

Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2011 November Paper 1 Variant 3: Calculator
2 hours

The total number of marks available is 80.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You must write down all the stages in your working.

1. Given that

$$\frac{(6x^{\frac{3}{2}}y^{\frac{4}{5}})^4}{2x^{\frac{1}{2}}y^{-1}} = ax^p y^q,$$

(3)

find the values of the constants a , p , and q .

Solution

Well,

$$\begin{aligned}\frac{(6x^{\frac{3}{2}}y^{\frac{4}{5}})^4}{2x^{\frac{1}{2}}y^{-1}} &= \frac{1\,296x^6y^{\frac{16}{5}}}{2x^{\frac{1}{2}}y^{-1}} \\ &= \underline{\underline{648x^{\frac{11}{2}}y^{\frac{21}{5}}}};\end{aligned}$$

hence, $\underline{\underline{a = 648}}$, $\underline{\underline{b = \frac{11}{2}}}$, and $\underline{\underline{c = \frac{21}{5}}}$.

2. Express in the form

$$\sqrt{\frac{1 - \cos^2 \theta}{4 \sec^2 \theta - 4}}$$

(4)

in the form

$$k \cos \theta,$$

where k is a constant to be found.

Solution

$$\begin{aligned}\sqrt{\frac{1 - \cos^2 \theta}{4 \sec^2 \theta - 4}} &\equiv \sqrt{\frac{\sin^2 \theta}{4(\tan^2 \theta + 1) - 4}} \\ &\equiv \sqrt{\frac{\sin^2 \theta}{4 \tan^2 \theta}} \\ &\equiv \sqrt{\frac{1}{4} \cos^2 \theta} \\ &\equiv \underline{\underline{\frac{1}{2} \cos \theta}};\end{aligned}$$

hence, $\underline{\underline{k = \frac{1}{2}}}$.

3. (a) Given that

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ -8 & -2 \end{pmatrix},$$

find \mathbf{A}^{-1} .

Solution

Well,

$$\det \mathbf{A} = -8 - (-24) = 16$$

so

$$\mathbf{A}^{-1} = \underline{\underline{\frac{1}{16} \begin{pmatrix} -2 & -3 \\ 8 & 4 \end{pmatrix}}}.$$

(b) Hence, find the matrix \mathbf{M} such that

$$\begin{pmatrix} 4 & 3 \\ -8 & -2 \end{pmatrix} \mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

Solution

Now,

$$\begin{aligned}\begin{pmatrix} 4 & 3 \\ -8 & -2 \end{pmatrix} \mathbf{M} &= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \Rightarrow \mathbf{M} = \begin{pmatrix} 4 & 3 \\ -8 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \\ \Rightarrow \mathbf{M} &= \frac{1}{16} \begin{pmatrix} -2 & -3 \\ 8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \\ \Rightarrow \mathbf{M} &= \frac{1}{16} \begin{pmatrix} -8 & -17 \\ 16 & 44 \end{pmatrix}.\end{aligned}$$

4. (a) Sets A and B are such that

(2)

- $n(A) = 11$,
- $n(B) = 13$, and
- $n(A \cup B) = 18$.

Find $n(A \cap B)$.

Solution

Well,

$$\begin{aligned}n(A \text{ only}) &= 18 - 11 = 7, \\ n(B \text{ only}) &= 18 - 13 = 5, \\ n(A \cap B) &= 18 - 7 - 5 = \underline{\underline{6}}.\end{aligned}$$

(b) Sets \mathcal{E} , X , and Y are such that

- $\mathcal{E} = \{\theta : 0 \leq \theta \leq 2\pi\}$,
- $X = \{\theta : \sin \theta = -0.5\}$, and
- $Y = \{\theta : \sec^2 \theta = \frac{4}{3}\}$.

(i) Find, in terms of π , the elements of the set X .

(1)

Solution

Well,

$$\sin \theta = -0.5 \Rightarrow \theta = \underline{\underline{\frac{7}{6}\pi, \frac{11}{6}\pi}}.$$

(ii) Find, in terms of π , the elements of the set Y .

