, the x-axis, and the curve is shaded in the figure. This region is rotated through 360 degrees about the x-axis.

(a) Use calculus to find the exact value of the volume of the solid generated. (5)

Solution

$$\begin{aligned}\text{Volume} &= \pi \int_{-\frac{1}{4}}^{\frac{1}{2}} \frac{1}{9(1+2x)^2}\,\mathrm{d}x \\ &= \pi \left[-\tfrac{1}{18}(1+2x)^{-1}\right]_{x=-\frac{1}{4}}^{\frac{1}{2}} \\ &= \pi \left(-\tfrac{1}{36} - (-\tfrac{1}{9})\right) \\ &= \underline{\underline{\tfrac{1}{12}\pi}}.\end{aligned}$$

Here we shows a paperweight with axis of symmetry AB where $AB = 3$ cm. A is a point on top surface of the paperweight and B is a point on bottom surface of the paperweight. The paperweight is geometrically similar to the solid in part (a).

(b) Find the volume of this paperweight. (2)

Solution

$\frac{1}{2} - (-\frac{1}{4}) = \frac{3}{4}$ so the we have a volume that is $3 \div \frac{3}{4} = 4$ times as large. Hence, the volume of the paperweight is

$$\tfrac{1}{12}\pi \times 4^3 = \underline{\underline{\tfrac{16}{3}\pi \text{ cm}^3}}.$$

9.

$$I = \int_0^5 e^{\sqrt{3x+1}}\,dx.$$

(a) Use the substitution $t = \sqrt{3x+1}$ to show that I may be expressed as (5)

$$\int_a^b kte^t\,dx,$$

giving the values of a, b, k.

Solution

$$t = \sqrt{3x+1} \Rightarrow \frac{dt}{dx} = \frac{3}{2\sqrt{3x+1}} = \frac{3}{2t} \Rightarrow \frac{2t}{3}\,dt = dx$$

and

$$x = 0 \Rightarrow t = 1 \text{ and } x = 5 \Rightarrow t = 4.$$

Hence

$$\int_0^5 e^{\sqrt{3x+1}}\,dx = \underline{\underline{\int_1^4 \tfrac{2}{3}te^t\,du}};$$

hence, $\underline{\underline{a = 1}}$, $\underline{\underline{b = 4}}$, and $\underline{\underline{k = \tfrac{2}{3}}}$.

(b) Use integration by parts to evaluate this integral, and hence find the value of I correct to 4 significant figures, showing all the steps in your working. (5)

Solution

$$u = \tfrac{2}{3}t \Rightarrow \frac{du}{dx} = \tfrac{2}{3} \text{ and } \frac{dv}{dx} = e^t \Rightarrow v = e^t.$$

$$\begin{aligned}\int_0^5 e^{\sqrt{3x+1}}\,dx &= \int_1^4 \tfrac{2}{3}te^t\,du \\ &= \left[\tfrac{2}{3}te^t\right]_{t=1}^4 - \int_1^4 \tfrac{2}{3}e^t\,du \\ &= \left[\tfrac{2}{3}te^t - \tfrac{2}{3}e^t\right]_{t=1}^4 \\ &= \left(\tfrac{8}{3}e^4 - \tfrac{2}{3}e^4\right) - \left(\tfrac{2}{3}e - \tfrac{2}{3}e\right) \\ &= 2e^4 \\ &= 109.196\,300\,1 \text{ (FCD)} \\ &= \underline{\underline{109.2 \text{ (4 sf)}}}.\end{aligned}$$

10. Use the substitution $u = 2^x$ to find the exact value of (6)

$$\int_0^1 \frac{2^x}{(2^x+1)^2}\,\mathrm{d}x.$$

Solution

$$u = 2^x \Rightarrow \ln u = x \ln 2 \Rightarrow \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = \ln 2 \Rightarrow \mathrm{d}u = 2^x \ln 2\,\mathrm{d}x$$

and

$$x = 0 \Rightarrow u = 1 \text{ and } x = 1 \Rightarrow u = 2.$$

Hence,

$$\begin{aligned}\int_0^1 \frac{2^x}{(2^x+1)^2}\,\mathrm{d}x &= \frac{1}{\ln 2}\int_1^2 \frac{1}{(u+1)^2}\,\mathrm{d}u \\ &= \frac{1}{\ln 2}\left[-(u+1)^{-1}\right]_{u=1}^2 \\ &= \frac{1}{\ln 2}\left(-\frac{1}{3} - \left(-\frac{1}{2}\right)\right) \\ &= \underline{\underline{\frac{1}{6\ln 2}}}.\end{aligned}$$

11. (a) Find $\displaystyle\int x\cos 2x\,\mathrm{d}x$. (4)

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x \Rightarrow v = \tfrac{1}{2}\sin 2x.$$

$$\begin{aligned}\int x\cos 2x\,\mathrm{d}x &= \tfrac{1}{2}x\sin 2x - \tfrac{1}{2}\int \sin 2x\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x + c.}}\end{aligned}$$

(b) Hence, using the identity $\cos 2x = 2\cos^2 x - 1$, deduce $\displaystyle\int x\cos^2 x\,\mathrm{d}x$. (3)

Solution

$$\begin{aligned}\int x\cos^2 x\,\mathrm{d}x &= \int x(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,\mathrm{d}x \\ &= \int \tfrac{1}{2}x\,\mathrm{d}x + \tfrac{1}{2}\int x\cos 2x\,\mathrm{d}x \\ &= \tfrac{1}{4}x^2 + \tfrac{1}{2}(\tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x) + c \\ &= \underline{\underline{\tfrac{1}{4}x^2 + \tfrac{1}{4}x\sin 2x + \tfrac{1}{8}\cos 2x + c}}.\end{aligned}$$

12.

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv A + \frac{B}{2x+1} + \frac{C}{2x-1}.$$

(a) Find the values of A, B, and C. (4)

Solution

$$\begin{aligned}\frac{2(4x^2+1)}{(2x+1)(2x-1)} &\equiv A + \frac{B}{2x+1} + \frac{C}{2x-1} \\ &\equiv \frac{A(2x+1)(2x-1) + B(2x-1) + C(2x+1)}{(2x+1)(2x-1)}\end{aligned}$$

and so

$$2(4x^2+1) \equiv A(2x+1)(2x-1) + B(2x-1) + C(2x+1).$$

$\underline{x = -\tfrac{1}{2}}$: $4 = -2B \Rightarrow B = -2$.
$\underline{x = \tfrac{1}{2}}$: $4 = 2C \Rightarrow C = 2$.
$\underline{x = 0}$: $2 = -A + B + C \Rightarrow A = 2$.
Hence

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv \underline{\underline{2 - \frac{2}{2x+1} + \frac{2}{2x-1}}},$$

(b) Hence show that the exact value of $\displaystyle\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)}\,\mathrm{d}x$ is $2 + \ln k$, giving the value of the constant k. (6)

Solution

$$\begin{aligned}\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)}\,\mathrm{d}x &= \int_1^2 \left(2 - \frac{2}{2x+1} + \frac{2}{2x-1}\right)\,\mathrm{d}x \\ &= \left[2x - \ln|2x+1| + \ln|2x-1|\right]_{x=1}^{2} \\ &= (4 - \ln 5 + \ln 3) - (2 - \ln 3 + \ln 1) \\ &= 2 - \ln 5 + 2\ln 3 \\ &= 2 - \ln 5 + \ln 9 \\ &= \underline{\underline{2 + \ln \tfrac{9}{5}}}.\end{aligned}$$

13. Figure 5 shows part of the curve with equation $y = \sqrt{\tan x}$. (4)

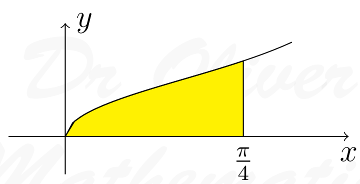

Figure 5: $y = \sqrt{\tan x}$

The region bounded R by the lines $x = \frac{\pi}{4}$, the x-axis, and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis to generate a solid of revolution. Use integration to find an exact value the volume of the solid generated.

Solution

$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{4}} \tan x\,\mathrm{d}x \\ &= \pi\left[\ln|\sec x|\right]_{x=0}^{\frac{\pi}{4}} \\ &= \pi\left(\ln\sqrt{2} - \ln 1\right) \\ &= \underline{\underline{\pi \ln \sqrt{2}}}.\end{aligned}$$

14. The curve shown in Figure 6 has equation $y = \dfrac{1}{2x+1}$.

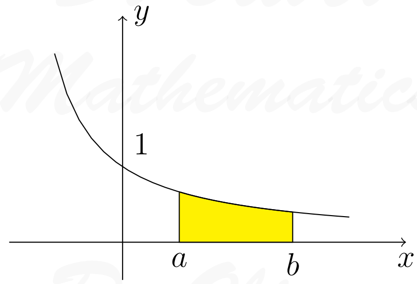

Figure 6: $y = \dfrac{1}{2x+1}$

The region bounded R by the lines $x = a$, $x = b$, the x-axis, and the curve is shaded in the figure. This region is rotated through 360 degrees about the x-axis to generate a solid of revolution. Find the volume of the solid generated. Express your answer as a single simplified fraction, in terms of a and b.

Solution

$$\begin{aligned}
\text{Volume} &= \pi \int_a^b \frac{1}{(2x+1)^2}\,\mathrm{d}x \\
&= \pi\left[-\tfrac{1}{2}(2x+1)^{-1}\right]_{x=a}^{b} \\
&= -\tfrac{1}{2}\pi\left(\frac{1}{2b+1} - \frac{1}{2a+1}\right) \\
&= -\tfrac{1}{2}\pi\left(\frac{(2a+1)-(2b+1)}{(2a+1)(2b+1)}\right) \\
&= -\tfrac{1}{2}\pi\left(\frac{2a-2b}{(2a+1)(2b+1)}\right) \\
&= \underline{\underline{\frac{(b-a)\pi}{(2a+1)(2b+1)}}}.
\end{aligned}$$

15. (a) Find $\displaystyle\int \ln\left(\frac{x}{2}\right)\,\mathrm{d}x$. (4)

Solution

$$u = \ln\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x.$$

Now,

$$\int \ln\left(\frac{x}{2}\right)\,\mathrm{d}x = x\ln\left(\frac{x}{2}\right) - \int 1\,\mathrm{d}x$$
$$= \underline{\underline{x\ln\left(\frac{x}{2}\right) - x + c.}}$$

(b) Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\,\mathrm{d}x$. (5)

Solution

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)\mathrm{d}x$$
$$= \left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_{x=\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= \left(\tfrac{\pi}{4} - 0\right) - \left(\tfrac{\pi}{8} - \tfrac{1}{4}\right)$$
$$= \underline{\underline{\tfrac{\pi}{8} + \tfrac{1}{4}}}.$$

16. The curve C shown in Figure 7 has parametric equation

$$x = \ln(t+2),\ y = \frac{1}{t+1},\ t > -1.$$

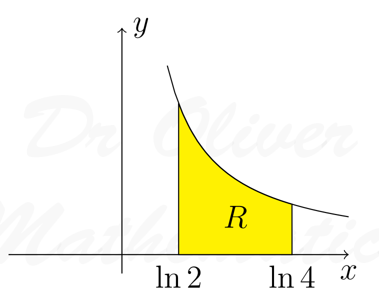

Figure 7: $x = \ln(t+2)$, $y = \dfrac{1}{t+1}$

The region bounded R by the lines $x = \ln 2$, $x = \ln 4$, the x-axis, and the curve is shaded in the figure.

