

Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2012 November Paper 1 Variant 1: Calculator
2 hours

The total number of marks available is 80.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You must write down all the stages in your working.

1. It is given that

$$\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{c} = \begin{pmatrix} 21 \\ 2 \end{pmatrix}.$$

- (a) Find

$$|\mathbf{a} + \mathbf{b} + \mathbf{c}|.$$

(2)

Solution

Well,

$$\begin{aligned} \mathbf{a} + \mathbf{b} + \mathbf{c} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 21 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 24 \\ 7 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} |\mathbf{a} + \mathbf{b} + \mathbf{c}| &= \sqrt{24^2 + 7^2} \\ &= \sqrt{625} \\ &= \underline{\underline{25}}. \end{aligned}$$

- (b) Find λ and μ such that

$$\lambda \mathbf{a} + \mu \mathbf{b} = \mathbf{c}.$$

(3)

Solution

Now,

$$\lambda \mathbf{a} + \mu \mathbf{b} = \mathbf{c} \Rightarrow \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 21 \\ 2 \end{pmatrix}$$

and so

$$4\lambda - \mu = 21 \quad (1)$$

$$3\lambda + 2\mu = 2 \quad (2).$$

Do $2 \times (1)$:

$$8\lambda - 2\mu = 42 \quad (3)$$

and add $(3) + (2)$:

$$11\lambda = 44 \Rightarrow \underline{\underline{\lambda = 4}}$$

insert into (1):

$$\begin{aligned} 4(4) + \mu &= 21 \Rightarrow 16 + \mu = 21 \\ &\Rightarrow \underline{\underline{\mu = -5}}. \end{aligned}$$

[Check: $3(4) + 2(-5) = 2 \checkmark$]

2. (a) Find the inverse of the matrix

(2)

$$\begin{pmatrix} 2 & -1 \\ -1 & 1.5 \end{pmatrix}.$$

Solution

Well,

$$\det = 3 - 1 = 2$$

and inverse is

$$\underline{\underline{\frac{1}{2} \begin{pmatrix} 1.5 & 1 \\ 1 & 2 \end{pmatrix}}}.$$

- (b) Hence find the matrix **A** such that

(3)

$$\begin{pmatrix} 2 & -1 \\ -1 & 1.5 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 1 & 6 \\ -0.5 & 4 \end{pmatrix}.$$

Solution

Now,

$$\begin{aligned}\begin{pmatrix} 2 & -1 \\ -1 & 1.5 \end{pmatrix} \mathbf{A} &= \begin{pmatrix} 1 & 6 \\ -0.5 & 4 \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 1.5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 6 \\ -0.5 & 4 \end{pmatrix} \\ \Rightarrow \mathbf{A} &= \frac{1}{2} \begin{pmatrix} 1.5 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ -0.5 & 4 \end{pmatrix} \\ \Rightarrow \mathbf{A} &= \frac{1}{2} \begin{pmatrix} 1 & 13 \\ 0 & 14 \end{pmatrix} \\ \Rightarrow \mathbf{A} &= \underline{\underline{\begin{pmatrix} 0.5 & 6.5 \\ 0 & 7 \end{pmatrix}}}.\end{aligned}$$

3. (a) Show that

(5)

$$\cot \theta + \frac{\sin \theta}{1 + \cos \theta} \equiv \operatorname{cosec} \theta.$$

Solution

Well,

$$\cot \theta + \frac{\sin \theta}{1 + \cos \theta} \equiv \cot \theta + \left(\frac{\sin \theta}{1 + \cos \theta} \times \frac{1 - \cos \theta}{1 - \cos \theta} \right)$$

$\times$	$1 - \cos \theta$
1	$1 - \cos \theta$
$+ \cos \theta$	$+ \cos \theta - \cos^2 \theta$

$$\begin{aligned}
&\equiv \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta(1 - \cos \theta)}{1 - \cos^2 \theta} \\
&\equiv \frac{\sin \theta \cos \theta}{\sin^2 \theta} + \frac{\sin \theta(1 - \cos \theta)}{\sin^2 \theta} \\
&\equiv \frac{\sin \theta \cos \theta - \sin \theta \cos \theta + \sin \theta}{\sin^2 \theta} \\
&\equiv \frac{\sin \theta}{\sin^2 \theta} \\
&\equiv \frac{1}{\sin \theta} \\
&\equiv \underline{\underline{\operatorname{cosec} \theta}},
\end{aligned}$$

as required.

