Dr Oliver Mathematics
Further Pure Mathematics
Induction
Past Examination Questions

This booklet consists of 30 questions across a variety of examination topics.
The total number of marks available is 227.

1.

$$f(n) \equiv 24 \times 2^{4n} + 3^{4n},\ n \in \mathbb{N}.$$

(a) Write down $f(n+1) - f(n)$. (5)

Solution

$$\begin{aligned} f(n+1) - f(n) &= (24 \times 2^{4(n+1)} + 3^{4(n+1)}) - (24 \times 2^{4n} + 3^{4n}) \\ &= (24 \times 2^{4n+4} + 3^{4n+4}) - (24 \times 2^{4n} + 3^{4n}) \\ &= 24 \times 2^{4n}(2^4 - 1) + 3^{4n}(3^4 - 1) \\ &= \underline{\underline{360 \times 2^{4n} + 80 \times 3^{4n}}}. \end{aligned}$$

(b) Prove, by induction, that $f(n)$ is divisible by 5 for all $n \in \mathbb{N}$. (4)

Solution

$\underline{n = 1}$: $f(0) = 24 \times 2^0 + 3^0 = 25 = 5 \times 5$, and so the solution is true for $n = 1$.
Suppose the solution is true for $n = k$, i.e.,

$$f(k) \equiv 24 \times 2^{4k} + 3^{4k}$$

is divisible by 5. Then

$$f(k+1) = f(k) + 360 \times 2^{4k} + 80 \times 3^{4k}$$

and both 360 and 80 are divisible by 5 and so the whole expression is divisible by 5.
Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.

2. Prove, by induction, that (6)

$$\sum_{r=1}^{n} r(r+3) = \tfrac{1}{3}n(n+1)(n+5)$$

for all $n \in \mathbb{N}$.

Solution

$\underline{n = 1}$: $1 \times 4 = 4$ and $\frac{1}{3} \times 1 \times 2 \times 6 = 4$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} r(r+3) = \tfrac{1}{3}k(k+1)(k+5).$$

Then,

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+3) &= (k+1)(k+4) + \sum_{r=1}^{k+1} r(r+3) \\ &= (k+1)(k+4) + \tfrac{1}{3}k(k+1)(k+5) \\ &= \tfrac{1}{3}(k+1)\left[3(k+4) + k(k+5)\right] \\ &= \tfrac{1}{3}(k+1)(k^2 + 8k + 12) \\ &= \tfrac{1}{3}(k+1)(k+2)(k+6),\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.

3. Prove, by induction, that $(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta$, $n \in \mathbb{N}$.

Solution

$\underline{n = 1}$: $(\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$(\cos\theta + \mathrm{i}\sin\theta)^k \equiv \cos k\theta + \mathrm{i}\sin k\theta.$$

Then,

$$\begin{aligned}&(\cos\theta + \mathrm{i}\sin\theta)^{k+1} \\ =&(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta)^k \\ =&(\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta) \\ =&(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + \mathrm{i}(\cos k\theta\sin\theta + \cos\theta\sin k\theta) \\ =&\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.

4. Prove that the expression (8)

$$7^n + 4^n + 1$$

is divisible by 6 for all $n \in \mathbb{N}$.

Solution

Let $f(n) \equiv 7^n + 4^n + 1$.

$f(1) = 7 + 4 + 1 = 12 = 6 \times 2$ and so the result is true for $n = 1$.

Suppose now that the result is true for some $n = k$, i.e., $f(k)$ is divisible by 6. Then

$$\begin{aligned} f(k+1) - f(k) &= (7^{k+1} + 4^{k+1} + 1) - (7^k + 4^k + 1) \\ &= (7^{k+1} - 7^k) - (4^{k+1} - 4^k) \\ &= 7^k(7-1) - 4^k(4-1) \\ &= 6 \times 7^k - 3 \times 4^k, \end{aligned}$$

and this is clearly divisible by 6: the first term is a multiple of 6 and the second term is 3 times and even number. So, by the inductive hypothesis, $f(k+1)$ is divisible by 6.

Hence, by mathematical induction, $f(n)$ is divisible by 6 all $n \in \mathbb{N}$, as required.

5. Prove that the expression (8)

$$4^n + 6n - 1$$

is divisible by 9 for all $n \in \mathbb{Z}^+$.