(a) Show that the region of R is given by the integral (4)

$$\int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t.$$

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t+2}$$

and

$$x = \ln 2 \Rightarrow t = 0 \text{ and } x = \ln 4 \Rightarrow t = 2.$$

Hence

$$\text{area} = \int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t.$$

(b) Hence find an exact value for this area. (4)

Solution

$$\frac{1}{(t+1)(t+2)} \equiv \frac{A}{t+1} + \frac{B}{t+2} \equiv \frac{A(t+2)+B(t+1)}{(t+1)(t+2)}$$

and hence

$$1 \equiv A(t+2) + B(t+1).$$

$\underline{t=-2}$: $1 = -B \Rightarrow B = -1$.
$\underline{t=-1}$: $1 = A$.
Hence

$$\frac{1}{(t+1)(t+2)} \equiv \frac{1}{t+1} - \frac{1}{t+2}$$

and

$$\begin{aligned}\int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t &= \int_0^2 \left[\frac{1}{t+1} - \frac{1}{t+2}\right]\,\mathrm{d}t \\ &= \left[\ln|t+1| - \ln|t+2|\right]_{t=0}^{2} \\ &= (\ln 3 - \ln 4) - (\ln 1 - \ln 2) \\ &= \ln\left(\frac{3\times 2}{4}\right) \\ &= \ln \tfrac{3}{2}\end{aligned}$$

17. Using the substitution $h = (20 - x)^2$, or otherwise, find the exact value of (6)

$$\int_0^{100} \frac{50}{20 - \sqrt{h}} \, \mathrm{d}h.$$

Solution

This is **not** a good substitution to use: we would have been better using $x = 20 - \sqrt{h}$ (why?). Well,

$$h = (20 - x)^2 \Rightarrow \frac{\mathrm{d}h}{\mathrm{d}x} = -2(20 - x) \Rightarrow \mathrm{d}h = -2(20 - x) \, \mathrm{d}x$$

and

$$h = 0 \Rightarrow x = 20 \text{ and } h = 100 \Rightarrow x = 10.$$

$$\begin{aligned}
\int_0^{100} \frac{50}{20 - \sqrt{h}} \, \mathrm{d}h &= \int_{20}^{10} \frac{-100(20 - x)}{x} \, \mathrm{d}x \\
&= \int_{20}^{10} \left(100 - \frac{2000}{x} \right) \mathrm{d}x \\
&= \left[100x - 2000 \ln |x| \right]_{t=20}^{10} \\
&= (1000 - 2000 \ln 10) - (2000 - 2000 \ln 20) \\
&= 2000 \ln 20 - 2000 \ln 10 - 1000 \\
&= 2000 \ln(\tfrac{20}{10}) - 1000 \\
&= \underline{\underline{2000 \ln 2 - 1000}}.
\end{aligned}$$

18. (a) Use integration by parts to find $\int x\mathrm{e}^x \, \mathrm{d}x$. (3)

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \mathrm{e}^x.$$

Now,

$$\begin{aligned}
\int x\mathrm{e}^x \, \mathrm{d}x &= x\mathrm{e}^x - \int \mathrm{e}^x \, \mathrm{d}x \\
&= \underline{\underline{x\mathrm{e}^x - \mathrm{e}^x + c}}.
\end{aligned}$$

(b) Hence find $\int x^2 \mathrm{e}^x \, \mathrm{d}x$. (3)

Solution

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \mathrm{e}^x.$$

Now,

$$\begin{aligned}\int x^2 \mathrm{e}^x \, \mathrm{d}x &= x^2 \mathrm{e}^x - \int 2x\mathrm{e}^x \, \mathrm{d}x \\ &= \underline{\underline{x^2 \mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + c}}.\end{aligned}$$

19. Find $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64 \sin^2 t \cos t \, \mathrm{d}t$, giving your answer in the form $a + b\sqrt{3}$, where a and b are constants to be determined. (4)

Solution

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64 \sin^2 t \cos t \, \mathrm{d}t &= \left[\tfrac{64}{3} \sin^3 t\right]_{t=\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \underline{\underline{\tfrac{64}{3} - 8\sqrt{3}}}.\end{aligned}$$

20. Figure 8 shows part of the curve $y = \dfrac{3}{\sqrt{1 + 4x}}$.

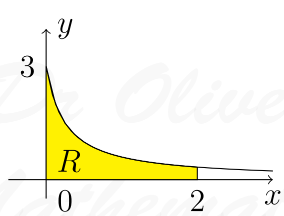

Figure 8: $y = \dfrac{3}{\sqrt{1 + 4x}}$

The region bounded R by the lines $x = 0$, $x = 2$, the x-axis, and the curve is shaded in the figure.

(a) Use integration to find the area of R. (4)

Solution

$$\int_0^2 \frac{3}{\sqrt{1+4x}}\,\mathrm{d}x = 3\left[\tfrac{1}{2}\sqrt{1+4x}\right]_{x=0}^{2}$$
$$= 3(\tfrac{3}{2} - \tfrac{1}{2})$$
$$= \underline{\underline{3}}.$$

The region R is rotated 360° about the x-axis.

(b) Use integration to find the exact value of the volume of the solid formed. (5)

Solution

$$\text{Volume} = \pi \int_0^2 \frac{9}{1+4x}\,\mathrm{d}x$$
$$= 9\pi\left[\tfrac{1}{4}\ln|1+4x|\right]_{x=0}^{2}$$
$$= 9\pi(\tfrac{1}{4}\ln 9 - \tfrac{1}{4}\ln 1)$$
$$= \underline{\underline{\tfrac{9}{4}\pi\ln 9}}.$$

21. (a) Find $\int \tan^2 x\,\mathrm{d}x$. (2)

Solution

$$\int \tan^2 x\,\mathrm{d}x = \int (\sec^2 x - 1)\,\mathrm{d}x = \underline{\underline{\tan x - x + c}}.$$

(b) Use integration by parts to find $\int \frac{1}{x^3}\ln x\,\mathrm{d}x$. (4)

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^3} \Rightarrow v = -\tfrac{1}{2}x^{-2}.$$

Now,

$$\int \frac{1}{x^3}\ln x\,\mathrm{d}x = -\tfrac{1}{2}x^{-2}\ln x - \int(-\tfrac{1}{2}x^{-3})\,\mathrm{d}x$$
$$= \underline{\underline{-\tfrac{1}{2}x^{-2}\ln x + \tfrac{1}{4}x^{-2} + c}}.$$

(c) Use the substitution $u = 1 + \mathrm{e}^x$ to show that (7)

$$\int \frac{\mathrm{e}^{3x}}{1+\mathrm{e}^x}\,\mathrm{d}x = \tfrac{1}{2}\mathrm{e}^{2x} - \mathrm{e}^x + \ln(1+\mathrm{e}^x) + k,$$

where k is a constant.

Solution

$$u = 1 + \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\,\mathrm{d}x.$$

$$\begin{aligned}\int \frac{\mathrm{e}^{3x}}{1+\mathrm{e}^x}\,\mathrm{d}x &= \int \frac{(u-1)^2}{u}\,\mathrm{d}u \\ &= \int \frac{(u^2-2u+1)}{u}\,\mathrm{d}u \\ &= \int \left(u - 2 + \frac{1}{u}\right)\mathrm{d}u \\ &= \tfrac{1}{2}u^2 - 2u + \ln u + c \\ &= \tfrac{1}{2}(1+\mathrm{e}^x)^2 - 2(1+\mathrm{e}^x) + \ln(1+\mathrm{e}^x) + c \\ &= \tfrac{1}{2}(1+2\mathrm{e}^x+\mathrm{e}^{2x}) - 2 - 2\mathrm{e}^x + \ln(1+\mathrm{e}^x) + c \\ &= \tfrac{1}{2}\mathrm{e}^{2x} - \mathrm{e}^x + \ln(1+\mathrm{e}^x) - \tfrac{3}{2} + c \\ &= \underline{\underline{\tfrac{1}{2}\mathrm{e}^{2x} - \mathrm{e}^x + \ln(1+\mathrm{e}^x) + k}}.\end{aligned}$$

22. Use integration to find the exact area of $y = 3\cos(\frac{x}{3})$, $0 \leqslant x \leqslant \frac{3\pi}{2}$. (3)

Solution

$$\begin{aligned}\int_0^{\frac{3\pi}{2}} 3\cos(\tfrac{x}{3})\,\mathrm{d}x &= \left[9\sin(\tfrac{x}{3})\right]_{x=0}^{\frac{3\pi}{2}} \\ &= 9 - 0 \\ &= \underline{\underline{9}}.\end{aligned}$$

23.

$$\mathrm{f}(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

(a) Find the values of the constants A, B, and C. (4)

Solution

$$\begin{aligned}&\frac{4-2x}{(2x+1)(x+1)(x+3)}\\ \equiv&\frac{A}{2x+1}+\frac{B}{x+1}+\frac{C}{x+3}\\ \equiv&\frac{A(x+1)(x+3)+B(2x+1)(x+3)+C(2x+1)(x+1)}{(2x+1)(x+1)(x+3)}\end{aligned}$$

and hence

$$4-2x \equiv A(x+1)(x+3)+B(2x+1)(x+3)+C(2x+1)(x+1).$$

$\underline{x=-3}$: $10=10C \Rightarrow C=1$.
$\underline{x=-1}$: $6=-2B \Rightarrow B=-3$.
$\underline{x=-\frac{1}{2}}$: $5=\frac{5}{2}A \Rightarrow A=4$.
Hence

$$\frac{4-2x}{(2x+1)(x+1)(x+3)}=\underline{\underline{\frac{4}{2x+1}-\frac{3}{x+1}+\frac{1}{x+3}}};$$

and thus $\underline{\underline{A=4}}$, $\underline{\underline{B=-3}}$, and $\underline{\underline{C=1}}$.

(b) Hence find $\int \mathrm{f}(x)\,\mathrm{d}x$. (3)

Solution

$$\int \mathrm{f}(x)\,\mathrm{d}x=\underline{\underline{2\ln|2x+1|-3\ln|x+1|+\ln|x+3|+c}}.$$

(c) Find $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$ in the form $\ln k$, where k is a constant. (3)

Solution

$$\begin{aligned}\int_0^2 \mathrm{f}(x)\,\mathrm{d}x &= \left[2\ln|2x+1| - 3\ln|x+1| + \ln|x+3|\right]_{x=0}^{2} \\ &= (2\ln 5 - 3\ln 3 + \ln 5) - (0 - 0 + \ln 3) \\ &= 3\ln 5 - 4\ln 3 \\ &= \ln\left(\frac{5^3}{3^4}\right) \\ &= \underline{\underline{\ln\left(\frac{125}{81}\right)}}.\end{aligned}$$

24. (a) Find $\int \sqrt{5-x}\,\mathrm{d}x$. (2)

Solution

$$\int \sqrt{5-x}\,\mathrm{d}x = \underline{\underline{-\tfrac{2}{3}(5-x)^{\frac{3}{2}} + c}}.$$

Figure 9 shows a sketch of the curve with equation

$$y = (x-1)\sqrt{5-x},\ 1 \leqslant x \leqslant 5.$$

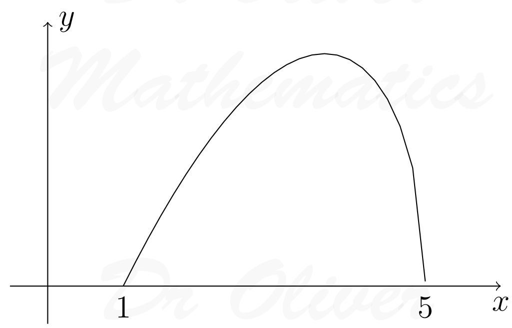

Figure 9: $y = (x-1)\sqrt{5-x}$

(b) Using integration by parts, or otherwise, find (4)

$$\int (x-1)\sqrt{5-x}\,\mathrm{d}x.$$

Solution

$$u = x - 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = (5-x)^{\frac{1}{2}} \Rightarrow v = -\tfrac{2}{3}(5-x)^{\frac{3}{2}}.$$

Now,

$$\begin{aligned}\int (x-1)\sqrt{5-x}\,\mathrm{d}x &= -\tfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \int -\tfrac{2}{3}(5-x)^{\frac{3}{2}}\,\mathrm{d}x \\ &= \underline{\underline{-\tfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \tfrac{4}{15}(5-x)^{\frac{5}{2}} + c}}.\end{aligned}$$

(c) Hence find (2)

$$\int_1^5 (x-1)\sqrt{5-x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_1^5 (x-1)\sqrt{5-x}\,\mathrm{d}x &= \left[-\tfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \tfrac{4}{15}(5-x)^{\frac{5}{2}}\right]_{x=1}^5 \\ &= (0-0) - \left(0 - \tfrac{4}{15} \times 4^{\frac{5}{2}}\right) \\ &= \underline{\underline{8\tfrac{8}{15}}}.\end{aligned}$$

25. (a) Using the identity $\cos 2\theta \equiv 1 - 2\sin^2\theta$, find $\displaystyle\int \sin^2\theta\,\mathrm{d}\theta$. (2)

Solution

$$\int \sin^2\theta\,\mathrm{d}\theta = \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)\mathrm{d}\theta = \underline{\underline{\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + c}}.$$

Figure 10 shows part of the curve C with parametric equations

$$x = \tan\theta,\ y = 2\sin 2\theta,\ 0 \leqslant \theta < \tfrac{\pi}{2}.$$

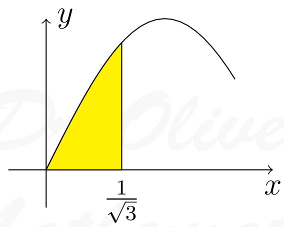

Figure 10: $x = \tan\theta,\ y = 2\sin 2\theta$

The region bounded R by the lines $x = \frac{1}{\sqrt{3}}$, the x-axis, and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis to generate a solid of revolution.