(2)

Solution

Now,

$$\begin{aligned}\sec^2 \theta = \frac{4}{3} &\Rightarrow \sec \theta = \pm \frac{2}{\sqrt{3}} \\ &\Rightarrow \cos \theta = \pm \frac{\sqrt{3}}{2}.\end{aligned}$$

$$\underline{\cos \theta = \frac{\sqrt{3}}{2} :}$$

$$\cos \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{1}{6}\pi, \frac{11}{6}\pi.$$

$$\underline{\cos \theta = -\frac{\sqrt{3}}{2} :}$$

$$\cos \theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{5}{6}\pi, \frac{7}{6}\pi.$$

Hence,

$$\underline{\underline{\theta = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi.}}$$

- (iii) Use set notation to describe the relationship between the sets
- X
- and
- Y
- . (1)

Solution

$$\underline{\underline{X \subset Y.}}$$

5. It is given that

$$\log_{10} p^3 q = 10a \text{ and } \log_{10} \left(\frac{p}{q^2} \right) = a.$$

- (a) Find, in terms of
- a
- , expressions for
- $\log_{10} p$
- and
- $\log_{10} q$
- . (5)

Solution

$$\begin{aligned}\log_{10} p^3 q = 10a &\Rightarrow \log_{10} p^3 + \log_{10} q = 10a \\ &\Rightarrow 3 \log_{10} p + \log_{10} q = 10a \\ &\Rightarrow 6 \log_{10} p + 2 \log_{10} q = 20a \quad (1)\end{aligned}$$

and

$$\begin{aligned}\log_{10} \left(\frac{p}{q^2} \right) = a &\Rightarrow \log_{10} p - \log_{10} q^2 = a \\ &\Rightarrow \log_{10} p - 2 \log_{10} q = a \quad (2).\end{aligned}$$

Add (1) + (2):

$$\begin{aligned}7 \log_{10} p = 21a &\Rightarrow \underline{\underline{\log_{10} p = 3a}} \\ &\Rightarrow \underline{\underline{\log_{10} q = a}}.\end{aligned}$$

(b) Find the value of $\log_p q$.

(1)

Solution

Finally,

$$\begin{aligned}\log_p q &= \frac{\log_{10} q}{\log_{10} p} \\ &= \frac{a}{3a} \\ &= \underline{\underline{\frac{1}{3}}}.\end{aligned}$$

6. A curve has equation

$$y = 6 \cos\left(\frac{1}{2}x\right) + 4 \sin\left(\frac{1}{2}x\right), \text{ for } 0 < x < 2\pi \text{ radians.}$$

(a) Find the x -coordinate of the stationary point on the curve.

(5)

Solution

Well,

$$y = 6 \cos\left(\frac{1}{2}x\right) + 4 \sin\left(\frac{1}{2}x\right) \Rightarrow \frac{dy}{dx} = -3 \sin\left(\frac{1}{2}x\right) + 2 \cos\left(\frac{1}{2}x\right)$$

and

$$\begin{aligned}\frac{dy}{dx} = 0 &\Rightarrow -3 \sin\left(\frac{1}{2}x\right) + 2 \cos\left(\frac{1}{2}x\right) = 0 \\ &\Rightarrow 3 \sin\left(\frac{1}{2}x\right) = 2 \cos\left(\frac{1}{2}x\right) = 0 \\ &\Rightarrow \tan\left(\frac{1}{2}x\right) = \frac{2}{3} \\ &\Rightarrow \frac{1}{2}x = 0.588\,002\,603\,5 \text{ (FCD)} \\ &\Rightarrow x = 1.176\,005\,207 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 1.18 \text{ (3 sf)}}}.\end{aligned}$$

(b) Determine the nature of this stationary point.