Solution

Let $\mathrm{f}(n) \equiv 4^n + 6n - 1$.
$\underline{n = 1}$: $\mathrm{f}(1) = 4 + 6 - 1 = 9$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e., $\mathrm{f}(k) = 4^k + 6k - 1$ is divisible by 9. Then

$$\begin{aligned} \mathrm{f}(k+1) &= 4^{k+1} + 6(k+1) - 1 \\ &= 4(4^k + 6k - 1) - 18k + 9 \\ &= 4(4^k + 6k - 1) - 9(2k - 1) \\ &= 4\,\mathrm{f}(k) - 9(2k - 1), \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

6. Prove that the expression (8)

$$3^{4n-1} + 2^{4n-1} + 5$$

is divisible by 10 for all positive integers n.

Solution

Let $f(n) = 3^{4n-1} + 2^{4n-1} + 5$.

$\underline{n = 1}$: $f(1) = 3^3 + 2^3 + 5 = 40 = 10 \times 4$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e., $f(k) = 3^{4k-1} + 2^{4k-1} + 5$ is divisible by 9. Then

$$\begin{aligned} f(k+1) - f(k) &= (3^{4(k+1)-1} + 2^{4(k+1)-1} + 5) - (3^{4k-1} + 2^{4k-1} + 5) \\ &= (3^{4k+3} - 3^{4k-1}) + (2^{4k+3} - 2^{4k-1}) \\ &= 3^{4k-1}(3^4 - 1) + 2^{4k-1}(2^4 - 1) \\ &= 80 \times 3^{4k-1} + 15 \times 2^{4k-1} \\ &= 80 \times 3^{4k-1} + 30 \times 2^{4k-2}, \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

7. (a) Express $\dfrac{6x + 10}{x + 3}$ in the form $p + \dfrac{q}{x + 3}$, where p and q are integers to be found. (1)

Solution

$$\frac{6x + 10}{x + 3} = \frac{6(x + 3) - 8}{x + 3} = \underline{\underline{6 - \frac{8}{x + 3}}}.$$

The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5.2$ and $u_{n+1} = \dfrac{6u_n + 10}{u_n + 3}$.

(b) Prove by induction that $u_n > 5$ for $n \in \mathbb{Z}^+$. (4)

Solution

$\underline{n = 1}$: $u_1 = 5.2 > 5$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e., $u_k > 5$. Then

$$\begin{aligned} u_{k+1} &= \frac{6u_k + 10}{u_k + 3} = 6 - \frac{8}{u_k + 3} \\ \Rightarrow \quad & \frac{8}{u_k + 3} < 1 \\ \Rightarrow \quad & 8 < u_k + 3 \\ \Rightarrow \quad & u_k > 5, \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

8. Prove, by induction, that, for $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} r2^r = 2\left[1 + (n-1)2^n\right].$$

Solution

$\underline{n = 1}$: $\sum_{r=1}^{1} r2^r = 1 \times 2^1 = 2$ and $2\left[1 + 0 \times 2^1\right] = 2$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} r2^r = 2\left[1 + (k-1)2^k\right].$$

Then

$$\begin{aligned} \sum_{r=1}^{k+1} r2^r &= \sum_{r=1}^{k} r2^r + (k+1)2^{k+1} \\ &= 2\left[1 + (k-1)2^k\right] + (k+1)2^{k+1} \\ &= 2\left[1 + (k-1)2^k\right] + 2(k+1)2^k \\ &= 2\left[1 + (k-1)2^k + (k+1)2^k\right] \\ &= 2\left[1 + 2k \times 2^k\right] \\ &= 2\left[1 + k2^{k+1}\right] \\ &= 2\left[1 + \left\{(k-1)+1\right\}2^{k+1}\right], \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

9. (5)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove by induction, that for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2+3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution

$\underline{n=1}$:

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2+3\times 1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\mathbf{A}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{A}^{k+1} &= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+3k)+k+2 \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+5k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k+1 & \frac{1}{2}\left[(k+1)^2+3(k+1)\right] \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

10. Prove by induction, for $n \in \mathbb{Z}^+$, $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$. (5)

Solution

$\underline{n=1}$: $\sum_{r=1}^{1}(2r-1)^2 = 1$ and $\frac{1}{3} \times 1 \times 1 \times 3 = 1$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k}(2r-1)^2 = \tfrac{1}{3}k(2k-1)(2k+1).$$

Then

$$\begin{aligned}\sum_{r=1}^{k+1}(2r-1)^2 &= \tfrac{1}{3}k(2k-1)(2k+1) + (2k+1)^2 \\ &= \tfrac{1}{3}(2k+1)\left[k(2k-1) + 3(2k+1)\right] \\ &= \tfrac{1}{3}(2k+1)(2k^2+5k+3) \\ &= \tfrac{1}{3}(k+1)(2k+1)(2k+3) \\ &= \tfrac{1}{3}(k+1)\left[2(k+1)-1\right]\left[2(k+1)+1\right]\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

11. De Moivre's theorem states that (5)

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta,\ n \in \mathbb{R}.$$

Use induction to prove de Moivre's theorem for $n \in \mathbb{Z}^+$.