(b) Show that the volume of the solid of revolution formed is given by the integral (5)

$$k\int_0^{\frac{\pi}{6}} \sin^2\theta \,\mathrm{d}\theta,$$

where k is a constant.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$$

and

$$x = 0 \Rightarrow \theta = 0 \text{ and } x = \tfrac{1}{\sqrt{3}} \Rightarrow \theta = \tfrac{\pi}{6}.$$

$$\begin{aligned}
\text{Volume} &= \pi\int y^2 \,\mathrm{d}x \\
&= \pi\int y^2 \frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta \\
&= \pi\int_0^{\frac{\pi}{6}} (2\sin 2\theta)^2(\sec^2\theta)\,\mathrm{d}\theta \\
&= \pi\int_0^{\frac{\pi}{6}} (4\sin\theta\cos\theta)^2(\sec^2\theta)\,\mathrm{d}\theta \\
&= \underline{\underline{16\pi\int_0^{\frac{\pi}{6}} \sin^2\theta\,\mathrm{d}\theta.}}
\end{aligned}$$

(c) Hence find the exact value for this volume, giving your answer in the form $p\pi^2 + q\pi\sqrt{3}$, where p and q are constants. (3)

Solution

$$\begin{aligned}\text{Volume} &= 16\pi\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{\theta=0}^{\frac{\pi}{6}} \\ &= 16\pi\left[\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) - (0-0)\right] \\ &= \underline{\underline{\frac{4}{3}\pi^2 - 2\sqrt{3}\pi}}.\end{aligned}$$

26.

$$y = x\ln x,\ 1 \leqslant x \leqslant 4.$$

(a) Use integration by parts to find $\int x \ln x \, \mathrm{d}x$. (4)

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x \Rightarrow v = \frac{1}{2}x^2.$$

Now,

$$\begin{aligned}\int x\ln x\,\mathrm{d}x &= \tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + c}}.\end{aligned}$$

(b) Hence find the exact area of this integral, giving your answer in the form $\frac{1}{4}(a\ln 2 + b)$, where a and b are integers. (3)

Solution

$$\begin{aligned}\int_1^4 x\ln x\,\mathrm{d}x &= \left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_{x=1}^{4} \\ &= (8\ln 4 - 4) - \left(0 - \tfrac{1}{4}\right) \\ &= 16\ln 2 - \tfrac{15}{4} \\ &= \underline{\underline{\tfrac{1}{4}(64\ln 2 - 15)}}.\end{aligned}$$

27. Find $\displaystyle\int \frac{9x+6}{x}\,\mathrm{d}x$, $x > 0$. (2)

Solution

$$\int \frac{9x+6}{x}\,\mathrm{d}x = \int \left(9 + \frac{6}{x}\right)\mathrm{d}x$$
$$= \underline{\underline{9x + 6\ln x + c}},$$

because we know that $x > 0$.

28. Using the substitution $x = 2\cos u$, or otherwise, find the exact value of (7)

$$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x.$$

Solution

This is **not** a good substitution to use: we would have been better using

$$u = \cos^{-1}\tfrac{1}{2}x$$

would have been better (why?). Oh well.

$$x = 2\cos u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -2\sin u \Rightarrow \mathrm{d}x = -2\sin u\,\mathrm{d}u$$

and

$$x = 1 \Rightarrow u = \frac{\pi}{3} \text{ and } x = \sqrt{2} \Rightarrow u = \frac{\pi}{4}.$$

Now,

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x &= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4\cos^2 u\sqrt{4-4\cos^2 u}}(-2\sin u)\,\mathrm{d}u \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{-2\sin u}{8\sin u\cos^2 u}\,\mathrm{d}u \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{4\cos^2 u}\,\mathrm{d}u \\ &= \tfrac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 u\,\mathrm{d}u \\ &= \tfrac{1}{4}\left[\tan u\right]_{u=\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \underline{\underline{\tfrac{1}{4}(\sqrt{3}-1)}}.\end{aligned}$$

29. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} \mathrm{e}^{\cos x + 1}\sin x\,\mathrm{d}x = e(e-1).$$

Solution

$$u = \cos x + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \Rightarrow \mathrm{d}u = -\sin x\,\mathrm{d}x$$

and

$$x = 0 \Rightarrow u = 2 \text{ and } x = \tfrac{\pi}{2} \Rightarrow u = 1.$$

Now,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \mathrm{e}^{\cos x+1}\sin x\,\mathrm{d}x &= \int_2^1 (-\mathrm{e}^u)\,\mathrm{d}u \\ &= \left[\mathrm{e}^u\right]_{u=2}^{1} \\ &= -\mathrm{e} - (-\mathrm{e}^2) \\ &= \underline{\underline{e(e-1)}}.\end{aligned}$$

30.

$$\mathrm{f}(\theta) = 4\cos^2\theta - 3\sin^2\theta.$$

(a) Show that (3)

$$\mathrm{f}(\theta) = \tfrac{1}{2} + \tfrac{7}{2}\cos 2\theta.$$

Solution

$$\begin{aligned} 4\cos^2\theta - 3\sin^2\theta &= 4\cos^2\theta - 3(1-\cos^2\theta) \\ &= 7\cos^2\theta - 3 \\ &= 7(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta) - 3 \\ &= \underline{\underline{\tfrac{1}{2} + \tfrac{7}{2}\cos 2\theta}}. \end{aligned}$$

(b) Hence, using calculus, find the exact value of (7)

$$\int_0^{\frac{\pi}{2}} \theta\,\mathrm{f}(\theta)\,\mathrm{d}\theta.$$

Solution

$$\theta\,\mathrm{f}(\theta) = \tfrac{1}{2}\theta + \tfrac{7}{2}\theta\cos 2\theta$$

and

$$u = \tfrac{7}{2}\theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \tfrac{7}{2} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2\theta \Rightarrow v = \tfrac{1}{2}\sin 2\theta.$$

Now,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \theta\,\mathrm{f}(\theta)\,\mathrm{d}\theta &= \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2}\theta + \tfrac{7}{2}\theta\cos 2\theta\right)\,\mathrm{d}\theta \\ &= \left[\tfrac{1}{4}\theta^2 + \tfrac{7}{4}\theta\sin 2\theta\right]_{\theta=0}^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \tfrac{7}{4}\sin 2\theta\,\mathrm{d}\theta \\ &= \left[\tfrac{1}{4}\theta^2 + \tfrac{7}{4}\theta\sin 2\theta + \tfrac{7}{8}\cos 2\theta\right]_{\theta=0}^{\frac{\pi}{2}} \\ &= \left(\tfrac{\pi^2}{16} + 0 - \tfrac{7}{8}\right) - \left(0 + 0 + \tfrac{7}{8}\right) \\ &= \underline{\underline{\tfrac{\pi^2}{16} - \tfrac{7}{4}}}. \end{aligned}$$

31. Use integration to find the exact value of

$$\int_0^{\frac{\pi}{2}} x\sin 2x\,\mathrm{d}x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \Rightarrow v = -\tfrac{1}{2}\cos 2x$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x \sin 2x \,\mathrm{d}x &= \left[-\tfrac{1}{2}x\cos 2x\right]_{x=0}^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\tfrac{1}{2}\cos 2x)\,\mathrm{d}x \\ &= (\tfrac{\pi}{4} - 0) + \left[\tfrac{1}{4}\sin 2x\right]_{x=0}^{\frac{\pi}{2}} \\ &= \tfrac{\pi}{4} - (0 - 0) \\ &= \underline{\underline{\tfrac{\pi}{4}}}.\end{aligned}$$

32. (a) Express $\dfrac{5}{(x-1)(3x+2)}$ in partial fractions. (3)

Solution

$$\begin{aligned}\frac{5}{(x-1)(3x+2)} &\equiv \frac{A}{x-1} + \frac{B}{3x+2} \\ &\equiv \frac{A(3x+2) + B(x-1)}{(x-1)(3x+2)}\end{aligned}$$

and so

$$5 \equiv A(3x+2) + B(x-1).$$

$\underline{x = 1}$: $5 = 5A \Rightarrow A = 1$.
$\underline{x = -\tfrac{2}{3}}$: $5 = -\tfrac{5}{3}B \Rightarrow B = -3$.
Hence

$$\frac{5}{(x-1)(3x+2)} \equiv \frac{1}{x-1} - \frac{3}{3x+2}.$$

(b) Hence find $\displaystyle\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x$, where $x > 1$. (3)

Solution

$$\begin{aligned}\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x &= \int \left(\frac{1}{x-1} - \frac{3}{3x+2}\right)\mathrm{d}x \\ &= \underline{\underline{\ln|x-1| - \ln|3x+2| + c}},\end{aligned}$$

as $x > 1$.

33. The curve C has parametric equations

$$x = \ln t,\ y = t^2 - 2,\ t > 0.$$

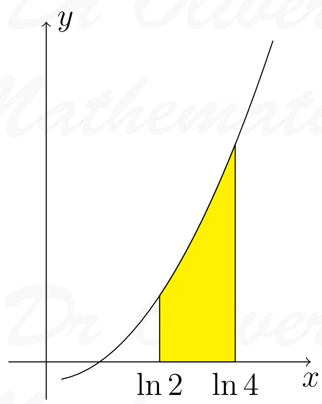

Figure 11: $x = \ln t$, $y = t^2 - 2$

The region bounded R by the lines $x = \ln 2$, $x = \ln 4$, the x-axis, and the curve is shaded in the figure. This region is rotated through 360° about the x-axis to generate a solid of revolution. Use calculus to find the exact volume of the solid generated.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t}$$

and

$$x = \ln 2 \Rightarrow t = 2 \text{ and } x = \ln 4 \Rightarrow t = 4.$$

$$\begin{aligned}
\text{Volume} &= \pi \int_{\ln 2}^{\ln 4} y^2 \, \mathrm{d}x \\
&= \pi \int_2^4 y^2 \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t \\
&= \pi \int_2^4 \frac{(t^2-2)^2}{t} \, \mathrm{d}t \\
&= \pi \int_2^4 \frac{(t^4-4t^2+4)}{t} \, \mathrm{d}t \\
&= \pi \int_2^4 \left(t^3 - 4t + \frac{4}{t} \right) \mathrm{d}t \\
&= \pi \left[\tfrac{1}{4}t^4 - 2t^2 + 4\ln t \right]_{t=2}^{4} \\
&= \pi \left((64 - 32 + 4\ln 4) - (4 - 8 + 4\ln 2) \right) \\
&= \underline{\underline{\pi(36 + 4\ln 2)}}.
\end{aligned}$$

34. Using the substitution $x = (u-1)^2 + 1$, or otherwise, and integrating, find the exact value of

$$\int_2^5 \frac{1}{4 + \sqrt{x-1}} \, \mathrm{d}x.$$

Solution

This is **not** a good substitution to use: we would have been better using

$$u = \sqrt{x-1} + 4$$

would have been better (why?).

$$x = (u-4)^2 + 1 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2(u-4) \Rightarrow \mathrm{d}x = 2(u-4)\,\mathrm{d}u$$

and

$$x = 2 \Rightarrow u = 5 \text{ and } x = 5 \Rightarrow u = 6.$$

Now,

$$\begin{aligned}\int_2^5 \frac{1}{4+\sqrt{x-1}}\,\mathrm{d}x &= \int_5^6 \frac{2(u-4)}{u}\,\mathrm{d}u \\ &= \int_5^6 \left(2-\frac{8}{u}\right)\mathrm{d}u \\ &= \left[2u - 8\ln|u|\right]_{u=5}^6 \\ &= (12-8\ln 6)-(10-8\ln 5) \\ &= \underline{\underline{2+8\ln\tfrac{5}{6}}}.\end{aligned}$$

35. Figure 12 shows a sketch of the curve with equation $y = x^3\ln(x^2+2)$, $x \geqslant 0$.

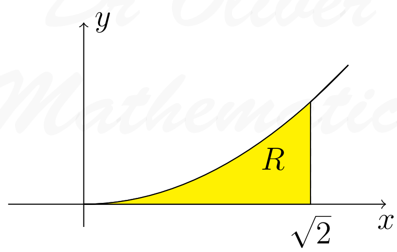

Figure 12: $y = x^3\ln(x^2+2)$

The finite region R, shown shaded in the figure, is bounded by the curve, the x-axis, and the line $x = \sqrt{2}$.

(a) Use the substitution $u = x^2+2$ to show that the area of R is (4)

$$\tfrac{1}{2}\int_2^4 (u-2)\ln 2\,\mathrm{d}u.$$

Solution

$$u = x^2+2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \tfrac{1}{2}\,\mathrm{d}u = x\,\mathrm{d}x$$

and

$$x = 0 \Rightarrow u = 2 \text{ and } x = \sqrt{2} \Rightarrow u = 4.$$

Now,

$$\int_0^{\sqrt{2}} x^3 \ln(x^2+2)\,\mathrm{d}x = \int_0^{\sqrt{2}} x^2 \ln(x^2+2)x\,\mathrm{d}x$$
$$= \underline{\underline{\tfrac{1}{2}\int_2^4 (u-2)\ln u\,\mathrm{d}u.}}$$

(b) Hence, or otherwise, find the exact area of R. (6)

Solution

$$w = \ln u \Rightarrow \frac{\mathrm{d}w}{\mathrm{d}u} = \frac{1}{u} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}u} = u-2 \Rightarrow v = \tfrac{1}{2}u^2 - 2u.$$

Hence,

$$\begin{aligned}\int_0^{\sqrt{2}} x^3 \ln(x^2+2)\,\mathrm{d}x &= \tfrac{1}{2}\int_2^4 (u-2)\ln u\,\mathrm{d}u \\ &= \tfrac{1}{2}\left\{\left[(\tfrac{1}{2}u^2-2u)\ln u\right]_{u=2}^4 - \int_2^4 \frac{1}{u}(\tfrac{1}{2}u^2-2u)\,\mathrm{d}u\right\} \\ &= \tfrac{1}{2}\left\{(0-(-2\ln 2)) - \int_2^4 (\tfrac{1}{2}u-2)\,\mathrm{d}u\right\} \\ &= \tfrac{1}{2}\left\{2\ln 2 - \left[\tfrac{1}{4}u^2-2u\right]_{u=2}^4\right\} \\ &= \tfrac{1}{2}\left\{2\ln 2 - (-4-(-3))\right\} \\ &= \tfrac{1}{2}(2\ln 2+1) \\ &= \underline{\underline{\ln 2 + \tfrac{1}{2}}}.\end{aligned}$$

36. Figure 13 shows part of the curve C with parametric equations

$$x = \tan t,\ y = \sin t,\ 0 \leqslant t < \tfrac{\pi}{2}.$$

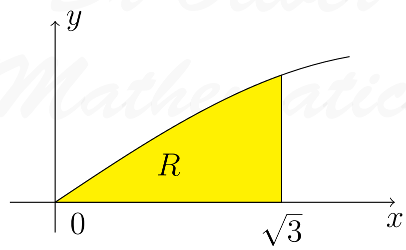

Figure 13: $x = \tan t$, $y = \sin t$

The point P lies on C and has coordinates $(\sqrt{3}, \frac{1}{3}\sqrt{3})$.