(2)

Solution

Now,

$$\frac{d^2y}{dx^2} = -\frac{3}{2} \cos\left(\frac{1}{2}x\right) - \sin\left(\frac{1}{2}x\right)$$

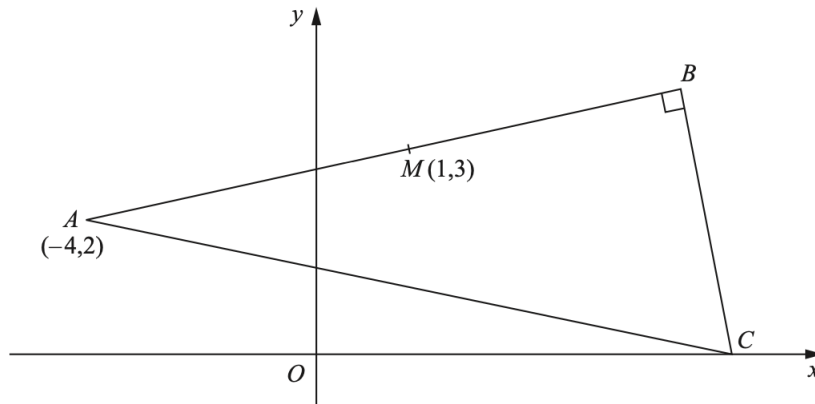
and

$$x = 1.176 \dots \Rightarrow \frac{d^2y}{dx^2} = -1.802 \dots < 0$$

so it is a maximum.

7. The figure shows a right-angled triangle ABC , where the point A has coordinates $(-4, 2)$, the angle B is 90° and the point C lies on the x -axis.

(7)



The point $M(1, 3)$ is the midpoint of AB .

Find the area of the triangle ABC .

Solution

Well,

$$\begin{aligned}\vec{OB} &= \vec{OA} + \vec{AB} \\ &= \vec{OA} + 2\vec{AM} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 1 - (-4) \\ 3 - 2 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} 10 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 4 \end{pmatrix};\end{aligned}$$

so, $B(6, 4)$.

Now,

$$\begin{aligned}m_{AB} &= \frac{4 - 2}{6 - (-4)} \\ &= \frac{2}{10} \\ &= \frac{1}{5}\end{aligned}$$

and

$$m_{BC} = -5.$$

Next, the equation of BC is

$$y - 4 = -5(x - 6)$$

and

$$\begin{aligned}y = 0 &\Rightarrow -4 = -5(x - 6) \\ &\Rightarrow \frac{4}{5} = x - 6 \\ &\Rightarrow x = \frac{34}{5};\end{aligned}$$

so, $C(\frac{34}{5}, 0)$.

In addition, we need the lengths of AB and BC :

$$\begin{aligned}AB &= \sqrt{[6 - (-4)]^2 + (4 - 2)^2} \\ &= \sqrt{10^2 + 2^2} \\ &= \sqrt{104}\end{aligned}$$

and

$$\begin{aligned}BC &= \sqrt{\left(6 - \frac{34}{5}\right)^2 + (4 - 0)^2} \\&= \sqrt{\left(\frac{4}{5}\right)^2 + 4^2} \\&= \sqrt{\frac{416}{25}}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area} &= \frac{1}{2} \times AB \times BC \\&= \frac{1}{2} \times \sqrt{104} \times \sqrt{\frac{416}{25}} \\&= \underline{\underline{20\frac{4}{5}}}.\end{aligned}$$

8. Vectors \mathbf{a} and \mathbf{b} are such that

$$\mathbf{a} = \begin{pmatrix} 3 + m \\ 5 - 2n \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 4 - 2n \\ 10 + 3m \end{pmatrix}.$$

(a) Given that

$$3\mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 + n \\ -5 \end{pmatrix},$$

(4)

find the value of m and of n .

Solution

Well,

$$\begin{aligned}3\mathbf{a} + \mathbf{b} &= \begin{pmatrix} 1 + n \\ -5 \end{pmatrix} \\ \Rightarrow 3 \begin{pmatrix} 3 + m \\ 5 - 2n \end{pmatrix} + \begin{pmatrix} 4 - 2n \\ 10 + 3m \end{pmatrix} &= \begin{pmatrix} 1 + n \\ -5 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 9 + 3m \\ 15 - 6n \end{pmatrix} + \begin{pmatrix} 4 - 2n \\ 10 + 3m \end{pmatrix} &= \begin{pmatrix} 1 + n \\ -5 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 13 + 3m - 2n \\ 25 - 6n + 3m \end{pmatrix} &= \begin{pmatrix} 1 + n \\ -5 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}3m - 3n &= -12 \quad (1) \\ 3m - 6n &= -30 \quad (2).\end{aligned}$$