Solution

$\underline{n=1}$: $(\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$(\cos\theta + \mathrm{i}\sin\theta)^k \equiv \cos k\theta + \mathrm{i}\sin k\theta.$$

Then,

$$\begin{aligned}
&(\cos\theta + \mathrm{i}\sin\theta)^{k+1} \\
=&(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta + \mathrm{i}\sin\theta)^{k} \\
=&(\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta) \\
=&(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + \mathrm{i}(\cos k\theta\sin\theta + \cos\theta\sin k\theta) \\
=&\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta
\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.

12. Prove by induction, for $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}.$$

Solution

$\underline{n=1}$: LHS $= \frac{1}{1\times 2} = \frac{1}{2}$ and RHS $= \frac{1}{1+1} = \frac{1}{2}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}.$$

Then,

$$\begin{aligned}
\sum_{r=1}^{k+1} \frac{1}{r(r+1)} &= \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

13. A series of positive integers $u_1, u_2, u_3, \ldots$ is defined by (5)

$$u_1 = 6 \text{ and } u_{n+1} = 6u_n - 5, \text{ for } n \geqslant 1.$$

Prove by induction that $u_n = 5 \times 6^{n-1} + 1$, for $n \geqslant 1$.

Solution

$\underline{n = 1}$: $u_1 = 5 \times 6^0 + 1 = 6$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$u_k = 5 \times 6^{k-1} + 1.$$

Then,

$$\begin{aligned} u_{k+1} &= 6u_k - 5 \\ &= 6(5 \times 6^{k-1} + 1) - 5 \\ &= 5 \times 6^k + 6 - 5 \\ &= 5 \times 6^k + 1 \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

14. Prove by induction, for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$, (5)

Solution

$\underline{n = 1}$: LHS $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and RHS $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}.$$

Then,

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}^{k+1} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}^{k}$$
$$= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos k\theta & -\sin k\theta\\ \sin k\theta & \cos k\theta\end{pmatrix}$$
$$= \begin{pmatrix}\cos k\theta\cos\theta - \sin k\theta\sin\theta & -\sin k\theta\cos\theta - \sin\theta\cos k\theta\\ \sin k\theta\cos\theta + \sin\theta\cos k\theta & \cos k\theta\cos\theta - \sin k\theta\sin\theta\end{pmatrix}$$
$$= \begin{pmatrix}\cos(k+1)\theta & -\sin(k+1)\theta\\ \sin(k+1)\theta & \cos(k+1)\theta\end{pmatrix}$$

and so the result is true for $n = k+1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) $(4n+3)5^n - 3$ is divisible by 16. (5)

Solution

Let $\mathrm{f}(n) = (4n+3)5^n - 3$.

$\underline{n=1}$: $(4\times 1 + 3)5^1 - 3 = 7\times 5 - 3 = 32 = 16\times 2$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$(4k+3)5^k - 3 \text{ is divisible by } 16.$$

Then,

$$\mathrm{f}(k+1) - \mathrm{f}(k) = \left[(4(k+1)+3)5^{k+1} - 3\right] - \left[(4k+3)5^k - 3\right]$$
$$= (4k+7)5^{k+1} - (4k+3)5^k$$
$$= 5^k\left[(20k+35) - (4k+3)\right]$$
$$= 5^k(16k+32)$$

which is divisible by 16 and so the result is true for $n = k+1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

15. Prove by induction, for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}^n = \begin{pmatrix}1 & 0 & 0\\ n & 1 & 0\\ n(n+2) & 2n & 1\end{pmatrix}$, (5)

Solution

$\underline{n=1}$: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1(1+2) & 2\times 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k(k+2) & 2k & 1 \end{pmatrix}.$$

Then,

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}^k \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k(k+2) & 2k & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ k+1 & 1 & 0 \\ 3+2k+k(k+2) & 2+2k & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ k+1 & 1 & 0 \\ k^2+4k+3 & 2(k+1) & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ k+1 & 1 & 0 \\ (k+1)(k+3) & 2(k+1) & 1 \end{pmatrix} \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) $(2^{3n+1} + 5)$ is divisible by 7. (5)

Solution

Let $\mathrm{f}(n) = 2^{3n+1} + 5$.