(a) Find the value of t at the point P. (2)

Solution

$$y = \tfrac{1}{3}\sqrt{3} \Rightarrow \tan t = \tfrac{1}{3}\sqrt{3} \Rightarrow t = \underline{\underline{\tfrac{\pi}{3}}}.$$

The region bounded R by the lines $x = \sqrt{3}$, the x-axis, and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis to generate a solid of revolution.

(b) Find the exact value for the solid of revolution, giving your answer in the form $p\pi\sqrt{3} + q\pi^2$, where p and q are constants. (7)

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sec^2 t \text{ and } x = 0 \Rightarrow t = 0.$$

Now,

$$\begin{aligned}
\text{Volume} &= \pi \int_0^{\sqrt{3}} y^2 \, \mathrm{d}x \\
&= \pi \int_0^{\frac{\pi}{3}} y^2 \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t \\
&= \pi \int_0^{\frac{\pi}{3}} \sin^2 t \sec^2 t \, \mathrm{d}t \\
&= \pi \int_0^{\frac{\pi}{3}} \tan^2 t \, \mathrm{d}t \\
&= \pi \int_0^{\frac{\pi}{3}} (\sec^2 t - 1) \, \mathrm{d}t \\
&= \pi \left[\tan t - t\right]_{t=0}^{\frac{\pi}{3}} \\
&= \pi \left[\left(\sqrt{3} - \tfrac{\pi}{3}\right) - (0 - 0)\right] \\
&= \underline{\underline{\sqrt{3}\pi - \tfrac{1}{3}\pi^2}}.
\end{aligned}$$

37. Find $\displaystyle\int (4y+3)^{-\frac{1}{2}} \, \mathrm{d}y$. (2)

Solution

$$\int (4y+3)^{-\frac{1}{2}} \, \mathrm{d}y = \underline{\underline{\tfrac{1}{2}(4y+3)^{\frac{1}{2}} + c}}.$$

38. (a) Use integration by parts to find $\displaystyle\int x \sin 3x \, \mathrm{d}x$. (3)

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 3x \Rightarrow v = -\tfrac{1}{3}\cos 3x.$$

Now,

$$\begin{aligned}
\int x \sin 3x \, \mathrm{d}x &= -\tfrac{1}{3} x \cos 3x + \tfrac{1}{3} \int \cos 3x \, \mathrm{d}x \\
&= \underline{\underline{-\tfrac{1}{3} x \cos 3x + \tfrac{1}{9} \sin 3x \, \mathrm{d}x}}.
\end{aligned}$$

(b) Using your answer to part (a), find $\int x^2 \cos 3x \, \mathrm{d}x$. (4)

Solution

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 3x \Rightarrow v = \tfrac{1}{3}\sin 3x.$$

Now,

$$\begin{aligned}\int x^2 \cos 3x \, \mathrm{d}x &= \tfrac{1}{3}x^2 \sin 3x - \int \tfrac{2}{3}x \sin 3x \, \mathrm{d}x \\ &= \tfrac{1}{3}x^2 \sin 3x - \tfrac{2}{3}\left(-\tfrac{1}{3}x \cos 3x + \tfrac{1}{9}\sin 3x\right) + c \\ &= \underline{\underline{\tfrac{1}{3}x^2 \sin 3x + \tfrac{2}{9}x \cos 3x - \tfrac{2}{27}\sin 3x + c}}.\end{aligned}$$

39. Figure 14 shows the curve with equation (5)

$$y = \sqrt{\frac{2x}{3x^2+4}}, \; x \geqslant 0.$$

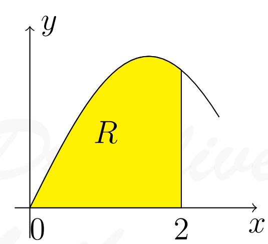

Figure 14: $y = \sqrt{\dfrac{2x}{3x^2+4}}$

The region bounded R by the lines $x = 2$, the x-axis, and the curve is shaded in the figure. This region is rotated through 360° about the x-axis to generate a solid of revolution. Find the exact value for the solid of revolution, giving your answer in the form $k \ln a$, where k and a are constants.

Solution

$$\begin{aligned}
\text{Volume} &= \pi \int_0^2 y^2 \, \mathrm{d}x \\
&= \pi \int_0^2 \frac{2x}{3x^2+4} \, \mathrm{d}x \\
&= \tfrac{1}{3}\pi \int_0^2 \frac{6x}{3x^2+4} \, \mathrm{d}x \\
&= \tfrac{1}{3} \left[\ln |3x^2+4| \right]_{x=0}^{2} \\
&= \tfrac{1}{3}(\ln 16 - \ln 4) \\
&= \underline{\underline{\tfrac{1}{3}\ln 3}}.
\end{aligned}$$

40. Use the substitution $u = 1 + \cos x$, or otherwise, show that (5)

$$\int \frac{2\sin 2x}{1+\cos x} \, \mathrm{d}x = 4\ln(1+\cos x) - 4\cos x + k,$$

where k is a constant.

Solution

$$u = 1 + \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \Rightarrow \mathrm{d}u = -\sin x \, \mathrm{d}x$$

and

$$\begin{aligned}
\int \frac{2\sin 2x}{1+\cos x} \, \mathrm{d}x &= \int \frac{4\sin x \cos x}{1+\cos x} \, \mathrm{d}x \\
&= \int \frac{-4(u-1)}{u} \, \mathrm{d}u \\
&= \int \left(\frac{4}{u} - 4 \right) \mathrm{d}u \\
&= 4\ln u - 4u + c \\
&= 4\ln(1+\cos x) - 4(1+\cos x) + c \\
&= \underline{\underline{4\ln(1+\cos x) - 4\cos x + k}}.
\end{aligned}$$

41.

$$\mathrm{f}(x) = \frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{3x-1} + \frac{C}{(3x-1)^2}.$$

(a) Find the value of the constants A, B, and C. (4)

Solution

$$\frac{1}{x(3x-1)^2} \equiv \frac{A}{x} + \frac{B}{3x-1} + \frac{C}{(3x-1)^2}$$
$$= \frac{A(3x-1)^2 + Bx(3x-1) + Cx}{x(3x-1)^2}$$

and so

$$1 \equiv A(3x-1)^2 + Bx(3x-1) + Cx.$$

$\underline{x = 0}$: $1 = A$.
$\underline{x = \frac{1}{3}}$: $1 = \frac{1}{3}C \Rightarrow C = 3$.
$\underline{x = 1}$: $1 = 4A + 2B + C \Rightarrow B = -3$.
Hence

$$\frac{1}{x(3x-1)^2} \equiv \underline{\underline{\frac{1}{x} - \frac{3}{3x-1} + \frac{3}{(3x-1)^2}}}.$$

(b) Hence find $\int \mathrm{f}(x)\,\mathrm{d}x$. (3)

Solution

$$\int \mathrm{f}(x)\,\mathrm{d}x = \underline{\underline{\ln|x| - \ln|3x-1| - (3x-1)^{-1} + c.}}$$

(c) Find $\int_1^2 \mathrm{f}(x)\,\mathrm{d}x$, leaving your answer in the form $a + \ln b$, where a and b are constants. (3)

Solution

$$\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = \left[\ln|x| - \ln|3x-1| - (3x-1)^{-1}\right]_{x=1}^2$$
$$= \left(\ln 2 - \ln 5 - \tfrac{1}{5}\right) - \left(0 - \ln 2 - \tfrac{1}{2}\right)$$
$$= 2\ln 2 - \ln 5 + \tfrac{3}{10}$$
$$= \underline{\underline{\tfrac{3}{10} + \ln\tfrac{4}{5}}}.$$

42. Figure 15 shows a sketch of part of the curve with equation $y = x^{\frac{1}{2}}\ln 2x$.

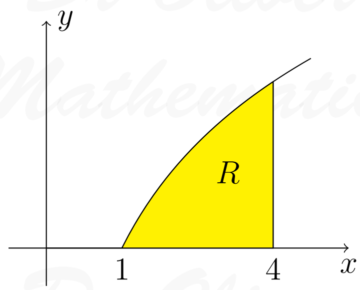

Figure 15: $y = x^{\frac{1}{2}} \ln 2x$

The finite region R, shown shaded in the figure, is bounded by the curve, the x-axis, and the lines $x = 1$ and $x = 4$.

(a) Find $\int x^{\frac{1}{2}} \ln 2x \, \mathrm{d}x$. (4)

Solution

$$u = \ln 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x^{\frac{1}{2}} \Rightarrow v = \tfrac{2}{3}x^{\frac{3}{2}}.$$

Now,

$$\begin{aligned} \int x^{\frac{1}{2}} \ln 2x \, \mathrm{d}x &= \tfrac{2}{3}x^{\frac{3}{2}} \ln 2x - \int \tfrac{2}{3}x^{\frac{1}{2}} \, \mathrm{d}x \\ &= \underline{\underline{\tfrac{2}{3}x^{\frac{3}{2}} \ln 2x - \tfrac{4}{9}x^{\frac{3}{2}} + c}}. \end{aligned}$$

(b) Hence find the exact area of R, giving your answer in the form $a \ln 2 + b$, where a and b are exact constants. (3)

Solution

$$\begin{aligned} \int_1^4 x^{\frac{1}{2}} \ln 2x \, \mathrm{d}x &= \left[\tfrac{2}{3}x^{\frac{3}{2}} \ln 2x - \tfrac{4}{9}x^{\frac{3}{2}} \right]_{x=1}^{4} \\ &= \left(\tfrac{16}{3} \ln 8 - \tfrac{32}{9}\right) - \left(\tfrac{2}{3} \ln 2 - \tfrac{32}{9}\right) \\ &= \tfrac{48}{3} \ln 2 - \tfrac{2}{3} \ln 2 - \tfrac{28}{9} \\ &= \underline{\underline{\tfrac{46}{3} \ln 2 - \tfrac{28}{9}}}. \end{aligned}$$

43. (a) Use integration to find (5)

$$\int \frac{1}{x^3} \ln x \, \mathrm{d}x.$$

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x^{-3} \Rightarrow v = -\tfrac{1}{2}x^{-2}.$$

Now,

$$\begin{aligned}\int \frac{1}{x^3}\ln x \,\mathrm{d}x &= -\tfrac{1}{2}x^{-2}\ln x + \int \tfrac{1}{2}x^{-3}\,\mathrm{d}x \\ &= \underline{\underline{-\tfrac{1}{2}x^{-2}\ln x - \tfrac{1}{4}x^{-2} + c}}.\end{aligned}$$

(b) Hence calculate (2)

$$\int_1^2 \frac{1}{x^3}\ln x \,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_1^2 \frac{1}{x^3}\ln x \,\mathrm{d}x &= \left[-\tfrac{1}{2}x^{-2}\ln x - \tfrac{1}{4}x^{-2}\right]_{x=1}^{2} \\ &= \left(-\tfrac{1}{8}\ln 2 - \tfrac{1}{16}\right) - \left(0 - \tfrac{1}{4}\right) \\ &= \underline{\underline{\tfrac{3}{16} - \tfrac{1}{8}\ln 2}}.\end{aligned}$$

44. Figure 16 shows a sketch of part of the curve with equation $y = \dfrac{x}{1+\sqrt{x}}$. (8)

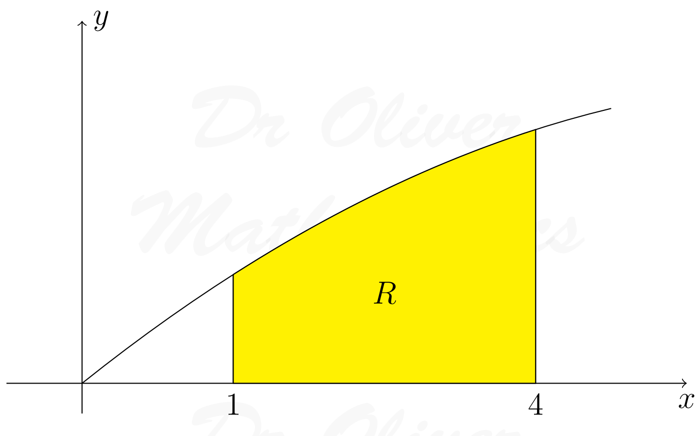

Figure 16: $y = \dfrac{x}{1+\sqrt{x}}$

The finite region R, shown shaded in the figure, is bounded by the curve, the x-axis, and the lines $x = 1$ and $x = 4$. Use the substitution $u = 1 + \sqrt{x}$, to find, by integrating, the exact area of R.