(b) Explain why the equation

$$\cot \theta + \frac{\sin \theta}{1 + \cos \theta} = \frac{1}{2}$$

(1)

has no solution.

Solution

Now,

$$\begin{aligned}
\cot \theta + \frac{\sin \theta}{1 + \cos \theta} = \frac{1}{2} &\Rightarrow \operatorname{cosec} \theta = \frac{1}{2} \\
&\Rightarrow \sin \theta = 2,
\end{aligned}$$

and the equation has no solution.

4. Given that

$$\log_a pq = 9 \text{ and } \log_a p^2 q = 15,$$

find the value of

(a) $\log_a p$ and of $\log_a q$,

(4)

Solution

Well,

$$\log_a pq = 9 \Rightarrow \log_a p + \log_a q = 9 \quad (1)$$

and

$$\begin{aligned} \log_a p^2 q = 15 &\Rightarrow \log_a p^2 + \log_a q = 15 \\ &\Rightarrow 2 \log_a p + \log_a q = 15 \quad (2). \end{aligned}$$

Do (2) – (1):

$$\underline{\underline{\log_a p = 6.}}$$

Insert into (1):

$$6 + \log_a q = 9 \Rightarrow \underline{\underline{\log_a q = 3.}}$$

[Check: $2 \times 6 + 3 = 15$ ✓](b) $\log_p a + \log_q a$.

(2)

Solution

Well,

$$\begin{aligned} \log_p a + \log_q a &= \frac{1}{\log_a p} + \frac{1}{\log_a q} \\ &= \frac{1}{6} + \frac{1}{\log_a 3} \\ &= \underline{\underline{\frac{1}{2}}}. \end{aligned}$$

5. The line

(7)

$$x - 2y = 6$$

intersects the curve

$$x^2 + xy + 10y + 4y^2 = 156$$

at the points A and B .Find the length of AB .**Solution**

Well,

$$x - 2y = 6 \Rightarrow x = 2y + 6$$

and insert it into the non-linear equation:

$$x^2 + xy + 10y + 4y^2 = 156$$

$$\Rightarrow (2y + 6)^2 + y(2y + 6) + 10y + 4y^2 = 156$$

$\times$	$2y$	$+6$
$2y$	$4y^2$	$+12y$
$+6$	$+12y$	$+36$

$$\Rightarrow (4y^2 + 24y + 36) + (2y^2 + 6y) + 10y + 4y^2 = 156$$

$$\Rightarrow 10y^2 + 40y - 120 = 0$$

$$\Rightarrow 10(y^2 + 4y - 12) = 0$$

$$\left. \begin{array}{l} \text{add to:} \quad +4 \\ \text{multiply to:} \quad -12 \end{array} \right\} +6, -2$$

e.g.,

$$\Rightarrow 10(y + 6)(y - 2) = 0$$

$$\Rightarrow y + 6 = 0 \text{ or } y - 2 = 0$$

$$\Rightarrow y = -6 \text{ or } y = 2$$

$$\Rightarrow x = -6 \text{ or } x = 10;$$

so, $A(-6, -6)$ and $B(10, 2)$.

