Dr Oliver Mathematics
Mathematics: Higher
2023 Paper 1: Non-Calculator
1 hour 15 minutes

The total number of marks available is 55.
You must write down all the stages in your working.

1. Given that (3)

$$y = x^{\frac{5}{3}} - \frac{10}{x^4}, \text{ where } x \neq 0,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = x^{\frac{5}{3}} - \frac{10}{x^4} \Rightarrow y = x^{\frac{5}{3}} - 10x^{-4}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{5}{3}x^{\frac{2}{3}} + 40x^{-5}.$$

2. P and Q are the points $(-2, 6)$ and $(10, 0)$. (4)

Find the equation of the perpendicular bisector of PQ.

Solution

Well, the midpoint is

$$\left(\frac{-2+10}{2}, \frac{6+0}{2}\right) = (4, 3).$$

Now,

$$m_{PQ} = \frac{6-0}{-2-10} = \tfrac{6}{-12} = -\tfrac{1}{2}$$

which means

$$m_{\text{normal}} = -\frac{1}{-\frac{1}{2}} = 2.$$

Finally, the equation of the perpendicular bisector of PQ is

$$y - 3 = 2(x - 4) \Rightarrow y - 3 = 2x - 8$$
$$\Rightarrow \underline{\underline{y = 2x - 5}}.$$

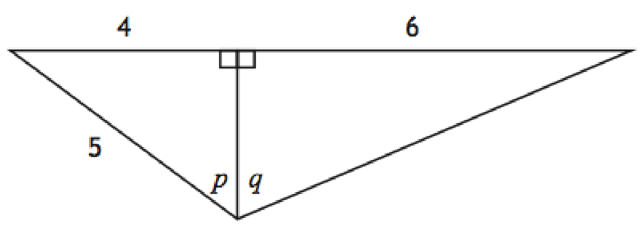
3. Solve (3)

$$\log_5 x - \log_5 3 = 2.$$

Solution

$$\log_5 x - \log_5 3 = 2 \Rightarrow \log_5 \left(\tfrac{1}{3}x\right) = 2$$
$$\Rightarrow \tfrac{1}{3}x = 5^2$$
$$\Rightarrow \tfrac{1}{3}x = 25$$
$$\Rightarrow \underline{\underline{x = 75}}.$$

4. The diagram shows two right-angled triangles with angles p and q as marked.



(a) Determine the value of:

(i) $\cos p$, (1)

Solution
Well, we have a 3-4-5 right-angled triangle so

$$\underline{\underline{\cos p = \tfrac{3}{5}}}.$$

(ii) $\cos q$. (1)

Solution
Well,

$$\begin{aligned} \text{opp}^2 + \text{adj}^2 = \text{hyp}^2 &\Rightarrow 3^2 + 6^2 = \text{hyp}^2 \\ &\Rightarrow 9 + 36 = \text{hyp}^2 \\ &\Rightarrow \text{hyp}^2 = 45 \\ &\Rightarrow \text{hyp} = \sqrt{45} \\ &\Rightarrow \text{hyp} = \sqrt{9 \times 5} \\ &\Rightarrow \text{hyp} = \sqrt{9} \times \sqrt{5} \\ &\Rightarrow \text{hyp} = 3\sqrt{5} \end{aligned}$$

and so

$$\underline{\underline{\cos q = \tfrac{3}{3\sqrt{5}} = \underline{\underline{\tfrac{\sqrt{5}}{5}}}}}.$$

(b) Hence determine the value of $\cos(p + q)$. (3)

Solution
Cosine rule:

$$\begin{aligned} & (4 + 6)^2 = 5^2 + (3\sqrt{5})^2 - 2 \times 5 \times 3\sqrt{5} \times \cos(p + q) \\ \Rightarrow\quad & 100 = 25 + 45 - 30\sqrt{5}\cos(p + q) \\ \Rightarrow\quad & 30\sqrt{5}\cos(p + q) = -30 \\ \Rightarrow\quad & \cos(p + q) = \underline{\underline{-\tfrac{\sqrt{5}}{5}}}. \end{aligned}$$

5. The equation (3)

$$2x^2 + (3p - 2)x + p = 0$$

has equal roots.

Determine the possible values of p.