Solution

$$u = 1 + \sqrt{x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \tfrac{1}{2}x^{-\frac{1}{2}} \Rightarrow 2\,\mathrm{d}u = x^{-\frac{1}{2}}\,\mathrm{d}x$$

and

$$x = 1 \Rightarrow u = 2 \text{ and } x = 4 \Rightarrow u = 3.$$

$$\begin{aligned}
\int_1^4 \frac{x}{1+\sqrt{x}}\,\mathrm{d}x &= \int_1^4 \frac{x^{\frac{3}{2}}}{1+\sqrt{x}}\frac{1}{x^{\frac{1}{2}}}\,\mathrm{d}x \\
&= \int_2^3 \frac{(u-1)^2}{u} 2(u-1)\,\mathrm{d}u \\
&= 2\int_2^3 \frac{(u-1)^3}{u}\,\mathrm{d}u \\
&= 2\int_2^3 \frac{(u^3-3u^2+3u-1)}{u}\,\mathrm{d}u \\
&= 2\int_2^3 \left(u^2 - 3u + 3 - \frac{1}{u}\right)\mathrm{d}u \\
&= 2\left[\tfrac{1}{3}u^3 - \tfrac{3}{2}u^2 + 3u - \ln|u|\right]_{u=2}^{3} \\
&= 2\left\{\left(9 - \tfrac{27}{2} + 9 - \ln 3\right) - \left(\tfrac{8}{3} - 6 + 6 - \ln 2\right)\right\} \\
&= 2\left(\tfrac{11}{6} - \ln 3 + \ln 2\right) \\
&= \underline{\underline{\tfrac{11}{3} - 2\ln\tfrac{2}{3}}}.
\end{aligned}$$

45. Figure 17 shows a sketch of part of the curve with parametric equations

$$x = 1 - \tfrac{1}{2}t,\ y = 2^t - 1.$$

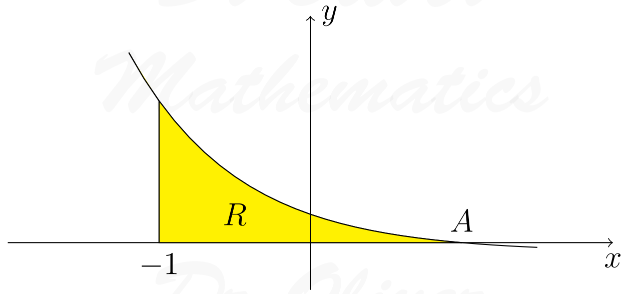

Figure 17: $x = 1 - \frac{1}{2}t$, $y = 2^t - 1$

(a) Find the x-coordinate of the point A. (2)

Solution

$$y = 0 \Rightarrow 2^t - 1 = 0 \Rightarrow t = 0 \Rightarrow \underline{\underline{x = 1}}.$$

The finite region R, shown shaded in the figure, is bounded by the curve C, the x-axis, and the line $x = -1$.

(b) Use integration to find exact area of R. (6)

Solution

$$x = 1 - \tfrac{1}{2}t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = -\tfrac{1}{2}$$

and

$$x = -1 \Rightarrow t = 4 \text{ and } x = 1 \Rightarrow t = 0.$$

Hence

$$\begin{aligned}
\text{Area of } R &= \int_4^0 (2^t - 1)(-\tfrac{1}{2})\,\mathrm{d}t \\
&= -\tfrac{1}{2}\left[\frac{2^t}{\ln 2} - t\right]_{t=4}^{0} \\
&= -\tfrac{1}{2}\left[\left(\frac{1}{\ln 2} - 0\right) - \left(\frac{16}{\ln 2} - 4\right)\right] \\
&= -\tfrac{1}{2}\left(\frac{15}{\ln 2} + 4\right) \\
&= \underline{\underline{\frac{15}{2\ln 2} - 2}}.
\end{aligned}$$

46. Figure 18 shows a sketch of part of the curve with equation $y = 1 - 2\cos x$. The curve

crosses the x-axis at A and B.

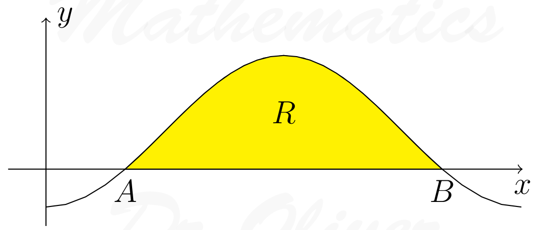

Figure 18: $y = 1 - 2\cos x$

(a) Find, in terms of π, the x-coordinate of the point A and of the point B. (3)

Solution

$$y = 0 \Rightarrow 1 - 2\cos x = 0 \Rightarrow \cos x = \tfrac{1}{2} \Rightarrow x = \underline{\underline{\tfrac{\pi}{3} \text{ or } \tfrac{5\pi}{3}}}.$$

The region bounded R by the lines the x-axis and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis to generate a solid of revolution.

(b) Find, by integration, the exact value for the volume of the solid generated. (6)

Solution

$$\begin{aligned}
\text{Volume} &= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos x)^2 \, \mathrm{d}x \\
&= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 4\cos x + 4\cos^2 x) \, \mathrm{d}x \\
&= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 4\cos x + 4(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)) \, \mathrm{d}x \\
&= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (3 - 4\cos x + 2\cos 2x) \, \mathrm{d}x \\
&= \pi \left[3x - 4\sin x + \sin 2x\right]_{x=\frac{\pi}{3}}^{\frac{5\pi}{3}} \\
&= \pi \left[\left(5\pi + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) - \left(\pi - 2\sqrt{3} + \tfrac{\sqrt{3}}{2}\right)\right] \\
&= \underline{\underline{\pi(4\pi + 3\sqrt{3})}}.
\end{aligned}$$

47. (a) Find $\displaystyle\int x^2 \mathrm{e}^x \, \mathrm{d}x$. (5)

Solution

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \mathrm{e}^x$$

and

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \mathrm{e}^x.$$

Now,

$$\begin{aligned}\int x^2 \mathrm{e}^x \, \mathrm{d}x &= x^2 \mathrm{e}^x - \int 2x\mathrm{e}^x \, \mathrm{d}x \\ &= x^2\mathrm{e}^x - (2x\mathrm{e}^x - \int 2\mathrm{e}^x \, \mathrm{d}x) \\ &= \underline{\underline{x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + c}}.\end{aligned}$$

(b) Hence find the exact value of $\displaystyle\int_0^1 x^2\mathrm{e}^x \, \mathrm{d}x$. (2)

Solution

$$\begin{aligned}\int_0^1 x^2\mathrm{e}^x \, \mathrm{d}x &= \left[x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x\right]_{x=0}^1 \\ &= (\mathrm{e} - 2\mathrm{e} + 2\mathrm{e}) - (0 - 0 + 2) \\ &= \underline{\underline{\mathrm{e} - 2}}.\end{aligned}$$

48. Figure 19 shows a sketch of part of the curve with equation (4)

$$y = \sec(\tfrac{1}{2}x), \; 0 \leqslant x \leqslant \tfrac{\pi}{2}.$$

The curve crosses the x-axis at A and B.

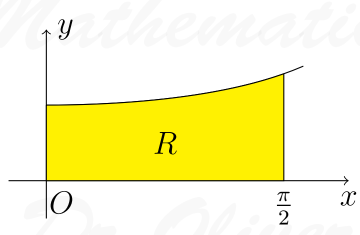

Figure 19: $y = \sec(\frac{1}{2}x)$

The region bounded R by the lines the x-axis, the y-axis, the line $x = \frac{\pi}{2}$, and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis to generate a solid of revolution. Find, by integration, the exact value for the volume of the solid generated.

Solution

$$\begin{aligned}\text{Volume} &= \pi \int_0^{\frac{\pi}{2}} \sec^2(\tfrac{1}{2}x)\,\mathrm{d}x \\ &= \pi \left[2\tan(\tfrac{1}{2}x)\right]_{x=0}^{\frac{\pi}{2}} \\ &= \pi(2 - 0) \\ &= \underline{\underline{2\pi}}.\end{aligned}$$

49. (a) Use the substitution $x = u^2$, $u > 0$, to show that (3)

$$\int \frac{1}{x(2\sqrt{x} - 1)}\,\mathrm{d}x = \int \frac{2}{u(2u - 1)}\,\mathrm{d}u.$$

Solution

This is **not** a good substitution to use: we would have been better using $u = \sqrt{x}$ would have been better (why?). Oh, well.

$$x = u^2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2u \Rightarrow \mathrm{d}x = 2u\,\mathrm{d}u$$

and

$$\begin{aligned}\int \frac{1}{x(2\sqrt{x} - 1)}\,\mathrm{d}x &= \int \frac{1}{u^2(2u - 1)} 2u\,\mathrm{d}u \\ &= \underline{\underline{\int \frac{2}{u(2u - 1)}\,\mathrm{d}u}}.\end{aligned}$$

(b) Hence show that (7)

$$\int_1^9 \frac{1}{x(2\sqrt{x} - 1)}\,\mathrm{d}x = 2\ln\left(\frac{a}{b}\right),$$

where a and b are integers to be determined.

Solution

$$\frac{2}{u(2u-1)} \equiv \frac{A}{u} + \frac{B}{2u-1} \equiv \frac{A(2u-1)+Bu}{u(2u-1)}$$

and

$$2 \equiv A(2u-1) + Bu.$$

$\underline{x = \frac{1}{2}}$: $2 = \frac{1}{2}B \Rightarrow B = 4$.
$\underline{x = 0}$: $2 = -A \Rightarrow A = -2$.
So,

$$\frac{2}{u(2u-1)} \equiv -\frac{2}{u} + \frac{4}{2u-1}.$$

$$x = 1 \Rightarrow u = 1 \text{ and } x = 9 \Rightarrow u = 3$$

and

$$\begin{aligned}\int_1^9 \frac{1}{x(2\sqrt{x}-1)}\,\mathrm{d}x &= \int_1^3 \left(-\frac{2}{u} + \frac{4}{2u-1}\right)\mathrm{d}x \\ &= \left[-2\ln|u| + 2\ln|2u-1|\right]_{u=1}^{3} \\ &= (2\ln 3 + 2\ln 5) - (0+0) \\ &= \underline{\underline{2\ln\tfrac{5}{3}}}.\end{aligned}$$

50. Using the substitution $u = 2 + \sqrt{2x+1}$, or other suitable substitutions, find the exact value of (8)

$$\int_0^4 \frac{1}{2+\sqrt{2x+1}}\,\mathrm{d}x,$$

giving your answer in the form $A + B\ln 2$, where A is an integer and B is a positive constant.

Solution

$$u = 2 + \sqrt{2x+1} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (2x+1)^{-\frac{1}{2}} \Rightarrow \mathrm{d}u = (2x+1)^{-\frac{1}{2}}\,\mathrm{d}x$$

and

$$x = 0 \Rightarrow u = 3 \text{ and } x = 4 \Rightarrow u = 5.$$

Now,

$$\begin{aligned}\int_0^4 \frac{1}{2+\sqrt{2x+1}}\,\mathrm{d}x &= \int_0^4 \frac{(2x+1)^{\frac{1}{2}}}{2+\sqrt{2x+1}}(2x+1)^{-\frac{1}{2}}\,\mathrm{d}x \\ &= \int_3^5 \frac{u-2}{u}\,\mathrm{d}u \\ &= \int_3^5 \left(1-\frac{2}{u}\right)\mathrm{d}u \\ &= \left[u - 2\ln|u|\right]_{u=3}^{5} \\ &= (5-2\ln 5)-(3-2\ln 3) \\ &= 2-2\ln 5+2\ln 3 \\ &= \underline{\underline{2+2\ln\tfrac{3}{5}}}.\end{aligned}$$

51. Figure 20 shows a sketch of part of the curve with equation $x = 4t\mathrm{e}^{-\frac{1}{3}t} + 3$. The region bounded R by the lines the x-axis, the t-axis, the line $t = 8$, and the curve is shaded in the figure. (6)

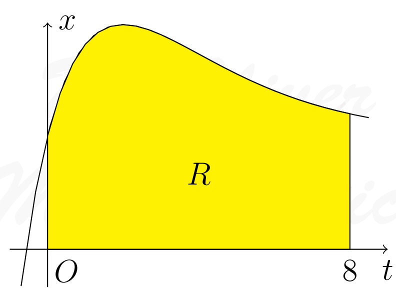

Figure 20: $x = 4t\mathrm{e}^{-\frac{1}{3}t} + 3$

Use calculus to find the exact value for the area of R.