Finally,

$$AB = \sqrt{[10 - (-6)]^2 + [2 - (-6)]^2}$$

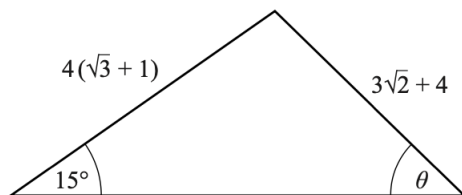
$$= \sqrt{16^2 + 8^2}$$

$$= \sqrt{320}$$

$$= \underline{\underline{8\sqrt{5}}}.$$

6. Here is a picture.

(5)



Using

$$\sin 15^\circ = \frac{\sqrt{2}}{4}(\sqrt{3} - 1),$$

and without using a calculator, find the value of $\sin \theta$ in the form $a + b\sqrt{2}$, where a and b are integers.

Solution

Sine rule:

$$\frac{\sin \theta}{4(\sqrt{3} + 1)} = \frac{\sin 15^\circ}{3\sqrt{2} + 4} \Rightarrow \sin \theta = \frac{4(\frac{\sqrt{2}}{4})(\sqrt{3} - 1)(\sqrt{3} + 1)}{3\sqrt{2} + 4}$$

$\times$	$\sqrt{3}$	$+1$
$\sqrt{3}$	3	$+\sqrt{3}$
-1	$-\sqrt{3}$	-1

$$\Rightarrow \sin \theta = \frac{\sqrt{2}(3 - 1)}{3\sqrt{2} + 4}$$

$$\Rightarrow \sin \theta = \frac{2\sqrt{2}}{3\sqrt{2} + 4}$$

$$\Rightarrow \sin \theta = \frac{2\sqrt{2}}{3\sqrt{2} + 4} \times \frac{3\sqrt{2} - 4}{3\sqrt{2} - 4}$$

$\times$	$3\sqrt{2}$	$+4$
$3\sqrt{2}$	18	$+12\sqrt{2}$
-4	$-12\sqrt{2}$	-16

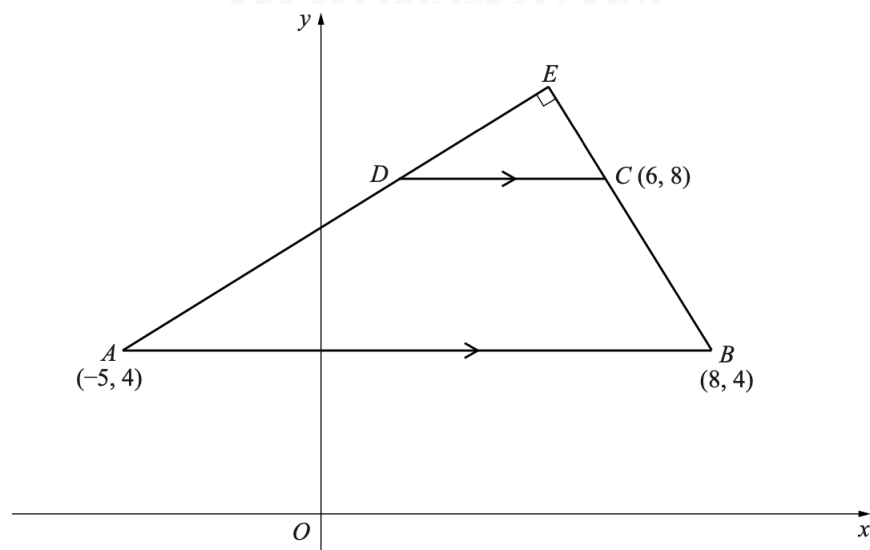
$$\Rightarrow \sin \theta = \frac{2\sqrt{2}(3\sqrt{2} - 4)}{2}$$

$$\Rightarrow \sin \theta = \sqrt{2}(3\sqrt{2} - 4)$$

$$\Rightarrow \sin \theta = \underline{\underline{6 - 4\sqrt{2}}};$$

hence, $a = 6$ and $b = -4$.

7. Solutions to this question by accurate drawing will not be accepted.



The vertices of the trapezium $ABCD$ are the points $A(-5, 4)$, $B(8, 4)$, $C(6, 8)$, and D .