Solution
Well, $a = 2$, $b = 3p - 2$, and $c = p$ and

$$b^2 - 4ac = 0 \Rightarrow (3p - 2)^2 - 4(2)(p) = 0$$

$\times$	$3p$	-2
$3p$	$9p^2$	$-6p$
-2	$-6p$	$+4$

$$\Rightarrow (9p^2 - 12p + 4) - 8p = 0$$
$$\Rightarrow 9p^2 - 20p + 4 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -20 \\ \text{multiply to:} & (+9)\times(+4) = +36\end{array}\right\} -18,\ -2$$

$$\Rightarrow 9p^2 - 18p - 2p + 4 = 0$$
$$\Rightarrow 9p(p-2) - 2(p-2) = 0$$
$$\Rightarrow (9p-2)(p-2) = 0$$
$$\Rightarrow 9p - 2 = 0 \text{ or } p - 2 = 0$$
$$\Rightarrow \underline{\underline{p = \tfrac{2}{9}}} \text{ or } \underline{\underline{p = 2}}.$$

6. Find (4)

$$\int (2x^5 - 6\sqrt{x})\,\mathrm{d}x,\ x \geqslant 0.$$

Solution

$$\int (2x^5 - 6\sqrt{x})\,\mathrm{d}x = \int (2x^5 - 6x^{\frac{1}{2}})\,\mathrm{d}x$$
$$= \underline{\underline{\tfrac{1}{3}x^6 - 4x^{\frac{3}{2}} + c}}.$$

7. (a) Evaluate (2)

$$\log_2 5 + \log_2 \tfrac{1}{40}.$$

Solution

$$\begin{aligned}
\log_2 5 + \log_2 \tfrac{1}{40} &= \log_2(5 \times \tfrac{1}{40}) \\
&= \log_2 \tfrac{1}{8} \\
&= \log_2(2^{-3}) \\
&= -3 \log_2 2 \\
&= \underline{\underline{-3}}.
\end{aligned}$$

(b) Given that $a \in \mathbb{R}$ and that $\log_8 a$ is negative, state the range of possible values of a. (1)

Solution

$\underline{\underline{0 < a < 1}}$.

8. A function, f, is defined on $\mathbb{R}$, the set of real numbers, by (6)

$$\mathrm{f}(x) = x^3 + 3x^2 - 9x + 5.$$

Find the coordinates of the stationary points of f and determine their nature.

Solution

$$\mathrm{f}(x) = x^3 + 3x^2 - 9x + 5 \Rightarrow \mathrm{f}'(x) = 3x^2 + 9x - 9$$

and

$$\begin{aligned}
\mathrm{f}'(x) = 0 &\Rightarrow 3x^2 + 6x - 9 = 0 \\
&\Rightarrow 3(x^2 + 2x - 3) = 0
\end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +2 \\ \text{multiply to:} & -3 \end{array}\right\} + 3, \, -1$$

$$\begin{aligned}
&\Rightarrow 3(x + 3)(x - 1) = 0 \\
&\Rightarrow x = -3 \text{ or } x = 1.
\end{aligned}$$

We need a 'table of signs':

	$x < -3$	$x = -3$	$-3 < x < 1$	$x = 1$	$x > 1$
$x + 3$	$-$	0	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	0	$+$
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$

Next,

$$\begin{aligned} f(-3) &= (-3)^3 + 3[(-3)^2] - 9(-3) + 5 \\ &= -27 + 27 + 27 + 5 \\ &= 32 \end{aligned}$$