Solution

$$u = 4t \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 4 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{e}^{-\frac{1}{3}t} \Rightarrow v = -3\mathrm{e}^{-\frac{1}{3}t}.$$

Now,

$$\int (4te^{-\frac{1}{3}t} + 3)\,\mathrm{d}t = 3t - 12te^{-\frac{1}{3}t} + \int 12e^{-\frac{1}{3}t}\,\mathrm{d}t$$
$$= 3t - 12te^{-\frac{1}{3}t} - 36e^{-\frac{1}{3}t} + c$$

and

$$\int_0^8 (4te^{-\frac{1}{3}t} + 3)\,\mathrm{d}t = \left[3t - 12te^{-\frac{1}{3}t} - 36e^{-\frac{1}{3}t}\right]_{t=0}^{8}$$
$$= \left(24 - 96e^{-\frac{8}{3}} - 36e^{-\frac{8}{3}}\right) - (0 - 0 - 36)$$
$$= \underline{\underline{60 - 132e^{-\frac{8}{3}}}}.$$

52. Figure 21 shows a sketch of part of the curve C with parametric equations (5)

$$x = 27\sec^3 t,\ y = 3\tan t,\ 0 \leqslant t \leqslant \tfrac{\pi}{3}.$$

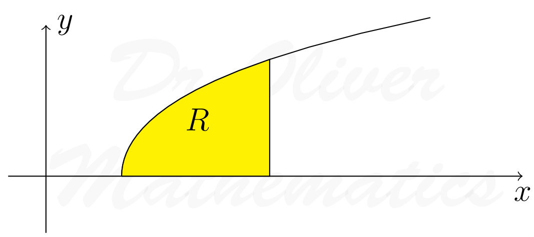

Figure 21: $x = 27\sec^3 t$, $y = 3\tan t$

(a) Show that the cartesian equation of C may be written in the form (3)

$$t = \left(x^{\frac{2}{3}} - 9\right)^{\frac{1}{2}},\ a \leqslant x \leqslant b,$$

stating the values of a and b.

Solution

Using $1 + \tan^2 t \equiv \sec^2 t$, we have

$$\begin{aligned} 1 + \left(\frac{y}{3}\right)^2 = \left(\left(\frac{x}{27}\right)^{\frac{1}{3}}\right)^2 &\Rightarrow 1 + \frac{y^2}{9} = \left(\frac{x^{\frac{1}{3}}}{3}\right)^2 \\ &\Rightarrow 1 + \frac{y^2}{9} = \frac{x^{\frac{2}{3}}}{9} \\ &\Rightarrow \frac{y^2}{9} = \frac{x^{\frac{2}{3}}}{9} - 1 \\ &\Rightarrow y^2 = x^{\frac{2}{3}} - 9 \\ &\Rightarrow \underline{\underline{y = (x^{\frac{2}{3}} - 9)^{\frac{1}{2}}}}, \end{aligned}$$

and $a = \underline{\underline{27}}$ and $b = \underline{\underline{216}}$

The region bounded R by the lines the x-axis, the line $x = 125$, and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis to generate a solid of revolution.

(b) Find, by integration, the exact value for the volume of the solid generated. (5)

Solution

$$\begin{aligned} \text{Volume} &= \pi \int_{27}^{125} (x^{\frac{2}{3}} - 9)\, \mathrm{d}x \\ &= \pi \left[\tfrac{3}{5}x^{\frac{5}{3}} - 9x\right]_{x=27}^{125} \\ &= \pi \left[(1875 - 1125) - \left(\tfrac{729}{5} - 243\right)\right] \\ &= \underline{\underline{\tfrac{4236\pi}{5}}}. \end{aligned}$$

53. U]se the substitution $u = \sqrt{x}$, or otherwise, to find the exact value of

$$\int_1^4 \frac{10}{2x + 5\sqrt{x}}\, \mathrm{d}x.$$

Solution

$$u = \sqrt{x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \tfrac{1}{2}x^{-\frac{1}{2}} \Rightarrow 2u\,\mathrm{d}u = \mathrm{d}x$$

and

$$x = 1 \Rightarrow u = 1 \text{ and } x = 4 \Rightarrow u = 2.$$

Now,

$$\begin{aligned}\int_1^4 \frac{10}{2x + 5\sqrt{x}}\,\mathrm{d}x &= \int_1^2 \frac{20u}{2u^2 + 5u}\,\mathrm{d}u \\ &= \int_1^2 \frac{20}{2u + 5}\,\mathrm{d}u \\ &= \left[10\ln|2u + 5|\right]_{u=1}^2 \\ &= \underline{\underline{10\ln 9 - 10\ln 7}} \text{ or } \underline{\underline{10\ln(\tfrac{9}{7})}}.\end{aligned}$$

54. (a) Find $\int x\mathrm{e}^{4x}\,\mathrm{d}x$. (3)

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{4x} \Rightarrow v = \tfrac{1}{4}\mathrm{e}^{4x}$$

and

$$\begin{aligned}\int x\mathrm{e}^{4x}\,\mathrm{d}x &= \tfrac{1}{4}x\mathrm{e}^{4x} - \int \tfrac{1}{4}\mathrm{e}^{4x}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{4}x\mathrm{e}^{4x} - \tfrac{1}{16}\mathrm{e}^{4x} + c}}.\end{aligned}$$

(b) Find $\int \frac{8}{(2x-1)^3}\,\mathrm{d}x$, $x > \frac{1}{2}$. (2)

Solution

$$\int \frac{8}{(2x-1)^3}\,\mathrm{d}x = \underline{\underline{-\frac{2}{(2x-1)^2} + c}}.$$

55. Figure 22 shows a sketch of part of the curve C with parametric equations

$$x = 3\tan\theta,\ y = 4\cos^2\theta,\ 0 \leqslant \theta < \tfrac{\pi}{2}.$$

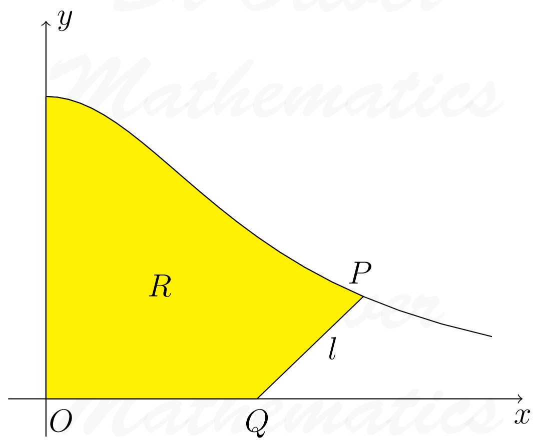

Figure 22: $x = 3\tan\theta$, $y = 4\cos^2\theta$

The point P lies on C and has coordinates $(3, 2)$. The line l is the normal to C at P. The normal cuts the x-axis at point Q.

(a) Find the x-coordinate of the point Q. (6)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-8\cos\theta\sin\theta}{3\sec^2\theta} = -\tfrac{8}{3}\cos^3\theta\sin\theta$$

and at $(3, 2)$ we have

$$\theta = \frac{\pi}{4} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}.$$

Now,

$$y - 2 = \tfrac{3}{2}(x - 3)$$

and

$$y = 0 \Rightarrow -2 = \tfrac{3}{2}(x-3) \Rightarrow -\tfrac{4}{3} = x - 3 \Rightarrow \underline{\underline{x = 1\tfrac{2}{3}}}.$$

The region bounded R by the lines the x-axis, the line y-axis, the line l, and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis to generate a solid of revolution.

(b) Find the exact value for the solid of revolution, giving your answer in the form $p\pi + q\pi^2$, where p and q are rational numbers to be determined. (9)

Solution

$$
\begin{aligned}
V_{\text{rev}} &= \pi \int y^2 \,\mathrm{d}x \\
&= \pi \int_0^{\frac{\pi}{4}} y^2 \frac{\mathrm{d}x}{\mathrm{d}\theta} \,\mathrm{d}\theta \\
&= \pi \int_0^{\frac{\pi}{4}} (4\cos^2\theta)^2 (3\sec^2\theta) \,\mathrm{d}\theta \\
&= \pi \int_0^{\frac{\pi}{4}} 48\cos^2\theta \,\mathrm{d}\theta \\
&= \pi \int_0^{\frac{\pi}{4}} 48(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta) \,\mathrm{d}\theta \\
&= \pi \int_0^{\frac{\pi}{4}} (24 + 24\cos 2\theta) \,\mathrm{d}\theta \\
&= \pi \left[24\theta + 12\sin 2\theta\right]_{\theta=0}^{\frac{\pi}{4}} \\
&= \pi \left\{(6\pi + 12) - (0 + 0)\right\} \\
&= \pi(6\pi + 12).
\end{aligned}
$$

Now,

$$V_{\text{cone}} = \tfrac{1}{3}\pi \times 2^2 \times (3 - \tfrac{5}{3}) = \tfrac{16}{9}\pi$$

and

$$V = V_{\text{rev}} - V_{\text{cone}} = (6\pi^2 + 12\pi) - \tfrac{16}{9}\pi = \underline{\underline{\tfrac{92}{9}\pi + 6\pi^2}}.$$

56. Figure 23 shows a sketch of part of the curve C with equation (5)

$$y = (2 - x)\mathrm{e}^{2x},\ x \in \mathbb{R}.$$

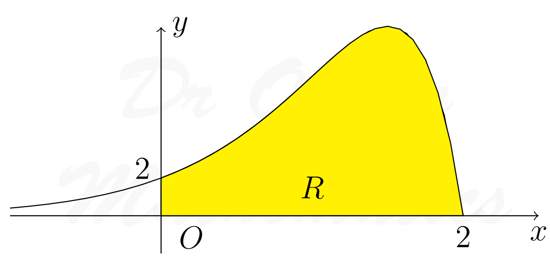

Figure 23: $y = (2 - x)\mathrm{e}^{2x}$

The region bounded R by the lines the x-axis, the line y-axis, and the curve is shaded in the figure. Use calculus, showing each step in your working, to obtain an exact value for the area of R. Give your answer in its simplest form.

Solution

$$u = 2 - x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x}$$

and

$$\begin{aligned}\int y\,\mathrm{d}x &= \tfrac{1}{2}(2-x)\mathrm{e}^{2x} + \int \tfrac{1}{2}\mathrm{e}^{2x}\,\mathrm{d}x \\ &= \tfrac{1}{2}(2-x)\mathrm{e}^{2x} + \tfrac{1}{4}\mathrm{e}^{2x} + c.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^2 y\,\mathrm{d}x &= \left[\tfrac{1}{2}(2-x)\mathrm{e}^{2x} + \tfrac{1}{4}\mathrm{e}^{2x}\right]_{x=0}^{2} \\ &= \left(0 + \tfrac{1}{4}\mathrm{e}^4\right) - \left(1 + \tfrac{1}{4}\right) \\ &= \underline{\underline{\tfrac{1}{4}\mathrm{e}^4 - \tfrac{5}{4}}}.\end{aligned}$$

57. (a) Express $\dfrac{25}{x^2(2x+1)}$ in partial fractions. (4)

Solution

$$\begin{aligned}\frac{25}{x^2(2x+1)} &\equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+1} \\ &\ \frac{Ax(2x+1) + B(2x+1) + Cx^2}{x^2(2x+1)}\end{aligned}$$

and so we have

$$25 \equiv Ax(2x+1) + B(2x+1) + Cx^2.$$

$\underline{\underline{x = 0}}$: $25 = B$.

$\underline{\underline{x = -\tfrac{1}{2}}}$: $25 = \tfrac{1}{4}C \Rightarrow C = 100$.

$\underline{\underline{x = 1}}$: $25 = 3A + 3B + C \Rightarrow A = -50$.

Hence

$$\frac{25}{x^2(2x+1)} \equiv \underline{\underline{-\frac{50}{x} + \frac{25}{x^2} + \frac{100}{2x+1}}}.$$

Figure 24 shows a sketch of part of the curve C with equation

$$y = \frac{5}{x\sqrt{2x+1}}.$$

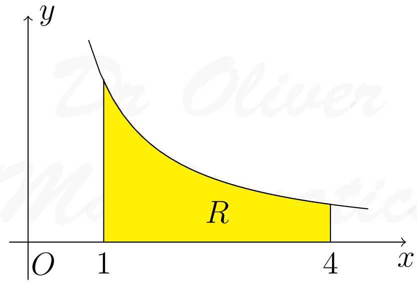

Figure 24: $y = \dfrac{5}{x\sqrt{2x+1}}$

The region bounded R by the lines the x-axis, the line $x = 1$, the line $x = 4$, and the curve is shaded in the figure. This region is rotated through 360° about the x-axis.

(b) Use calculus to find the exact value for the solid of revolution, giving your answer in the form $a + b\ln c$, where a, b, and c are constants. (6)

Solution

$$\begin{aligned}
\text{Volume} &= \pi \int_1^4 \frac{25}{x^2(2x+1)}\,\mathrm{d}x \\
&= \pi \int_1^4 \left(-\frac{50}{x} + \frac{25}{x^2} + \frac{100}{2x+1}\right)\mathrm{d}x \\
&= \pi \left[-50\ln|x| - 25x^{-1} + 50\ln|2x+1|\right]_{x=1}^{4} \\
&= \pi \left\{\left(-50\ln 4 - \tfrac{25}{4} + 50\ln 9\right) - (0 - 25 + 50\ln 3)\right\} \\
&= \pi \left(50\ln(\tfrac{9}{4\times 3}) + \tfrac{75}{4}\right) \\
&= \underline{\underline{\pi \left(50\ln\tfrac{3}{4} + \tfrac{75}{4}\right)}}.
\end{aligned}$$

58. Figure 25 shows a sketch of part of the curve with equation

$$y = 4x - xe^{\frac{1}{2}x},\ x \geqslant 0.$$

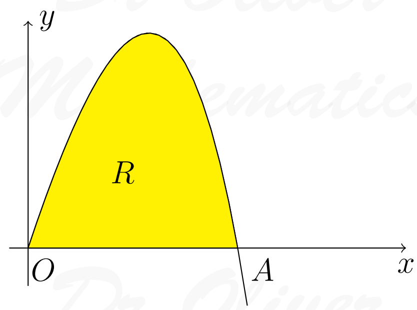

Figure 25: $y = 4x - xe^{\frac{1}{2}x}$

The curve meets the x-axis the the origin O and cuts the x-axis at the point A.

