Dr Oliver Mathematics
Mathematics: Advanced Higher
2014 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Given (3)
$$f(x) = \frac{x^2 - 1}{x^2 + 1},$$
obtain $f'(x)$ and simplify your answer.

   (b) Differentiate (3)
$$y = \tan^{-1}(3x^2).$$

2. (a) Write down and simplify the general term in the expression (1)
$$\left(\frac{2}{x} + \frac{1}{4x^2}\right)^{10}.$$

   (b) Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$. (4)

3. (a) Use Gaussian elimination on the system of equations below to give an expression for z in terms of λ: (4)
$$\begin{aligned} x + y + z &= 2 \\ 4x + 3y - \lambda z &= 4 \\ 5x + 6y + 8z &= 11. \end{aligned}$$

   (b) For what values of λ does this system have a solution? (1)

   (c) Determine the solution to this system of equations when $\lambda = 2$. (1)

4. Given (3)
$$x = \ln(1 + t^2) \text{ and } y = \ln(1 + 2t^2),$$
use parametric differentiation to find $\frac{dy}{dx}$ in terms of t.

5. Three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are given by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ where
$$\mathbf{u} = 5\mathbf{i} + 13\mathbf{j},\ \mathbf{v} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k},\ \text{ and } \mathbf{w} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}.$$

(a) Calculate (3)

$$\mathbf{u}.(\mathbf{v} \times \mathbf{w}).$$

(b) Interpret your result geometrically. (1)

6. Given (3)

$$\mathrm{e}^y = x^3 \cos^2 x,\ x > 0,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x} + b \tan x,$$

for some constants a and b and state the values of a and b.

7. Given $\mathbf{A}$ is the matrix (4)

$$\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix},$$

prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix},\ n \geqslant 1.$$

8. Find the solution $y = \mathrm{f}(x)$ to the differential equation (6)

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0,$$

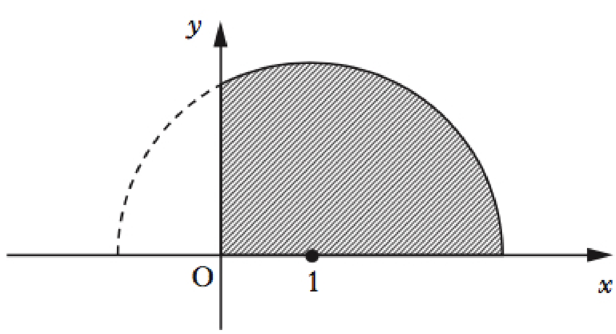
given that $y = 4$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$.

9. (a) Give the first three non-zero terms of the Maclaurin series for $\cos 3x$. (2)

(b) Write down the first four terms of the Maclaurin series for e^{2x}. (1)

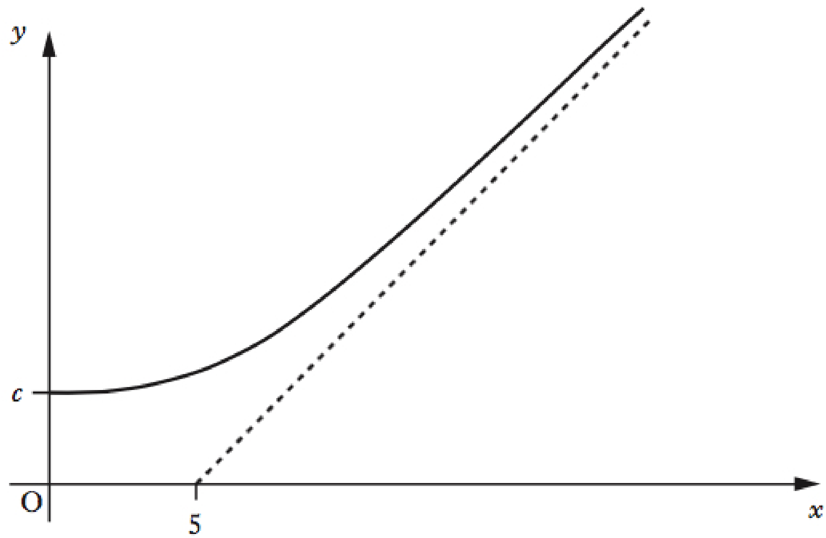
(c) Hence, or otherwise, determine the Maclaurin series for $\mathrm{e}^{2x} \cos 3x$ up to, and including, the term in x^3. (3)

10. A semi-circle with centre $(1, 0)$ and radius 2, lies on the x-axis as shown. (5)



Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis.

11. The function $\mathrm{f}(x)$ is defined for all $x \geqslant 0$.
The graph of $y = \mathrm{f}(x)$ intersects the y-axis at $(0, c)$, where $0 < c < 5$.
The graph of the function and its asymptote, $y = x - 5$, are shown below.



(a) Sketch the graph of $y = \mathrm{f}^{-1}(x)$. (4)
Clearly show any points of intersection and any asymptotes.

(b) What is the equation of the asymptote of the graph of $y = \mathrm{f}(x + 2)$? (1)

(c) Why does your diagram show that the equation $x = \mathrm{f}(\mathrm{f}(x))$ has at least one solution? (1)

12. Use the substitution $x = \tan\theta$ to determine the exact value of (6)

$$\int_0^1 \frac{1}{(1 + x^2)^{\frac{3}{2}}}\,\mathrm{d}x.$$

13. The fuel efficiency, $F(x)$, in km per litre, of a vehicle varies with its speed, s km per hour, and for a particular vehicle the relationship is thought to be (10)

$$F(x) = 15 + \mathrm{e}^x(\sin x - \cos x - \sqrt{2}),$$

where

$$x = \frac{\pi(s - 40)}{80},$$

for speeds in the range $40 \leqslant s \leqslant 120$ km per hour.

What is the greatest and least efficiency over the range and at what speeds do they occur?

14. (a) (i) Given the series (1)

$$1 + r + r^2 + r^3 + \dots,$$

write down the sum to infinity when $|r| < 1$.

(ii) Hence obtain an infinite geometric series for (2)

$$\frac{1}{2 - 3r}.$$

(iii) For what values of r is this series valid? (1)

(b) (i) Express (3)

$$\frac{1}{3r^2 - 5r + 2}$$

in partial fractions.

(ii) Hence, or otherwise, determine the first three terms of an infinite series for (2)

$$\frac{1}{3r^2 - 5r + 2}.$$

(iii) For what values of r does the series converge? (1)

15. (a) Use integration by parts to obtain an expression for (4)

$$\int \mathrm{e}^x \cos x \, \mathrm{d}x.$$

(b) Similarly, given (4)

$$I_n = \int \mathrm{e}^x \cos nx \, \mathrm{d}x, \text{ where } n \neq 0,$$

obtain an expression for I_n.

(c) Hence evaluate (2)

$$\int_0^{\frac{1}{2}\pi} \mathrm{e}^x \cos 8x \, \mathrm{d}x.$$

16. (a) Express -1 as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$. (3)

(b) Write down the four solutions to the equation $z^4 - 1 = 0$. (2)

(c) Plot the solutions of both equations on an Argand diagram. (1)

(d) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$. (2)

(e) Hence identify all the solutions to the equation (2)

$$z^6 + z^4 + z^2 + 1 = 0.$$