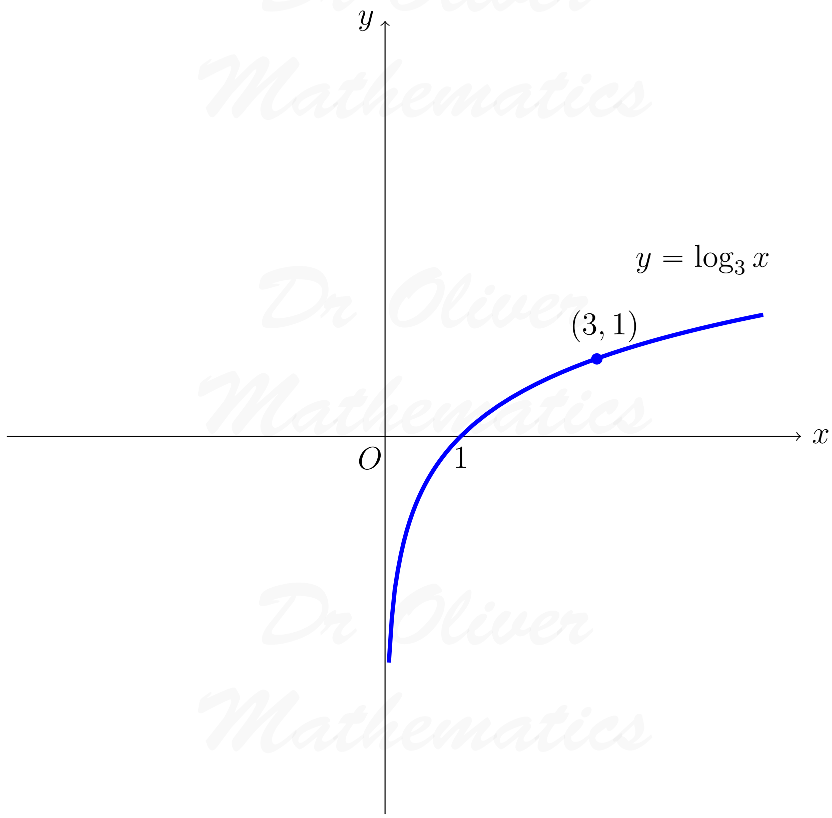
and

$$\begin{aligned} f(1) &= (1)^3 + 3[1^2] - 9(1) + 5 \\ &= 1 + 3 - 9 + 5 \\ &= 0. \end{aligned}$$

Finally, $(-3, 32)$ is a (local) maximum and $(1, 0)$ is a (local) minimum.

9. The diagram shows the graph of the function (3)

$$f(x) = \log_3 x, \text{ where } x > 0.$$



The inverse function, f^{-1}, exists.

Sketch the graph of

$$y = \mathrm{f}^{-1}(x) - 1.$$

Solution

Well,

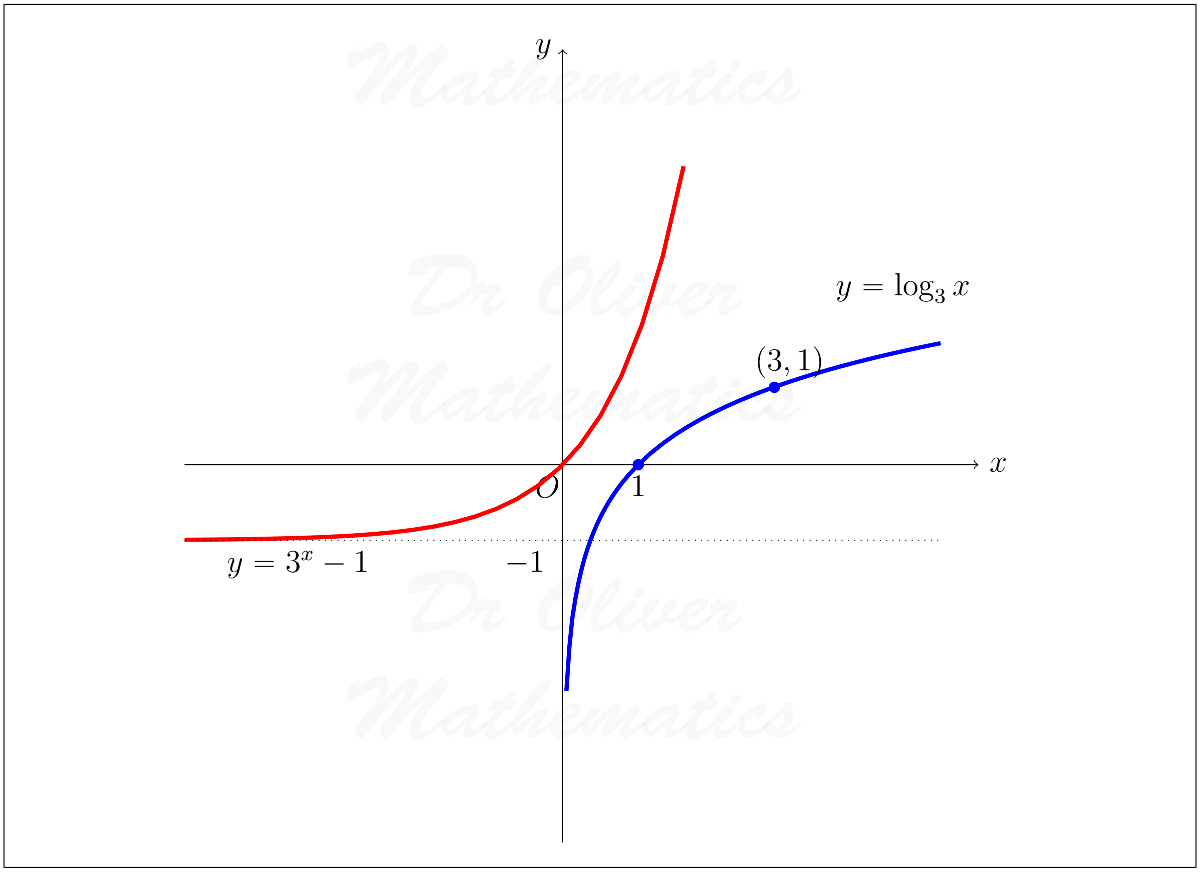
$$y = \log_3 x \Rightarrow x = 3^y$$

and so

$$\mathrm{f}^{-1}(x) = 3^x.$$

In particular,

$$y = 3^x - 1.$$



10. (a) Show that $(x+5)$ is a factor of (2)

$$x^4 + 3x^3 - 7x^2 + 9x - 30.$$

Solution

We use synthetic division:

-5	1	3	-7	$+9$	-30
	$\downarrow$	-5	10	-15	30
	1	-2	3	-6	0

So,

$$x^4 + 3x^3 - 7x^2 + 9x - 30 = (x+5)(x^3 - 2x^2 + 3x - 6)$$

and, hence, $(x+5)$ is a factor of $x^4 + 3x^3 - 7x^2 + 9x - 30$.

(b) Hence, or otherwise, solve (5)

$$x^4 + 3x^3 - 7x^2 + 9x - 30 = 0,\ x \in \mathbb{R}.$$

Solution

Well, we have in the factored cubic $(x^3 - 2x^2 + 3x - 6)$ the coefficient of x^3 (1) and the coefficient of x (3) are **double** the same as the coefficient of x^2 (-2) and the constant term (-6). Now,

$$\begin{aligned} x^3 - 2x^2 + 3x - 6 &= (x^3 + 3x) - (2x^2 + 6) \\ &= x(x^2 + 3) - 2(x^2 + 3) \\ &= (x - 2)(x^2 + 3). \end{aligned}$$

Finally,

$$\begin{aligned} x^4 + 3x^3 - 7x^2 + 9x - 30 = 0 &\Rightarrow (x + 5)(x - 2)(x^2 + 3) = 0 \\ &\Rightarrow x + 5 = \ \text{or}\ x - 2 = 0 \\ &\Rightarrow \underline{\underline{x = -5}}\ \text{or}\ \underline{\underline{x = 2}}. \end{aligned}$$

11. (a) Evaluate (3)

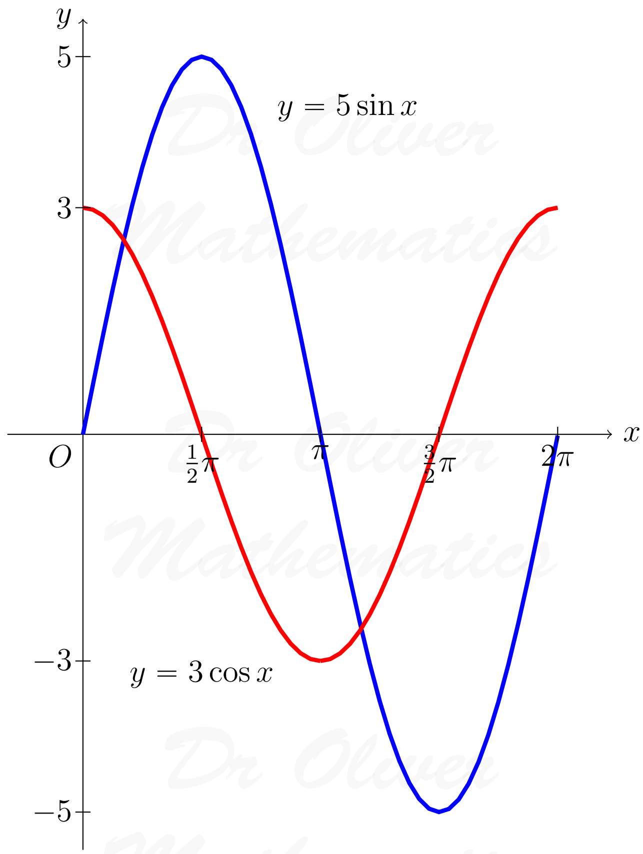
$$\int_{\frac{1}{2}\pi}^{\pi} (5\sin x - 3\cos x)\,\mathrm{d}x.$$

Solution

$$\begin{aligned} \int_{\frac{1}{2}\pi}^{\pi} (5\sin x - 3\cos x)\,\mathrm{d}x &= \left[-5\cos x - 3\sin x\right]_{x=\frac{1}{2}\pi}^{\pi} \\ &= (5 - 0) - (0 - 3) \\ &= \underline{\underline{8}}. \end{aligned}$$