(a) Find, in terms of $\ln 2$, the x-coordinate of the point A. (2)

Solution

$$\begin{aligned} 4x - xe^{\frac{1}{2}x} = 0 &\Rightarrow x(4 - e^{\frac{1}{2}x}) = 0 \\ &\Rightarrow x = 0 \text{ or } e^{\frac{1}{2}x} = 4 \\ &\Rightarrow x = 0 \text{ or } \tfrac{1}{2}x = \ln 4 \\ &\Rightarrow x = 0 \text{ or } x = 2\ln 4 \\ &\Rightarrow x = 0 \text{ or } \underline{\underline{x = 4\ln 2}}. \end{aligned}$$

(b) Find $\int xe^{\frac{1}{2}x}\,dx$. (3)

Solution

$$u = x \Rightarrow \frac{du}{dx} = 1 \text{ and } \frac{dv}{dx} = e^{\frac{1}{2}x} \Rightarrow v = 2e^{\frac{1}{2}x}$$

and

$$\begin{aligned} \int xe^{\frac{1}{2}x}\,dx &= 2xe^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x}\,dx \\ &= \underline{\underline{2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x} + c}}. \end{aligned}$$

The region bounded R by the lines the x-axis and the curve with equation

$$y = 4x - xe^{\frac{1}{2}x},\ x \geqslant 0.$$

(c) Find, by integration, the exact value for the area of R. Give your answer in terms of $\ln 2$. (3)

Solution

$$\begin{aligned}\int_0^{4\ln 2} (4x - xe^{\frac{1}{2}x})\,dx &= \left[2x^2 - 2xe^{\frac{1}{2}x} + 4e^{\frac{1}{2}x}\right]_{x=0}^{4\ln 2} \\ &= \left(2(4\ln 2)^2 - 2(4\ln 2)e^{2\ln 2} + 4e^{2\ln 2}\right) - (0 - 0 + 4) \\ &= 32(\ln 2)^2 - 32\ln 2 + 16 - 4 \\ &= \underline{\underline{32(\ln 2)^2 - 32\ln 2 + 12}}.\end{aligned}$$

59. Figure 26 shows a sketch of part of the curve with equation

$$y = \sqrt{(3-x)(x+1)},\ 0 \leqslant x \leqslant 3.$$

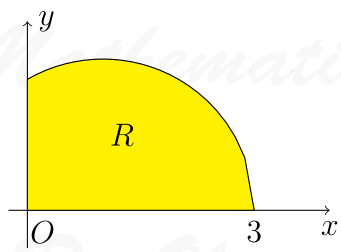

Figure 26: $y = \sqrt{(3-x)(x+1)}$

The region bounded R by the lines the x-axis, the y-axis, and the curve.

(a) Use the substitution $x = 1 + 2\sin\theta$ to show that (5)

$$\int_0^3 \sqrt{(3-x)(x+1)}\,dx = k\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta,$$

where k is a constant to be determined.

Solution

This is **not** a good substitution to use: what will be better?

$$x = 1 + 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta\,d\theta$$

and

$$x = 0 \Rightarrow \theta = -\frac{\pi}{6} \text{ and } x = 3 \Rightarrow \theta = \frac{\pi}{2}.$$

$$\begin{aligned}
\int_0^3 \sqrt{(3-x)(x+1)}\,\mathrm{d}x &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{(3-(1+2\sin\theta))((1+2\sin\theta)+1)}\,2\cos\theta\,\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{(2-2\sin\theta))(2\sin\theta+2)}\,2\cos\theta\,\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}\,2\cos\theta\,\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (2\cos\theta)(2\cos\theta)\,\mathrm{d}\theta \\
&= \underline{\underline{4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta.}}
\end{aligned}$$

(b) Hence find, by integration, the exact value of R. (3)

Solution

$$\begin{aligned}
4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta &= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,\mathrm{d}\theta \\
&= 2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\theta=-\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= 2\left\{\left(\tfrac{\pi}{2}+0\right) - \left(-\tfrac{\pi}{6}+\tfrac{\sqrt{3}}{4}\right)\right\} \\
&= 2\left(\tfrac{2\pi}{3}+\tfrac{\sqrt{3}}{4}\right) \\
&= \underline{\underline{\tfrac{4\pi}{3}+\tfrac{\sqrt{3}}{2}}}.
\end{aligned}$$

60. Figure 27 shows a sketch of part of the curve with equation

$$y = 3^x.$$

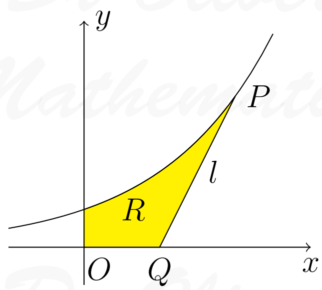

Figure 27: $y = 3^x$

The point P lies on C and has coordinates $(2, 9)$. The line l is a tangent to C at P. The line l cuts the x-axis at the Q.

(a) Find the exact value of the x-coordinate of Q. (4)

Solution

$$y = 3^x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3^x(\ln 3)$$

and so we have

$$\begin{aligned} y - 9 = 3^2(\ln 3)(x-2) &\Rightarrow -9 = 3^2(\ln 3)(x-2) \\ &\Rightarrow -1 = (\ln 3)(x-2) \\ &\Rightarrow -\frac{1}{\ln 3} = x - 2 \\ &\Rightarrow \underline{\underline{x = 2 - \frac{1}{\ln 3}}}. \end{aligned}$$

The region bounded R by the lines the x-axis, the line y-axis, the line l, and the curve is shaded in the figure. This region is rotated through 360° about the x-axis.

(b) Use integration to find the exact value of the volume of the solid generated. Give your answer in the form $\frac{p}{q}$, where p and q are exact constants. (6)

Solution

$$V_{\text{cone}} = \tfrac{1}{3} \times 9^2 \times \frac{1}{\ln 3} = \frac{27\pi}{\ln 3}$$

and

$$\begin{aligned} V_{\text{rev}} &= \pi \int_0^2 3^{2x} \, \mathrm{d}x \\ &= \pi \int_0^2 9^x \, \mathrm{d}x \\ &= \pi \left[\frac{9^x}{\ln 9} \right]_{x=0}^{2} \\ &= \pi \left(\frac{81}{\ln 9} - \frac{1}{\ln 9} \right) \\ &= \frac{80\pi}{\ln 9} \\ &= \frac{80\pi}{2 \ln 3} \\ &= \frac{40\pi}{\ln 3}; \end{aligned}$$

finally,

$$V = V_{\text{rev}} - V_{\text{cone}} = \frac{40\pi}{\ln 3} - \frac{27\pi}{\ln 3} = \underline{\underline{\frac{13\pi}{\ln 3}}}.$$

61. Figure 28 shows a sketch of part of the curve with equation

$$y = x^2 \ln x, \; x \geqslant 1.$$

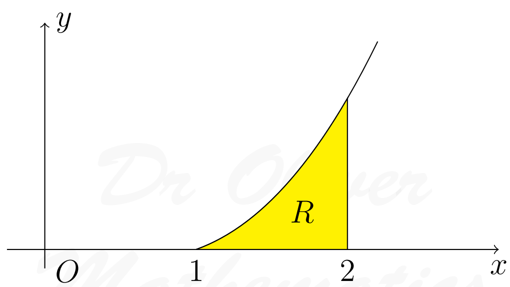

Figure 28: $y = x^2 \ln x$

The region bounded R by the lines the x-axis, the line $x = 2$-axis, and the curve is shaded in the figure. Use integration to find the exact value of the area of R.

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \tfrac{1}{3}x^3$$

and

$$\begin{aligned}\text{Area} &= \left[\tfrac{1}{3}x^3 \ln x\right]_{x=1}^{2} - \int_1^2 \tfrac{1}{3}x^2 \,\mathrm{d}x \\ &= \left(\tfrac{8}{3}\ln 2 - 0\right) - \left[\tfrac{1}{9}x^3\right]_{x=1}^{2} \\ &= \tfrac{8}{3}\ln 2 - \left(\tfrac{8}{9} - \tfrac{1}{9}\right) \\ &= \underline{\underline{\tfrac{8}{3}\ln 2 - \tfrac{7}{9}}}.\end{aligned}$$

62. (a) Given that $y > 0$, find (6)

$$\int \frac{3y-4}{y(3y+2)}\,\mathrm{d}y.$$

Solution

$$\begin{aligned}\frac{3y-4}{y(3y+2)} &\equiv \frac{A}{y} + \frac{B}{3y+2} \\ &\equiv \frac{A(3y+2)+By}{y(3y+2)}\end{aligned}$$

and so

$$3y - 4 \equiv A(3y+2) + By.$$

$\underline{\underline{y = 0}}$: $-4 = 2A \Rightarrow A = -2$.

$\underline{\underline{y = -\tfrac{2}{3}}}$: $-6 = -\tfrac{2}{3}B \Rightarrow B = 9$.

Hence

$$\frac{3y-4}{y(3y+2)} \equiv -\frac{2}{y} + \frac{9}{3y+2}$$

and

$$\begin{aligned}\int \frac{3y-4}{y(3y+2)}\,\mathrm{d}y &= \int \left(-\frac{2}{y} + \frac{9}{3y+2}\right)\,\mathrm{d}y \\ &= \underline{\underline{-2\ln y + 3\ln(3y+2) + c}}.\end{aligned}$$

(b) Use the substitution $x = 4\sin^2\theta$ to show that (5)

$$\int_0^3 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x = \lambda \int_0^{\frac{\pi}{3}} \sin^2\theta\,\mathrm{d}\theta,$$

where λ is a constant to be determined.

Solution

This is **not** a good substitution to use: what will be better?

$$x = 4\sin^2\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 8\sin\theta\cos\theta \Rightarrow \mathrm{d}x = 8\sin\theta\cos\theta\,\mathrm{d}\theta$$

and

$$x = 0 \Rightarrow \theta = 0 \text{ and } x = 3 \Rightarrow \theta = \tfrac{\pi}{3}.$$

Now,

$$\begin{aligned}
\int_0^3 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x &= \int_0^{\frac{\pi}{3}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\,8\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} \sqrt{\frac{4\sin^2\theta}{4\cos^2\theta}}\,8\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} \sqrt{\tan^2\theta}\,8\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} 8\tan\theta\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= \underline{\underline{8\int_0^{\frac{\pi}{3}} \sin^2\theta\,\mathrm{d}\theta.}}
\end{aligned}$$

(c) Hence use integration to find (4)

$$\int_0^3 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x,$$

giving your answer in the form $a\pi + b$, where a and b are exact constants.

Solution

$$
\begin{aligned}
\int_0^3 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x &= 8\int_0^{\frac{\pi}{3}} \sin^2\theta\,\mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{3}} (1-\cos 2\theta)\,\mathrm{d}\theta \\
&= 4\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\theta=0}^{\frac{\pi}{3}} \\
&= 4\left\{\left(\tfrac{\pi}{3} - \tfrac{\sqrt{3}}{4}\right) - (0-0)\right\} \\
&= \underline{\underline{\tfrac{4\pi}{3} - \sqrt{3}}}.
\end{aligned}
$$

63. (a) Find (2)

$$\int (2x-1)^{\frac{3}{2}}\,\mathrm{d}x,$$

giving your answer in its simplest form.

Solution

$$\int (2x-1)^{\frac{3}{2}}\,\mathrm{d}x = \underline{\underline{\tfrac{1}{5}(2x-1)^{\frac{5}{2}} + c}}.$$

Figure 29 shows a sketch of part of the curve with equation

$$y = (2x-1)^{\frac{3}{4}},\ x \geqslant \tfrac{1}{2}.$$

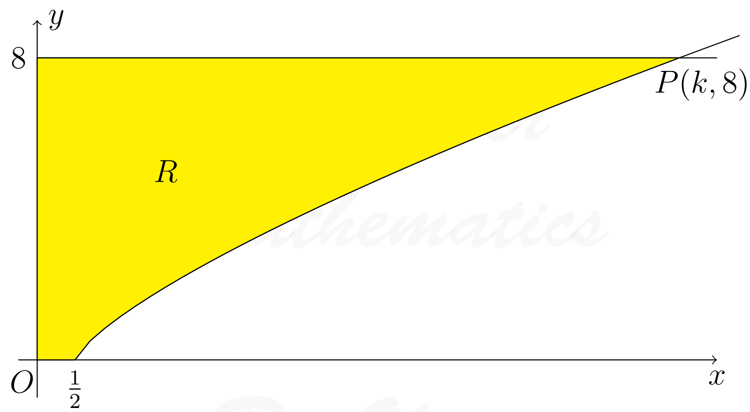

Figure 29: $(2x-1)^{\frac{3}{4}}$

The curve C cuts the line $y = 8$ at the point P with coordinates $(k, 8)$, where k is a constant.