- The line AB is parallel to the line DC .
- The lines AD and BC are extended to meet at E .
- Angle $AEB = 90^\circ$.

(a) Find the coordinates of D and of E .

(6)

Solution

Well,

$$\begin{aligned} m_{BC} &= \frac{8-4}{6-8} \\ &= \frac{4}{-2} \\ &= -2 \end{aligned}$$

and the equation of BC is

$$\begin{aligned} y-4 &= -2(x-8) \Rightarrow y-4 = -2x+16 \\ &\Rightarrow y = -2x+20 \quad (1). \end{aligned}$$

Now,

$$m_{AD} = \frac{1}{2}$$

and the equation of AD is

$$\begin{aligned} y-4 &= \frac{1}{2}(x+5) \Rightarrow y-4 = \frac{1}{2}x + \frac{5}{2} \\ &\Rightarrow y = \frac{1}{2}x + \frac{13}{2} \quad (2). \end{aligned}$$

Next, the lines AD and BC are extended to meet at E so

$$\begin{aligned} \frac{1}{2}x + \frac{13}{2} &= -2x+20 \Rightarrow \frac{5}{2}x = \frac{13}{2} \\ &\Rightarrow x = 5\frac{2}{5} \\ &\Rightarrow y = 9\frac{1}{5}; \end{aligned}$$

so, $E(5\frac{2}{5}, 9\frac{1}{5})$.

Finally, the y -coordinate of D is plainly 8:

$$\begin{aligned} y=8 &\Rightarrow 8 = \frac{1}{2}x + \frac{13}{2} \\ &\Rightarrow \frac{3}{2} = \frac{1}{2}x \\ &\Rightarrow x = 3; \end{aligned}$$

so, $D(3, 8)$

(b) Find the area of the trapezium $ABCD$.

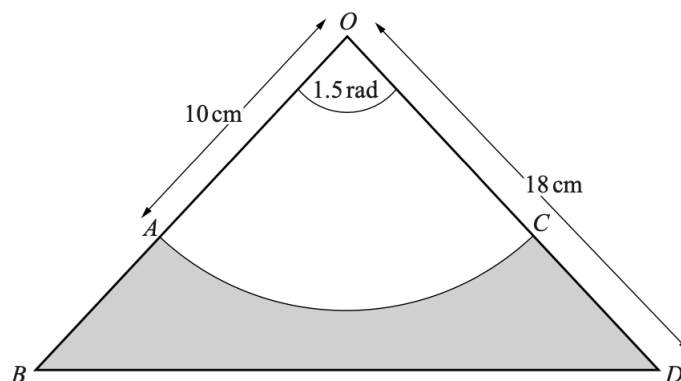
(2)

Solution

Well,

$$\begin{aligned}\text{area} &= \frac{1}{2} \times (13 + 3) \times 4 \\ &= \underline{\underline{32}}.\end{aligned}$$

8. The diagram shows an isosceles triangle OBD in which $OB = OD = 18$ cm and angle $BOD = 1.5$ radians.



An arc of the circle, centre O and radius 10 cm, meets OB at A and OD at C .

- (a) Find the area of the shaded region.

(3)

Solution

$$\begin{aligned}\text{Area} &= \text{triangle} - \text{sector} \\ &= \left(\frac{1}{2} \times 18^2 \times \sin 1.5\right) - \left(\frac{1}{2} \times 10^2 \times 1.5\right) \\ &= 86.59418783 \text{ (FCD)} \\ &= \underline{\underline{86.6 \text{ cm}^2}} \text{ (3 sf)}.\end{aligned}$$

- (b) Find the perimeter of the shaded region.

