Dr Oliver Mathematics
General Leibniz Rule

1 General Leibniz Rule

1.1 The Rule

If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are n-times differentiable functions, then the product $(\mathrm{f}\,\mathrm{g})^{(n)}(x)$ n-times differentiable and the nth derivative is given by

$$\boxed{(\mathrm{f}\,\mathrm{g})^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{f}^{(n-k)}(x)\,\mathrm{g}^{(k)}(x)}$$

1.2 $n = 1$

$$\begin{aligned}(\mathrm{f}\,\mathrm{g})'(x) &= \sum_{k=0}^{1} \binom{n}{k} \mathrm{f}^{(n-k)}(x)\,\mathrm{g}^{(k)}(x) \\ &= \mathrm{f}'(x)\,\mathrm{g}(x) + \mathrm{f}(x)\,\mathrm{g}'(x).\end{aligned}$$

1.3 $n = 2$

$$\begin{aligned}(\mathrm{f}\,\mathrm{g})''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}'(x)\,\mathrm{g}(x) + \mathrm{f}(x)\,\mathrm{g}'(x)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}'(x)\,\mathrm{g}(x)\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\,\mathrm{g}'(x)\right] \\ &= \left[\mathrm{f}''(x)\,\mathrm{g}(x) + \mathrm{f}'(x)\,\mathrm{g}'(x)\right] + \left[\mathrm{f}'(x)\,\mathrm{g}'(x) + \mathrm{f}(x)\,\mathrm{g}''(x)\right] \\ &= \mathrm{f}''(x)\,\mathrm{g}(x) + 2\,\mathrm{f}'(x)\,\mathrm{g}'(x) + \mathrm{f}(x)\,\mathrm{g}''(x).\end{aligned}$$

1.4 $n = 3$

$$(\mathrm{f}\,\mathrm{g})'''(x) = \mathrm{f}'''(x)\,\mathrm{g}(x) + 3\,\mathrm{f}''(x)\,\mathrm{g}'(x) + 3\,\mathrm{f}'(x)\,\mathrm{g}''(x) + \mathrm{f}(x)\,\mathrm{g}'''(x).$$

1.5 $n = 4$

$$(\mathrm{f}\,\mathrm{g})^{(4)}(x) = \mathrm{f}^{(4)}(x)\,\mathrm{g}(x) + 4\,\mathrm{f}'''(x)\,\mathrm{g}'(x) + 6\,\mathrm{f}''(x)\,\mathrm{g}''(x) + 4\,\mathrm{f}'(x)\,\mathrm{g}'''(x) + \mathrm{f}(x)\,\mathrm{g}^{(4)}(x).$$

We may prove this by induction — but we are not going to. Instead, let's look at a few examples.

2 Examples

1. Find the second derivative of $x^3 \ln x$.

Solution

It helps to draw a table.

Original	x^3	$\ln x$
1st derivative	$3x^2$	$\frac{1}{x}$
2nd derivative	$6x$	$-\frac{1}{x^2}$

We will work *down* one column (x^3) and work *up* the other ($\ln x$).

$$\begin{aligned}(x^3 \ln x)'' &= (x^3)\left(-\frac{1}{x^2}\right) + 2(3x^2)\left(\frac{1}{x}\right) + (6x)(\ln x)\\ &= -x + 6x + 6x\ln x\\ &= 5x + 6x\ln x\\ &= \underline{\underline{x(5 + 6\ln x)}}.\end{aligned}$$

2. Find the third derivative of $x^4 \sin 2x$.

Solution

Original	x^4	$\sin 2x$
1st derivative	$4x^3$	$2\cos 2x$
2nd derivative	$12x^2$	$-4\sin 2x$
3rd derivative	$24x$	$-8\cos 2x$

We will work *down* one column (x^4) and work *up* the other ($\sin 2x$).

$$\begin{aligned}(x^4 \sin 2x)''' &= (x^4)(-8\cos 2x) + 3(4x^3)(-4\sin 2x) \\ &\quad + 3(12x^2)(2\cos 2x) + (24x)(\sin 2x) \\ &= \underline{\underline{-8x^4\cos 2x - 48x^3\sin 2x + 72x^2\cos 2x + 24x\sin 2x}}.\end{aligned}$$

3. Find the fourth derivative of $3x^2e^{2x}$.

Solution

Original	$3x^2$	e^{2x}
1st derivative	$6x$	$2e^{2x}$
2nd derivative	6	$4e^{2x}$
3rd derivative	0	$8e^{2x}$
4th derivative	0	$16e^{2x}$

We will, as always, work *down* one column ($3x^2$) and work *up* the other (e^{2x}).

$$\begin{aligned}(3x^2e^{2x})^{(4)} &= (3x^2)(16e^{2x}) + 4(6x)(8e^{2x}) + 6(6)(4e^{2x}) + 4(0)(2e^{2x}) + (0)(e^{2x}) \\ &= 48x^2e^{2x} + 192xe^{2x} + 144e^{2x} \\ &= \underline{\underline{(48x^2 + 192x + 144)e^{2x}}}.\end{aligned}$$