(b) Find the value of k. (2)

Solution

$$(2x-1)^{\frac{3}{4}} = 8 \Rightarrow 2x - 1 = 16 \Rightarrow \underline{\underline{x = 8.5}}.$$

The region bounded R by the lines the x-axis, the line y-axis, the line $y = 8$, and the curve is shaded in the figure. This region is rotated through 2π radians about the x-axis.

(c) Use integration to find the exact value of the volume of the solid generated. (4)

Solution

$$\begin{aligned} V_{\text{rev}} &= \pi \int_{0.5}^{8.5} (2x-1)^{\frac{3}{2}}\, \mathrm{d}x \\ &= \pi \left[\tfrac{1}{5}(2x-1)^{\frac{5}{2}} \right]_{x=0.5}^{8.5} \\ &= \pi \left(\tfrac{1024}{5} - 0 \right) \\ &= \tfrac{1024\pi}{5} \end{aligned}$$

and

$$V_{\text{cylinder}} = \pi \times 8^2 \times \tfrac{17}{2} = 544\pi$$

and so

$$\begin{aligned} V &= V_{\text{cylinder}} - V_{\text{rev}} \\ &= 544\pi - \tfrac{1024\pi}{5} \\ &= \underline{\underline{\tfrac{1696\pi}{5}}}. \end{aligned}$$

64. Figure 30 shows a sketch of part of the curve with equation

$$y = \frac{6}{e^x + 2}, \ x \in \mathbb{R}.$$

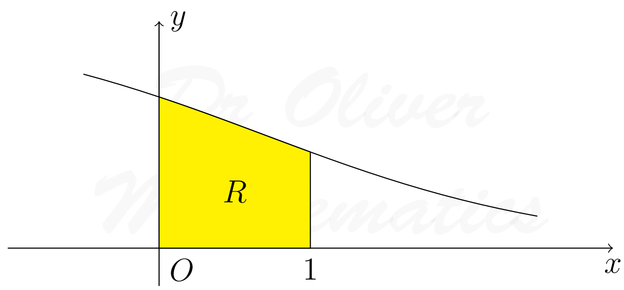

Figure 30: $y = \dfrac{6}{e^x + 2}$

The finite region R, shown shaded in Figure 30, is bounded by the curve, the y-axis, the x-axis, and the line with equation $x = 1$.

(a) Use the substitution $u = e^x$ to show that the area of R can be given by (2)

$$\int_a^b \frac{6}{u(u+2)} \, du,$$

where a and b are constants to be determined.

Solution

$$u = e^x \Rightarrow \frac{du}{dx} = e^x = u \Rightarrow dx = \frac{1}{u} \, du$$

with

$$x = 0 \Rightarrow u = 1 \text{ and } x = 1 \Rightarrow u = e.$$

Now,

$$\int_0^1 \frac{6}{e^x + 2} \, dx = \underline{\underline{\int_1^e \frac{6}{u(u+2)} \, du}}.$$

(b) Hence use calculus to find the exact area of R. (6)

Solution

$$\frac{6}{u(u+2)} \equiv \frac{A}{u} + \frac{B}{u+2}$$
$$\equiv \frac{A(u+2)+Bu}{u(u+2)}$$

and so

$$6 \equiv A(u+2) + Bu.$$

$\underline{u=-2}$: $6 = -2B \Rightarrow B = -3$.
$\underline{u=0}$: $6 = 2A \Rightarrow A = 3$.
Hence

$$\frac{6}{u(u+2)} \equiv \frac{3}{u} + \frac{3}{u+2}$$

and

$$\begin{aligned}\int_0^1 \frac{6}{\mathrm{e}^x+2}\,\mathrm{d}x &= \int_1^{\mathrm{e}} \frac{6}{u(u+2)}\,\mathrm{d}u \\ &= \int_1^{\mathrm{e}} \left(\frac{3}{u} + \frac{3}{u+2}\right)\mathrm{d}u \\ &= \left[3\ln|u| - 3\ln|u+2|\right]_{x=1}^{\mathrm{e}} \\ &= (3\ln e - 3\ln(\mathrm{e}+2)) - (3\ln 1 - 3\ln 3) \\ &= 3 - 3\ln(\mathrm{e}+2) + 3\ln 3 \\ &= \underline{\underline{3 + 3\ln\tfrac{3}{\mathrm{e}+2}}}.\end{aligned}$$

65. The finite region S, shown shaded in Figure 31, is bounded by the y-axis, the x-axis, the line with equation $x = \ln 4$, and the curve with equation (7)

$$y = \mathrm{e}^x + 2\mathrm{e}^{-x},\ x \geqslant 0.$$

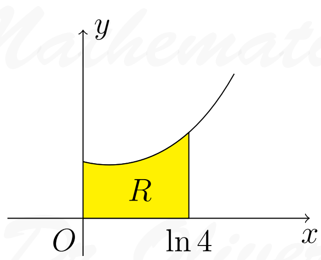

Figure 31: $y = \mathrm{e}^x + 2\mathrm{e}^{-x}$

The region S is rotated through 2π radians about the x-axis. Use integration to find the exact value of the volume of the solid generated. Give your answer in its simplest form.

Solution

$$\begin{aligned}
\int_0^{\ln 4} \pi(\mathrm{e}^x + 2\mathrm{e}^{-x})^2 \, \mathrm{d}x &= \pi \int_0^{\ln 4} (\mathrm{e}^{2x} + 4 + 4\mathrm{e}^{-2x}) \, \mathrm{d}x \\
&= \pi \left[\tfrac{1}{2}\mathrm{e}^{2x} + 4x - 2\mathrm{e}^{-2x} \right]_{x=0}^{\ln 4} \\
&= \pi \left[(\tfrac{1}{2}\mathrm{e}^{2\ln 4} + 4\ln 4 - 2\mathrm{e}^{-2\ln 4}) - (\tfrac{1}{2} + 0 - 2) \right] \\
&= \pi \left[(8 + 4\ln 4 - \tfrac{1}{8}) + \tfrac{3}{2} \right] \\
&= \underline{\underline{\pi(\tfrac{75}{8} + 4\ln 4)}}.
\end{aligned}$$

66. Figure 32 shows a sketch of part of the curve C with parametric equations

$$x = 3\theta \sin\theta, y = \sec^3\theta, \ 0 \leqslant \theta < \tfrac{\pi}{2}.$$

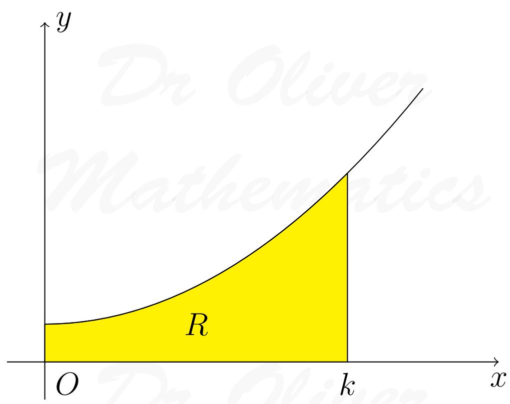

Figure 32: $x = 3\theta \sin\theta, y = \sec^3\theta$

The point $P(k, 8)$ lies on C, where k is a constant.

(a) Find the exact value of k. (2)

Solution

$$\sec^3\theta = 8 \Rightarrow \sec\theta = 2$$
$$\Rightarrow \cos\theta = \tfrac{1}{2}$$
$$\Rightarrow \theta = \tfrac{\pi}{3}$$
$$\Rightarrow \underline{\underline{x = \tfrac{\pi\sqrt{3}}{2}}}.$$

The finite region R, shown shaded in Figure 32, is bounded by the curve C, the y-axis, the x-axis, and the line with equation $x = k$.

(b) Show that the area of R can be expressed in the form (4)

$$\lambda \int_{\alpha}^{\beta} (\theta \sec^2\theta + \tan\theta \sec^2\theta)\,\mathrm{d}\theta,$$

where α, β, and λ are constants to be determined.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sin\theta + 3\theta\cos\theta$$

and

$$x = 0 \Rightarrow \theta = 0 \text{ and } x = \tfrac{\pi\sqrt{3}}{2} \Rightarrow \theta = \tfrac{\pi}{3}.$$

Now,

$$\int_0^{\frac{\pi\sqrt{3}}{2}} y\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} y\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{3}} \sec^3\theta(3\sin\theta + 3\theta\cos\theta)\,\mathrm{d}\theta$$
$$= \underline{\underline{3\int_0^{\frac{\pi}{3}} (\theta\sec^2\theta + \tan\theta\sec^2\theta)\,\mathrm{d}\theta}}.$$

(c) Hence use integration to find the exact value of the area of R. (6)

Solution

$$u = \theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sec^2\theta \Rightarrow v = \tan\theta,$$
$$\int \theta\sec^2\theta\,\mathrm{d}\theta = \theta\tan\theta - \int \sec^2\theta\,\mathrm{d}\theta = \theta\tan\theta - \tan\theta + c,$$

and

$$\int \tan\theta \sec^2\theta \, \mathrm{d}\theta = \tfrac{1}{2}\tan^2\theta + d.$$

Now,

$$\begin{aligned}
\text{Area} &= 3\int_0^{\frac{\pi}{3}} (\theta \sec^2\theta + \sec^2\theta)\, \mathrm{d}\theta \\
&= 3\left[\theta\tan\theta - \ln|\sec\theta| + \tfrac{1}{2}\tan^2\theta\right]_{\theta=0}^{\frac{\pi}{3}} \\
&= 3\left[(\tfrac{\pi\sqrt{3}}{3} - \ln 2 + \tfrac{3}{2}) - (0 - 0 + 0)\right] \\
&= \underline{\underline{\pi\sqrt{3} - 3\ln 2 + \tfrac{9}{2}}}.
\end{aligned}$$

67. Figure 33 shows a vertical cylindrical tank of height 200 cm containing water. Water is leaking from a hole P on the side of the tank.

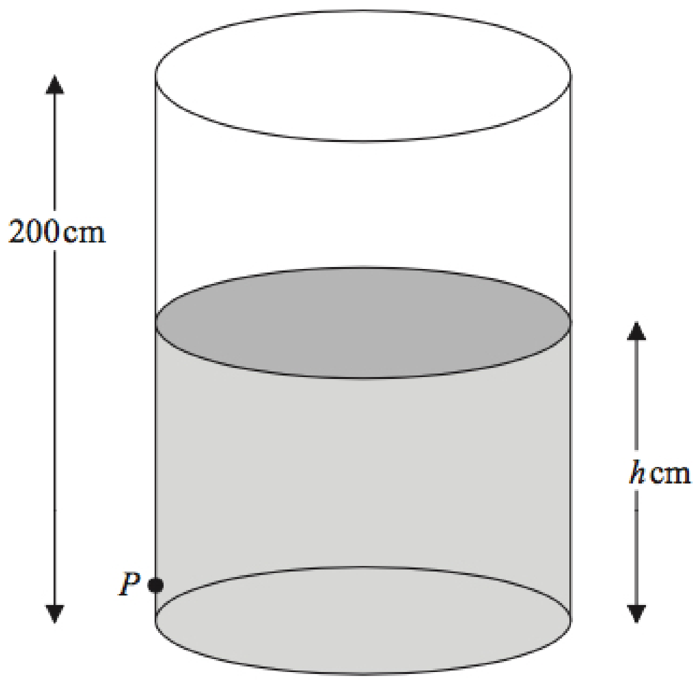

Figure 33: a leaky tank

At time t minutes after the leaking starts, the height of water in the tank is h cm. The height h cm of the water in the tank satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(h-9)^{\frac{1}{2}},\ 9 < h \leqslant 200,$$

where k is a constant. Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k. (2)

Solution

$$\begin{aligned} -1.1 = k(130-9)^{\frac{1}{2}} &\Rightarrow 11k = -1.1 \\ &\Rightarrow \underline{\underline{k = -\tfrac{1}{10}}}. \end{aligned}$$

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$. (6)

Solution

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}t} = -\tfrac{1}{10}(h-9)^{\frac{1}{2}} &\Rightarrow (h-9)^{-\frac{1}{2}}\,\mathrm{d}h = -\tfrac{1}{10}\,\mathrm{d}t \\ &\Rightarrow \int (h-9)^{-\frac{1}{2}}\,\mathrm{d}h = -\int \tfrac{1}{10}\,\mathrm{d}t \\ &\Rightarrow 2(h-9)^{\frac{1}{2}} = -\tfrac{1}{10}t + c. \end{aligned}$$

Now,

$$2(200-9)^{\frac{1}{2}} = 0 + c \Rightarrow c = 2(191)^{\frac{1}{2}}$$

and

$$2(h-9)^{\frac{1}{2}} = -\tfrac{1}{10}t + 2(191)^{\frac{1}{2}}.$$

Finally,

$$h = 50 \Rightarrow 2\sqrt{41} = -\tfrac{1}{10}t + 2\sqrt{191} \Rightarrow \underline{\underline{t = 20(\sqrt{191}-\sqrt{41})}}.$$