$\underline{n=1}$: $2^4 + 7 = 21 = 7 \times 3$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$2^{3k+1} + 5 \text{ is divisible by } 7.$$

Then,

$$\begin{aligned} f(k+1) - f(k) &= (2^{3(k+1)+1} + 5) - (2^{3k+1} + 5) \\ &= 2^{3k+4} - 2^{3k+1} \\ &= 2^{3k+4} - 2^{3k+1} \\ &= 2^{3k+1}(2^3 - 1) \\ &= 7 \times 2^{3k+1} \end{aligned}$$

which is divisible by 7 and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

16. Prove by induction, for $n \in \mathbb{Z}^+$,

(a) $f(n) \equiv 5^n + 8n + 3$ is divisible by 4, (7)

Solution

$\underline{n = 1}$: $f(1) = 5 + 8 + 3 = 16 = 4 \times 4$ which is divisible by 4 and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$f(k) \equiv 5^k + 8k + 3 \text{ is divisible by } 4.$$

Then,

$$\begin{aligned} f(k+1) - f(k) &= (5^{k+1} + 8(k+1) + 3) - (5^k + 8k + 3) \\ &= 5^{k+1} - 5^k + 8(k+1) - 8k \\ &= 5^k(5 - 1) + 8 \\ &= 4 \times 5^k + 8 \\ &= 4(5^k + 2) \end{aligned}$$

which is divisible by 4 and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$. (7)

Solution

$\underline{n=1}$: $\begin{pmatrix} 2\times 1+1 & -2\times 1 \\ 2\times 1 & 1-2\times 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}.$$

Then,

$$\begin{aligned}
\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^k \\
&= \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix} \\
&= \begin{pmatrix} 3(2k+1)-4k & -6k-2(1-2k) \\ 2(2k+1)-2k & -4k-(1-2k) \end{pmatrix} \\
&= \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix} \\
&= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}
\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

17. A sequence of numbers is defined by (4)

$$u_1 = 2,$$
$$u_{n+1} = 5u_n - 4, \; n \geqslant 1.$$

Prove by induction that, $n \in \mathbb{Z}^+$, $u_n = 5^{n-1} + 1$.

Solution

$\underline{n=1}$: $u_1 = 5^0 + 1 = 2$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$u_k = 5^{k-1} + 1.$$

Then,

$$\begin{aligned} u_{k+1} &= 5u_k - 4 \\ &= 5(5^{k-1} + 1) - 4 \\ &= 5^k + 5 - 4 \\ &= 5^k + 1 \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

18.

$$\mathrm{f}(n) \equiv 2^n + 6^n.$$

(a) Show that $\mathrm{f}(k+1) = 6\,\mathrm{f}(k) - 4 \times 2^k$. (3)

Solution

$$\begin{aligned} \mathrm{f}(k+1) &= 2^{k+1} + 6^{k+1} \\ &= 2 \times 2^k + 6 \times 6^k \\ &= 6(2^k + 6^k) - 4 \times 2^k \\ &= \underline{\underline{6\,\mathrm{f}(k+1) - 4 \times 2^k}}. \end{aligned}$$

(b) Hence, or otherwise, prove by induction that, $n \in \mathbb{Z}^+$, $\mathrm{f}(n)$ is divisible by 8. (4)

Solution

$\underline{n = 1}$: $\mathrm{f}(1) = 2^1 + 6^1 = 8$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\mathrm{f}(k) = 2^k + 6^k \text{ is divisible by 8.}$$

Then,

$$\begin{aligned} \mathrm{f}(k+1) &= 6\,\mathrm{f}(k) - 4 \times 2^k \\ &= 6\,\mathrm{f}(k) - 8 \times 2^{k-1} \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

19. A series of positive integers u_1, u_2, u_3, ... is defined by (5)

$$u_1 = 2 \text{ and } u_{n+1} = 4u_n + 2, \text{ for } n \geqslant 1.$$

Prove by induction that $u_n = \frac{2}{3}(4^n - 1)$.