(4)

Solution

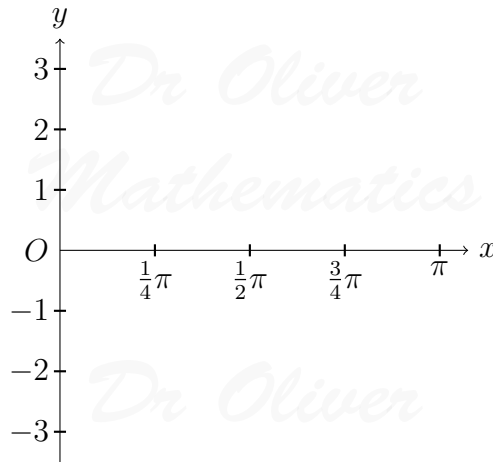
Now,

$$\begin{aligned}BD^2 &= OB^2 + OD^2 - 2 \times OB \times OD \times \cos BOD \\ \Rightarrow BD^2 &= 18^2 + 18^2 - 2 \times 18 \times 18 \times \cos 1.5 \\ \Rightarrow BD^2 &= 602.162\,293\,3 \text{ (FCD)} \\ \Rightarrow BD &= 24.538\,995\,36 \text{ (FCD)}.\end{aligned}$$

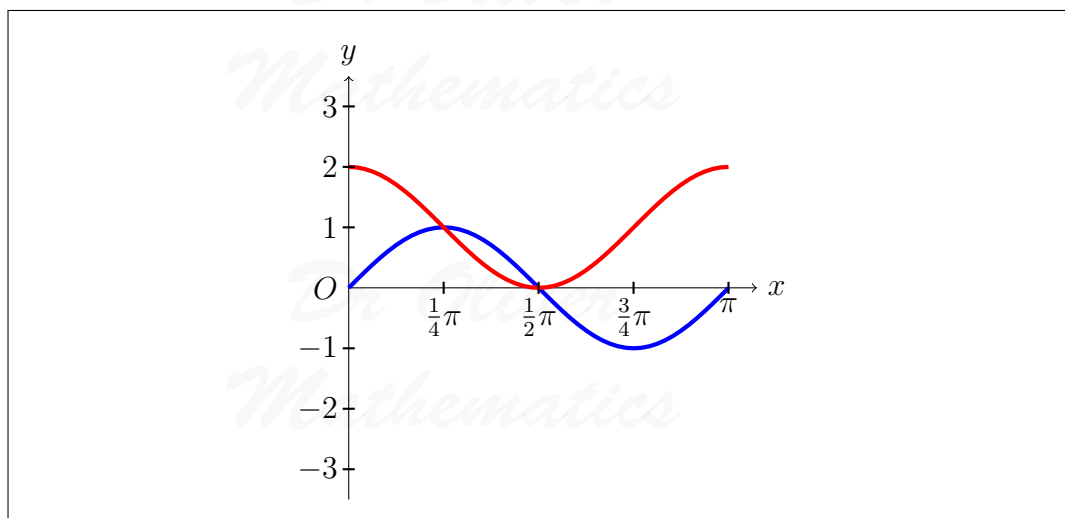
Next,

$$\begin{aligned}\text{perimeter} &= AB + BD + CD + AC \\ &= 8 + 24.538\ldots + 8 + (1.5 \times 10) \\ &= 55.538\,995\,36 \text{ (FCD)} \\ &= \underline{\underline{55.5 \text{ cm (3 sf)}}}.\end{aligned}$$

9. (a) (i) Using the axes below, sketch for $0 \leq x \leq \pi$, the graphs of (4)
 $y = \sin 2x$ and $y = 1 + \cos 2x$.



Solution



- (ii) Write down the solutions of the equation (2)

$$\sin 2x - \cos 2x = 1, \text{ for } 0 \leq x \leq \pi.$$

Solution

Well,

$$\underline{\underline{x = \frac{1}{4}\pi \text{ or } x = \frac{1}{2}\pi.}}$$

- (b) (i) Write down the amplitude and period of (2)

$$5 \cos 4x - 3.$$

Solution

The amplitude is 5 and the period is

$$\frac{2\pi}{4} = \underline{\underline{\frac{1}{2}\pi.}}$$

- (ii) Write down the period of (1)

$$4 \tan 3x.$$

Solution

$$\underline{\underline{\frac{1}{3}\pi.}}$$

10. A function f is such that

$$f(x) = 4x^3 + 4x^2 + ax + b.$$

It is given that $(2x - 1)$ is a factor of both $f(x)$ and $f'(x)$.