Subtract (1) – (2):

$$\begin{aligned}3n &= 18 \Rightarrow \underline{n = 6} \\ \Rightarrow 3m - 3(6) &= -12 \\ \Rightarrow 3m &= 6 \\ \Rightarrow \underline{m = 2}.\end{aligned}$$

[Check: $3 \times 2 - 6 \times 6 = -30 \checkmark$]

- (b) Show that the magnitude of \mathbf{b} is $k\sqrt{5}$, where k is an integer to be found. (2)

Solution

Now,

$$\mathbf{b} = \begin{pmatrix} -8 \\ 16 \end{pmatrix}$$

and the magnitude of \mathbf{b} is

$$\begin{aligned}\sqrt{(-8)^2 + 16^2} &= \sqrt{64 + 256} \\ &= \sqrt{320} \\ &= \sqrt{64 \times 5} \\ &= \sqrt{64} \times \sqrt{5} \\ &= \underline{8\sqrt{5}};\end{aligned}$$

hence, $\underline{k = 8}$.

- (c) Find the unit vector in the direction of \mathbf{b} . (1)

Solution

Finally, the unit vector in the direction of \mathbf{b} is

$$\frac{1}{8\sqrt{5}} \begin{pmatrix} -8 \\ 16 \end{pmatrix} = \underline{\underline{\frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}}.$$

9. The function f is defined, for $0^\circ \leq x \leq 360^\circ$, by

$$f(x) = 2 \sin 3x - 1.$$

- (a) State the amplitude and period of f . (2)

Solution

The amplitude is 2 and the period is

$$\frac{360}{3} = \underline{\underline{120^\circ}}.$$

- (b) State the maximum value of f and the corresponding values of x . (3)

Solution

The maximum value is

$$2 - 1 = \underline{\underline{1}}.$$

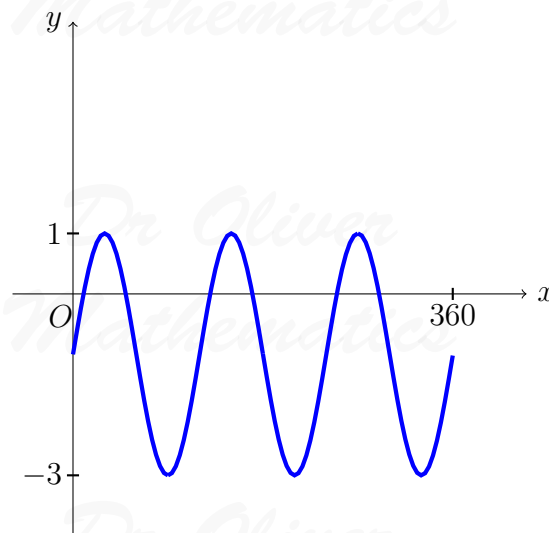
Now,

$$0^\circ \leq x \leq 360^\circ \Rightarrow 0^\circ \leq 3x \leq 1080^\circ$$

and the corresponding values of x is

$$\begin{aligned} \sin 3x = 1 &\Rightarrow 3x = 90, 450, 810 \\ &\Rightarrow \underline{\underline{x = 30, 150, 270.}} \end{aligned}$$

- (c) Sketch the graph of f . (2)

Solution

10. (a) Differentiate

$$\tan(3x + 2)$$

(2)

with respect to x .

Solution

$$\begin{aligned}\frac{d}{dx} [\tan(3x + 2)] &= \sec^2(3x + 2) \times 3 \\ &= \underline{\underline{3 \sec^2(3x + 2)}}.\end{aligned}$$

(b) Differentiate

$$(\sqrt{x} + 1)^{\frac{2}{3}}$$

(3)

with respect to x .

Solution

Well,

$$\sqrt{x} = x^{\frac{1}{2}}$$

and

$$\begin{aligned}\frac{d}{dx} \left[(\sqrt{x} + 1)^{\frac{2}{3}} \right] &= \frac{2}{3}(\sqrt{x} + 1)^{-\frac{1}{3}} \times \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{3}(\sqrt{x} + 1)^{-\frac{1}{3}}x^{-\frac{1}{2}} \\ &= \underline{\underline{\frac{(\sqrt{x} + 1)^{-\frac{1}{3}}}{3\sqrt{x}}}}.\end{aligned}$$

(c) Differentiate

$$\frac{\ln(x^3 - 1)}{2x + 3}$$

(3)

with respect to x .