Solution

$\underline{n = 1}$: $u_1 = \frac{2}{3}(4^1 - 1) = 2$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$u_k = \tfrac{2}{3}(4^k - 1).$$

Then,

$$\begin{aligned} u_{k+1} &= 4u_k + 2 \\ &= 4 \times \tfrac{2}{3}(4^n - 1) + 2 \\ &= \tfrac{2}{3} \times 4^{n+1} - \tfrac{8}{3} + 2 \\ &= \tfrac{2}{3} \times 4^{n+1} - \tfrac{2}{3} \\ &= \tfrac{2}{3}(4^{k+1} - 1) \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

20. (a) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$. (6)

Solution

$\underline{n = 1}$: $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}.$$

Then,

$$\begin{aligned}
\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k} \\
&= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 6 \times 3^k + 3(3^k - 1) & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 9 \times 3^k - 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3 \times 3^{k+1} - 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}
\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) $f(n) \equiv 7^{2n-1} + 5$ is divisible by 12. (6)

Solution

$\underline{n = 1}$: $\mathrm{f}(1) = 7^1 + 5 = 12$ which is divisible by 12 and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\mathrm{f}(k) \equiv 7^{2k-1} + 5 \text{ is divisible by } 12.$$

Then,

$$\begin{aligned}
\mathrm{f}(k+1) - \mathrm{f}(k) &= (7^{2k+1} + 5) - (7^{2k-1} + 5) \\
&= 7^{2k+1} - 7^{2k-1} \\
&= 7^{2k-1}(7^2 - 1) \\
&= 48 \times 7^{2k-1} \\
&= 12 \times 4 \times 7^{2k-1}
\end{aligned}$$

which is divisible by 12 and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

21. A sequence can be described by the recurrence formula

$$u_1 = 1 \text{ and } u_{n+1} = 2u_n + 1, \text{ for } n \geqslant 1.$$

(a) Find u_2 and u_3. (2)

Solution

$u_2 = \underline{\underline{3}}$ and $u_3 = \underline{\underline{7}}$

(b) Prove by induction that $u_n = 2^n - 1$. (5)

Solution

$\underline{n = 1}$: $u_1 = 1 = 2 \times 1 - 1$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$u_k = 2^k + 1.$$

Then,

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

22. Prove by induction, for $n \in \mathbb{Z}^+$, (6)

$$\mathrm{f}(n) \equiv 2^{2n-1} + 3^{2n-1} \text{ is divisible by 5.}$$

Solution

$\underline{n = 1}$: $\mathrm{f}(1) = 2^1 + 3^1 = 5$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\mathrm{f}(k) = 2^{2k-1} + 3^{2k-1} \text{ is divisible by 5.}$$

Then,

$$\begin{aligned} \mathrm{f}(k+1) - \mathrm{f}(k) &= (2^{2k+1} + 3^{2k+1}) - (2^{2k-1} + 3^{2k-1}) \\ &= (2^{2k+1} - 2^{2k-1}) - (3^{2k+1} - 3^{2k-1}) \\ &= 2^{2k-1}(2^2 - 1) - 3^{2k-1}(3^2 - 1) \\ &= 3 \times 2^{2k-1} + 8 \times 3^{2k-1} \\ &= 3(2^{2k-1} + 3^{2k-1}) + 5 \times 3^{2k-1} \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

23. (a) Prove by induction that, for $n \in \mathbb{Z}^+$, (6)

$$\sum_{r=1}^{n} r(r+3) = \tfrac{1}{3}n(n+1)(n+5).$$

Solution

$\underline{n = 1}$: LHS $= 1 \times 4 = 4$ and RHS $= \frac{1}{3} \times 1 \times 2 \times 6 = 4$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} r(r+3) = \tfrac{1}{3}k(k+1)(k+5).$$

Then,

$$\begin{aligned} \sum_{r=1}^{k+1} r(r+3) &= \sum_{r=1}^{k} r(r+3) + (k+1)(k+4) \\ &= \tfrac{1}{3}k(k+1)(k+5) + (k+1)(k+4) \\ &= \tfrac{1}{3}(k+1)[k(k+5) + 3(k+4)] \\ &= \tfrac{1}{3}(k+1)(k^2 + 8k + 12) \\ &= \tfrac{1}{3}(k+1)(k+2)(k+6) \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) A sequence can be described by the recurrence formula (5)

$$u_1 = 1 \text{ and } u_{n+1} = u_n + n(3n+1), \text{ for } n \geqslant 1.$$

Prove by induction that

$$u_n = n^2(n-1) + 1.$$

Solution

$\underline{n=1}$: $u_1 = 1 = 1^2 \times 0 + 1 = 1$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$u_k = k^2(k-1) + 1.$$