(a) Show that $b = 2$ and find the value of a .

(5)

Solution

Well,

$$f(x) = 4x^3 + 4x^2 + ax + b \Rightarrow f'(x) = 12x^2 + 8x + a$$

and we do synthetic division twice:

$$\begin{array}{r|rrr} \frac{1}{2} & 12 & 8 & a \\ & \downarrow & 6 & 7 \\ \hline & 12 & 14 & a+7 \end{array}$$

So

$$a + 7 = 0 \Rightarrow \underline{\underline{a = -7.}}$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 4 & 4 & -7 & b \\ & \downarrow & 2 & 3 & -2 \\ \hline & 4 & 6 & a+3 & -2+b \end{array}$$

So

$$-2 + b = 0 \Rightarrow \underline{\underline{b = 2.}}$$

Using the values of a and b from part (a),

(b) find the remainder when $f(x)$ is divided by $(x + 3)$,

(2)

Solution

$$\begin{array}{r|rrrr} -3 & 4 & 4 & -7 & 2 \\ & \downarrow & -12 & 24 & -51 \\ \hline & 4 & -8 & 17 & -49 \end{array}$$

Hence, the remainder is -49.

(c) express $f(x)$ in the form

(2)

$$f(x) = (2x - 1)(px^2 + qx + r),$$

where p , q , and r are integers to be found,

Solution

Well,

$$\begin{array}{r|rrrr} \frac{1}{2} & 4 & 4 & -7 & 2 \\ & \downarrow & 2 & 3 & -2 \\ \hline & 4 & 6 & -4 & 0 \end{array}$$

and so

$$\begin{aligned} f(x) &= (x - \frac{1}{2})(4x^2 + 6x - 4) \\ &= \underline{\underline{(2x - 1)(2x^2 + 3x - 2)}}; \end{aligned}$$

hence, $a = 2$, $b = 3$, and $c = -2$.

(d) find the values of x for which $f(x) = 0$.

(2)

Solution

$$\left. \begin{array}{l} \text{add to:} \quad \quad \quad +3 \\ \text{multiply to: } (+2) \times (-2) = -4 \end{array} \right\} -1, +4$$

E.g.,

$$\begin{aligned} 2x^2 + 3x - 2 &= 2x^2 + 4x - x - 2 \\ &= 2x(x + 2) - 1(x - 2) \\ &= (2x - 1)(x + 2) \end{aligned}$$

and so

$$\begin{aligned}f(x) = 0 &\Rightarrow (2x - 1)(2x^2 + 3x - 2) = 0 \\&\Rightarrow (2x - 1)^2(x + 2) = 0 \\&\Rightarrow 2x - 1 = 0 \text{ (repeated twice) or } x + 2 = 0 \\&\Rightarrow \underline{\underline{x = \frac{1}{2} \text{ (repeated twice) or } x = -2.}}\end{aligned}$$

EITHER

11. A curve is such that

$$y = \frac{5x^2}{1 + x^2}.$$

(a) Show that

$$\frac{dy}{dx} = \frac{kx}{(1 + x^2)^2},$$

(4)

where k is an integer to be found.

Solution

Quotient rule:

$$\begin{aligned}u = 5x^2 &\Rightarrow \frac{du}{dx} = 10x \\v = 1 + x^2 &\Rightarrow \frac{dv}{dx} = 2x\end{aligned}$$

and so

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 + x^2)(10x) - (2x)(5x^2)}{(1 + x^2)^2} \\&= \frac{10x + 10x^3 - 10x^3}{(1 + x^2)^2} \\&= \underline{\underline{\frac{10x}{(1 + x^2)^2}}};\end{aligned}$$

hence, $k = 10$.