Solution

Quotient rule:

$$\begin{aligned}u = \ln(x^3 - 1) &\Rightarrow \frac{du}{dx} = \frac{3x^2}{x^3 - 1} \\ v = 2x + 3 &\Rightarrow \frac{dv}{dx} = 2\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx} \left[\frac{\ln(x^3 - 1)}{2x + 3} \right] &= \frac{(2x + 3) \left(\frac{3x^2}{x^3 - 1} \right) - (2)(\ln(x^3 - 1))}{(2x + 3)^2} \\ &= \frac{(2x + 3)(3x^2) - (2 \ln(x^3 - 1))(x^3 - 1)}{(x^3 - 1)(2x + 3)^2} \\ &= \frac{3x^2(2x + 3) - 2(x^3 - 1) \ln(x^3 - 1)}{(x^3 - 1)(2x + 3)^2}.\end{aligned}$$

11. A particle moves in a straight line so that, t s after leaving a fixed point O , its velocity $v \text{ ms}^{-1}$ is given by

$$v = 3e^{2t} + 4t.$$

- (a) Find the initial velocity of the particle. (1)

Solution

Well,

$$t = 0 \Rightarrow \underline{v = 3 \text{ ms}^{-1}}.$$

- (b) Find the initial acceleration of the particle. (3)

Solution

Now,

$$v = 3e^{2t} + 4t \Rightarrow a = 6e^{2t} + 4$$

and

$$t = 0 \Rightarrow a = 6 + 4 = \underline{10 \text{ ms}^{-2}}.$$

- (c) Find the distance travelled by the particle in the third second. (4)

Solution

Well,

$$v = 3e^{2t} + 4t \Rightarrow s = \frac{3}{2}e^{2t} + 2t^2 + c,$$

for some constant c . Now,

$$\begin{aligned}\text{distance travelled} &= s(3) - s(2) \\ &= \left(\frac{3}{2}e^6 + 18 + c\right) - \left(\frac{3}{2}e^2 + 8 + c\right) \\ &= \underline{\underline{\frac{3}{2}e^4(e^2 - 1) + 10 \text{ or } 533 \text{ m (3 sf)}}}}.\end{aligned}$$

EITHER

12. A function f is such that

$$f(x) = \ln(5x - 10), \text{ for } x > 2.$$

(a) State the range of f .

(1)

Solution

$$\underline{\underline{-\infty < f(x) < \infty.}}$$

(b) Find $f^{-1}(x)$.

(3)

Solution

Well,

$$\begin{aligned}y = \ln(5x - 10) &\Rightarrow e^y = 5x - 10 \\ &\Rightarrow e^y + 10 = 5x \\ &\Rightarrow \frac{1}{5}(e^y + 10) = x\end{aligned}$$

and so

$$f^{-1}(x) = \underline{\underline{\frac{1}{5}(e^x + 10)}}.$$

(c) State the range of f^{-1} .

(1)

Solution

$$\underline{\underline{2 < f^{-1}(x) < \infty.}}$$

(d) Solve

(2)

$$f(x) = 0.$$

Solution

Now,

$$\begin{aligned}f(x) = 0 &\Rightarrow \ln(5x - 10) = 0 \\&\Rightarrow 5x - 10 = e^0 \\&\Rightarrow 5x = 11 \\&\Rightarrow \underline{\underline{x = 2\frac{1}{5}}}.\end{aligned}$$

A function g is such that

$$g(x) = 2x - \ln 2, \text{ for } x \in \mathbb{R}.$$

(e) Solve

$$g f(x) = f(x^2).$$

(5)

Solution

Well,

$$\begin{aligned}g f(x) &= g(f(x)) \\&= g(\ln(5x - 10)) \\&= 2 \ln(5x - 10) - \ln 2\end{aligned}$$

and

$$f(x^2) = \ln(5x^2 - 10).$$

Now,

$$\begin{aligned}g f(x) = f(x^2) &\Rightarrow 2 \ln(5x - 10) - \ln 2 = \ln(5x^2 - 10) \\&\Rightarrow \ln(5x - 10)^2 - \ln(5x^2 - 10) = \ln 2\end{aligned}$$