Then,

$$\begin{aligned} u_{k+1} &= u_k + k(3k+1) \\ &= k^2(k-1) + 1 + k(3k+1) \\ &= k^3 - k^2 + 1 + 3k^2 + k \\ &= k^3 + 2k^2 + k + 1 \\ &= (k+1)^2 k + 1 \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

24. (a) A sequence can be described by the recurrence formula (5)

$$u_1 = 8 \text{ and } u_{n+1} = 4u_n - 9n, \text{ for } n \geqslant 1.$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 4^n + 3n + 1.$$

Solution

$\underline{n=1}$: $u_1 = 4^1 + 3 \times 1 + 1 = 8$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$u_k = 4^k + 3k + 1.$$

Then,

$$\begin{aligned}
u_{k+1} &= 4u_k - 9k \\
&= 4(4^k + 3k + 1) - 9k \\
&= 4^{k+1} + 12k + 4 - 9k \\
&= 4^{k+1} + 3k + 4 \\
&= 4^{k+1} + 3(k+1) + 1
\end{aligned}$$

and so the result is true for $n = k + 1$.
Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) Prove by induction that, for $m \in \mathbb{Z}^+$, (5)

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^m = \begin{pmatrix} 2m+1 & -4m \\ m & 1-2m \end{pmatrix}.$$

Solution

$\underline{n = 1}$: $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 1 & -4 \times 1 \\ 1 & 1 - 2 \times 1 \end{pmatrix}$ and so the solution is true for $n = 1$.
Suppose the solution is true for $n = k$, i.e.,

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -4k \\ k & 1-2k \end{pmatrix}.$$

Then,

$$\begin{aligned}
\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k \\
&= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2k+1 & -4k \\ k & 1-2k \end{pmatrix} \\
&= \begin{pmatrix} 3(2k+1) - 4k & -12k - 4(1-2k) \\ 2k+1-k & -4k - (1-2k) \end{pmatrix} \\
&= \begin{pmatrix} 2k+3 & -4k-4 \\ k+1 & -1-2k \end{pmatrix} \\
&= \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ k+1 & 1-2(k+1) \end{pmatrix}
\end{aligned}$$

and so the result is true for $n = k + 1$.
Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

25. Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$\sum_{r=1}^{n} r(2r-1) = \tfrac{1}{6}n(n+1)(4n-1).$$

Solution

$\underline{n=1}$: LHS $= 1 \times 1 = 1$ and RHS $= \frac{1}{6} \times 1 \times 2 \times 3 = 1$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} r(2r-1) = \tfrac{1}{6}k(k+1)(4k-1).$$

Then,

$$\begin{aligned}\sum_{r=1}^{k+1} r(2r-1) &= \sum_{r=1}^{k} r(2r-1) + (k+1)(2k+1) \\ &= \tfrac{1}{6}k(k+1)(4k-1) + (k+1)(2k+1) \\ &= \tfrac{1}{6}(k+1)[k(4k-1) + 6(2k+1)] \\ &= \tfrac{1}{6}(k+1)(4k^2 + 11k + 6) \\ &= \tfrac{1}{6}(k+1)(k+2)(4k+3) \\ &= \tfrac{1}{6}(k+1)[(k+1)+1][4(k+1)-1]\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

26. Prove by induction that, $n \in \mathbb{Z}^+$,

$$\mathrm{f}(n) \equiv 8^n - 2^n$$

is divisible by 6.

Solution

$\underline{n=1}$: $\mathrm{f}(1) = 8^1 - 2^1 = 6$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\mathrm{f}(k) \equiv 8^k - 2^k \text{ is divisible by 6.}$$

Then,

$$\begin{aligned}
\mathrm{f}(k+1) - \mathrm{f}(k) &= (8^{k+1} - 2^{k+1}) - (8^k - 2^k) \\
&= (8^{k+1} - 8^k) - (2^{k+1} - 2^k) \\
&= 8^k(8-1) - 2^k(2-1) \\
&= 7 \times 8^k - 2^k \\
&= 6 \times 8^k + (8^k - 2^k) \\
&= 6 \times 8^k + \mathrm{f}(k)
\end{aligned}$$

and so the result is true for $n = k+1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

27. (a) Prove by induction that, for $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n2^n.$$

Solution

$\underline{n=1}$: $2 \times 1 = 2$ and $1 \times 2 = 2$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} (r+1)2^{r-1} = k2^k.$$