(b) Find the coordinates of the stationary point on the curve and determine the nature of this stationary point.

(3)

Solution

Well,

$$\begin{aligned}\frac{dy}{dx} = 0 &\Rightarrow \frac{10x}{(1+x^2)^2} = 0 \\ &\Rightarrow 10x = 0 \\ &\Rightarrow x = 0 \\ &\Rightarrow y = 0;\end{aligned}$$

hence, the stationary point is $(0, 0)$.

	$x = 0^-$	$x = 0$	$x = 0^+$
$\frac{dy}{dx}$	-	0	+

It is a minimum turning point.

(c) By using your result from part (a), find

(4)

$$\int \frac{x}{(1+x^2)^2} dx$$

and hence evaluate

$$\int_{-1}^2 \frac{x}{(1+x^2)^2} dx.$$

Solution

Well,

$$\begin{aligned}\int \frac{x}{(1+x^2)^2} dx &= \frac{1}{10} \int \frac{10x}{(1+x^2)^2} dx \\ &= \frac{1}{10} \times \frac{5x^2}{1+x^2} + c \\ &= \frac{x^2}{2(1+x^2)} + c\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^2 \frac{x}{(1+x^2)^2} dx &= \left[\frac{x^2}{2(1+x^2)} \right]_{x=-1}^2 \\ &= \frac{2}{5} - \frac{1}{5} \\ &= \underline{\underline{\frac{3}{10}}}.\end{aligned}$$

OR

12. A curve is such that

$$y = \frac{Ax^2 + B}{x^2 - 2},$$

where A and B are constants.

(a) Show that

$$\frac{dy}{dx} = -\frac{2x(2A+B)}{(x^2-2)^2}.$$

(4)

Solution

Quotient rule:

$$u = Ax^2 + B \Rightarrow \frac{du}{dx} = 2Ax$$

$$v = x^2 - 2 \Rightarrow \frac{dv}{dx} = 2x$$

and so

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2-2)(2Ax) - (2x)(Ax^2+B)}{(x^2-2)^2} \\ &= \frac{2Ax^3 - 4Ax - 2Ax^3 - 2Bx}{(x^2-2)^2} \\ &= \underline{\underline{-\frac{2x(2A+B)}{(x^2-2)^2}}}.\end{aligned}$$

It is given that $y = -3$ and $\frac{dy}{dx} = -10$ when $x = 1$.

(b) Find the value of A and of B .

(3)

Solution

Well,

$$\begin{aligned}x = 1, y = -3 &\Rightarrow \frac{A + B}{-1} = -3 \\&\Rightarrow A + B = 3 \quad (1)\end{aligned}$$

and solution Well,

$$\begin{aligned}x = 1, \frac{dy}{dx} = -10 &\Rightarrow -\frac{2(2A + B)}{(-1)^2} = -10 \\&\Rightarrow 4A + 2B = 10 \\&\Rightarrow 2A + B = 5 \quad (2).\end{aligned}$$

Do (2) – (1):

$$\underline{\underline{A = 2}}$$

and substitute into (1):

$$2 + B = 3 \Rightarrow \underline{\underline{B = 1}}.$$

- (c) Using your values of A and B , find the coordinates of the stationary point on the curve, and determine the nature of this stationary point. (4)

Solution

Now,

$$\begin{aligned}\frac{dy}{dx} = 0 &\Rightarrow -\frac{10x}{(x^2 - 2)^2} = 0 \\&\Rightarrow 10x = 0 \\&\Rightarrow x = 0 \\&\Rightarrow y = -\frac{1}{2};\end{aligned}$$

hence, the stationary point is $(0, -\frac{1}{2})$.

	$x = 0^-$	$x = 0$	$x = 0^+$
$\frac{dy}{dx}$	+	0	-

It is a maximum turning point.