\times	$5x$	-10
$5x$	$25x^2$	$-50x$
-10	$-50x$	$+100$

$$\Rightarrow \ln(25x^2 - 100x + 100) - \ln(5x^2 - 10) = \ln 2$$

$$\Rightarrow \ln \left(\frac{25x^2 - 100x + 100}{5x^2 - 10} \right) = \ln 2$$

$$\Rightarrow \frac{25x^2 - 100x + 100}{5x^2 - 10} = 2$$

$$\Rightarrow 25x^2 - 100x + 100 = 2(5x^2 - 10)$$

$$\Rightarrow 25x^2 - 100x + 100 = 10x^2 - 20$$

$$\Rightarrow 15x^2 - 100x + 120 = 0$$

$$\Rightarrow 5(3x^2 - 20x + 24) = 0$$

$a = 3$, $b = -20$, and $c = 24$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{20 \pm \sqrt{(-20)^2 - 4 \times 3 \times 24}}{2 \times 3} \\ &= \frac{20 \pm \sqrt{112}}{6} \\ &= 1.569\,499\,126, 5.097\,167\,541 \text{ (FCD);} \end{aligned}$$

but $x > 2$ so

$$\underline{\underline{x = 5.10 \text{ (3 sf)}}}.$$

OR

13. A function f is such that

$$f(x) = 4e^{-x} + 2, \text{ for } x \in \mathbb{R}.$$

(a) State the range of f .

(1)

Solution

$$\underline{\underline{f(x) > 2.}}$$

(b) Solve

$$f(x) = 26.$$

(2)

Solution

Well,

$$f(x) = 26 \Rightarrow 4e^{-x} + 2 = 26$$

$$\Rightarrow 4e^{-x} = 24$$

$$\Rightarrow e^{-x} = 6$$

$$\Rightarrow e^x = \frac{1}{6}$$

$$\Rightarrow \underline{\underline{x = \ln \frac{1}{6}}}.$$

(c) Find $f^{-1}(x)$.

(3)

Solution

Now,

$$y = 4e^{-x} + 2 \Rightarrow y - 2 = 4e^{-x}$$

$$\Rightarrow \frac{y - 2}{4} = e^{-x}$$

$$\Rightarrow \frac{4}{y - 2} = e^x$$

$$\Rightarrow \ln \left(\frac{4}{y - 2} \right) = x$$

and so

$$\underline{\underline{f^{-1}(x) = \ln \left(\frac{4}{x - 2} \right)}}.$$

(d) State the domain of f^{-1} .

(1)

Solution

$$\underline{\underline{2 < f^{-1}(x) < \infty}}.$$

A function g is such that

$$g(x) = 2e^x - 4, \text{ for } x \in \mathbb{R}.$$

(e) Using the substitution

$$t = e^x$$

(5)

or otherwise, solve

$$g(x) = f(x).$$

Solution

Well,

$$g(x) = f(x) \Rightarrow 2e^x - 4 = 4e^{-x} + 2$$

$$\Rightarrow 2t - 6 = \frac{4}{t}$$

$$\Rightarrow 2t^2 - 6t = 4$$

$$\Rightarrow 2t^2 - 6t - 4 = 0$$

$$\Rightarrow 2(t^2 - 3t - 2) = 0$$

$a = 1$, $b = -3$, and $c = -2$:

$$\Rightarrow t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow t = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$

$$\Rightarrow t = \frac{3 \pm \sqrt{17}}{2}$$

$$\Rightarrow t = -0.561\,552\,812\,8, 3.561\,552\,813 \text{ (FCD);}$$

but $x > 2$:

$$\Rightarrow e^x = 3.561\,552\,813 \text{ (FCD)}$$

$$\Rightarrow x = 1.270\,196\,633 \text{ (FCD)}$$

$$\Rightarrow \underline{\underline{x = 1.27 \text{ (3 sf)}}}.$$