Then,

$$\begin{aligned}
\sum_{r=1}^{k+1} (r+1)2^{r-1} &= \sum_{r=1}^{k} (r+1)2^{r-1} + (k+2)2^k \\
&= k2^k + (k+2)2^k \\
&= (2k+2)2^k \\
&= 2(k+1)2^k \\
&= (k+1)2^{k+1}
\end{aligned}$$

and so the result is true for $n = k+1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) A sequence can be described by the recurrence formula (7)

$$\begin{aligned} u_1 &= 0, \\ u_2 &= 32, \\ u_{n+2} &= 6u_{n+1} - 8u_n,\ n \geqslant 1. \end{aligned}$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 4^{n+1} - 2^{n+3}.$$

Solution

$\underline{n = 1}$: $u_1 = 0$ and $u_1 = 4^2 - 2^4 = 16 - 16 = 0$ and so the solution is true for $n = 1$.

$\underline{n = 2}$: $u_2 = 8$ and $u_2 = 4^3 - 2^5 = 64 - 32 = 32$ and so the solution is true for $n = 2$.

$\underline{n = 3}$: $u_3 = 6u_2 - 8u_1 = 6 \times 32 - 8 \times 0 = 192$ and $u_3 = 4^4 - 2^5 = 256 - 64 = 192$ and so the solution is true for $n = 3$.

Suppose the solution is true for $n \leqslant k$, i.e.,

$$u_k = 4^{k+1} - 2^{k+3}.$$

Then,

$$\begin{aligned} u_{k+2} &= 6u_{k+1} - 8u_k \\ &= 6(4^{k+2} - 2^{k+4}) - 8(4^{k+1} - 2^{k+3}) \\ &= 6 \times 4^{k+2} - 6 \times 2^{k+4} - 8 \times 4^{k+1} + 8 \times 2^{k+3} \\ &= 6 \times 4^{k+2} - 3 \times 2^{k+5} - 2 \times 4^{k+2} + 2 \times 2^{k+5} \\ &= 4 \times 4^{k+2} - 2^{k+5} \\ &= 4^{k+3} - 2^{k+5} \\ &= 4^{(k+2)+1} - 2^{(k+2)+3} \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

28. (a) Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}.$$

Solution

$\underline{n=1}$: $\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^1-1) & 5^1 \end{pmatrix}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix}.$$

Then,

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k \\
&= \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -1-\frac{5}{4}(5^k-1) & 5^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -1-\frac{1}{4}\times 5^{k+1}+\frac{5}{4} & 5^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}\times 5^{k+1}+\frac{1}{4} & 5^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1}-1) & 5^{k+1} \end{pmatrix}
\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$\sum_{r=1}^{n}(2r-1)^2 = \tfrac{1}{3}n(4n^2-1).$$

Solution

$\underline{n=1}$: $(2\times 1-1)^2 = 1$ and $\frac{1}{3}\times 1\times 3 = 1$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k}(2r-1)^2 = \tfrac{1}{3}k(4k^2-1).$$

Then,

$$\begin{aligned}\sum_{r=1}^{k+1}(2r-1)^2 &= \sum_{r=1}^{k}(2r-1)^2 + (2k+1)^2\\ &= \tfrac{1}{3}k(4k^2-1) + (2k+1)^2\\ &= \tfrac{1}{3}[k(4k^2-1) + 3(2k+1)^2]\\ &= \tfrac{1}{3}[(4k^3-k) + 3(4k^2+4k+1)]\\ &= \tfrac{1}{3}[(4k^3-k) + (12k^2+12k+3)]\\ &= \tfrac{1}{3}(4k^3+12k^2+11k+3)\\ &= \tfrac{1}{3}(k+1)(4k^2+8k+3)\\ &= \tfrac{1}{3}(k+1)[4(k^2+2k+1)-1]\\ &= \tfrac{1}{3}(k+1)[4(k+1)^2-1]\end{aligned}$$

and so the result is true for $n = k+1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

29. (a) Prove by induction that, $n \in \mathbb{Z}^+$, (5)

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

Solution

$\underline{n=1}$: $\frac{2\times1+1}{1^2\times2^2} = \frac{3}{4}$ and $1 - \frac{1}{2^2} = \frac{3}{4}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+1)^2}.$$

Then,

$$\begin{aligned}\sum_{r=1}^{k+1}\frac{2r+1}{r^2(r+1)^2} &= \sum_{r=1}^{k}\frac{2r+1}{r^2(r+1)^2}+\frac{2k+3}{(k+1)^2(k+2)^2}\\ &= 1-\frac{1}{(k+1)^2}+\frac{2k+3}{(k+1)^2(k+2)^2}\\ &= 1-\frac{(k+2)^2}{(k+1)^2(k+2)^2}+\frac{2k+3}{(k+1)^2(k+2)^2}\\ &= 1-\frac{(k+2)^2-(2k+3)}{(k+1)^2(k+2)^2}\\ &= 1-\frac{(k^2+4k+4)-(2k+3)}{(k+1)^2(k+2)^2}\\ &= 1-\frac{n^2+2k+1}{(k+1)^2(k+2)^2}\\ &= 1-\frac{(k+1)^2}{(k+1)^2(k+2)^2}\\ &= 1-\frac{1}{(k+2)^2},\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) A sequence of positive rational numbers is defined by (5)

$$\begin{aligned}u_1 &= 3,\\ u_{n+1} &= \tfrac{1}{3}u_n + \tfrac{8}{9}, n \in \mathbb{Z}^+.\end{aligned}$$

Prove by induction that, $n \in \mathbb{Z}^+$,

$$u_n = 5 \times \left(\tfrac{1}{3}\right)^n + \tfrac{4}{3}.$$

Solution

$\underline{n=1}$: $u_1 = 5 \times \left(\frac{1}{3}\right)^1 + \frac{4}{3} = 3$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$u_k = 5 \times \left(\tfrac{1}{3}\right)^k + \tfrac{4}{3}.$$

Then,

$$\begin{aligned} u_{k+1} &= \tfrac{1}{3}u_k + \tfrac{8}{9} \\ &= 5 \times \left(\tfrac{1}{3}\right)^{k+1} + \tfrac{4}{9} + \tfrac{8}{9} \\ &= 5 \times \left(\tfrac{1}{3}\right)^{k+1} + \tfrac{4}{3}, \end{aligned}$$

and so the result is true for $n = k + 1$.
Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

30. (a) A sequence of numbers is defined by (6)

$$\begin{aligned} u_1 &= 6, \\ u_2 &= 27, \\ u_{n+2} &= 6u_{n+1} - 9u_n,\ n \geqslant 1. \end{aligned}$$

Prove by induction that, $n \in \mathbb{Z}^+$,

$$u_n = 3^n(n+1).$$

Solution
$\underline{n = 1}$: $u_1 = 3^1(1+1) = 6$.
$\underline{n = 2}$: $u_2 = 3^2(2+1) = 27$ and so the solution is true for $n = 1$ and $n = 2$.
Suppose the solution is true for $n \leqslant k + 1$, i.e.,

$$u_{k+1} = 6u_k - 9u_{k-1}.$$

Then

$$\begin{aligned} u_{k+2} &= 6u_{k+1} - 9u_k \\ &= 6[3^{k+1}(k+1+1)] - 9[3^k(k+1)] \\ &= 2[3^{k+2}(k+2)] - 3^{k+2}(k+1) \\ &= 3^{k+2}[2(k+2) - (k+1)] \\ &= 3^{k+2}(2k+4-k-1) \\ &= 3^{k+2}[(k+2)+1] \end{aligned}$$

and so the result is true for $n = k + 2$.
Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

(b) Prove by induction that, $n \in \mathbb{Z}^+$, (6)

$$f(n) = 3^{3n-2} + 2^{3n+1}$$

is divisible by 19.

Solution

<u>$n = 1$</u> : $f(1) = 3^1 + 2^4 = 19$ and so the solution is true for $n = 1$.
Suppose the solution is true for $n = k$, i.e.,

$$f(k) = 3^{3k-2} + 2^{3k+1} \text{ is divisible by 19.}$$

Then

$$\begin{aligned} f(k+1) - f(k) &= (3^{3(k+1)-2} + 2^{3(k+1)+1}) - (3^{3k-2} + 2^{3k+1}) \\ &= (3^{3k+1} + 2^{3k+4}) - (3^{3k-2} + 2^{3k+1}) \\ &= (3^{3k+1} - 3^{3k-2}) - (2^{3k+4} - 2^{3k+1}) \\ &= 3^{3k-2}(3^3 - 1) - 2^{3k+1}(2^3 - 1) \\ &= 3^{3k-2} \times 26 - 2^{3k+1} \times 7 \\ &= 3^{3k-2} \times 19 - 7(3^{3k-2} - 2^{3k+1}) \\ &= 3^{3k-2} \times 19 - 7\,f(k) \end{aligned}$$

and so the result is true for $n = k + 1$.
Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.