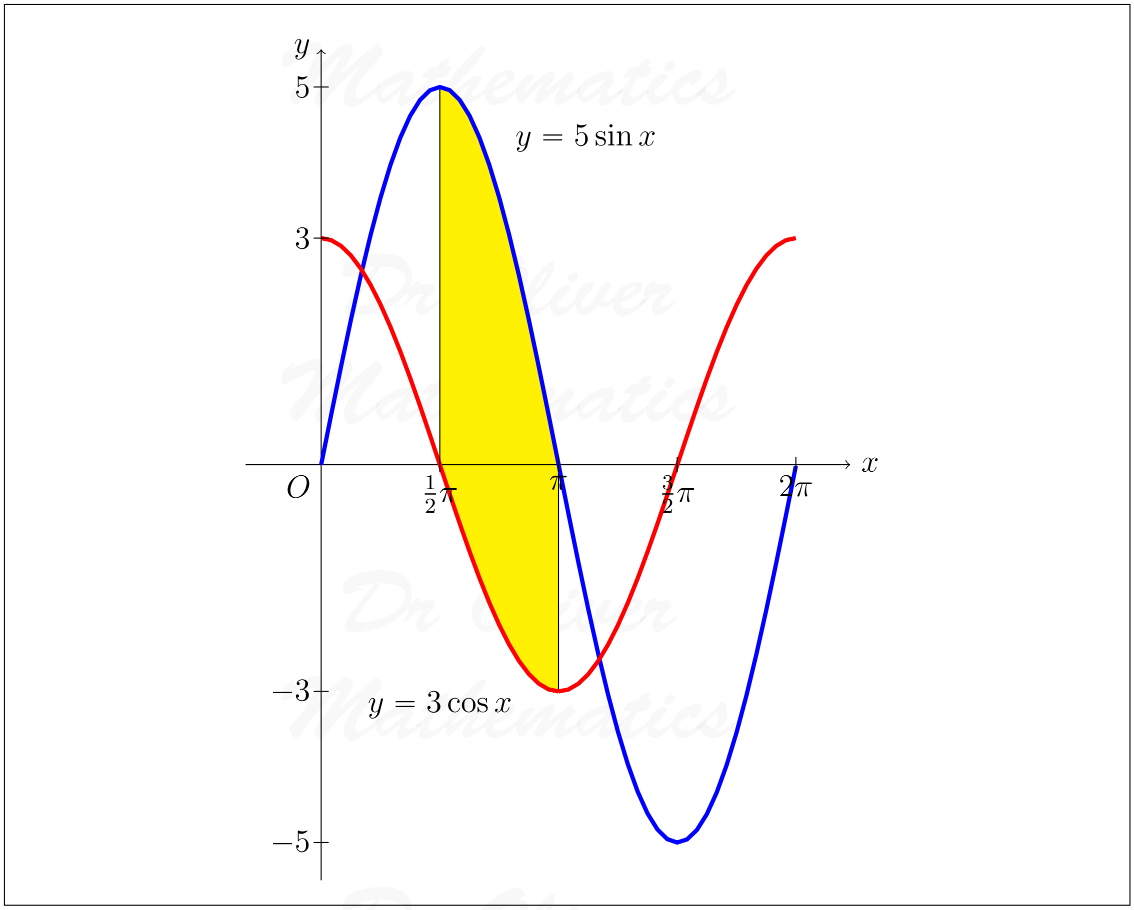
The diagram below shows the graphs with equations

$$y = 5\sin x \text{ and } y = 3\cos x,\ 0 \leqslant x \leqslant 2\pi.$$



(b) Shade the area represented by the integral in (a). (1)

Solution



12. Express (3)

$$-2x^2 - 12x + 7$$

in the form

$$a(x + b)^2 + c.$$

Solution

$$\begin{aligned} -2x^2 - 12x + 7 &= -2[x^2 + 6x] + 7 \\ &= -2[(x^2 + 6x + 9) - 9] + 7 \\ &= -2[(x + 3)^2 - 9] + 7 \\ &= -2(x + 3)^2 + 18 + 7 \\ &= \underline{\underline{-2(x + 3)^2 + 25}}; \end{aligned}$$

hence, $\underline{\underline{a = -2}}$, $\underline{\underline{b = 3}}$, and $\underline{\underline{c = 25}}$.

13. Functions f and g are defined by:

- $\mathrm{f}(x) = 2\sin x,$ where $0 < x < \frac{1}{2}\pi$ and
- $\mathrm{g}(x) = 2x,$ where $0 < x < \frac{1}{4}\pi$.

(a) (i) Evaluate (1)

$$\mathrm{f}(\mathrm{g}(\tfrac{1}{6}\pi)).$$

Solution

$$\begin{aligned} \mathrm{f}(\mathrm{g}(\tfrac{1}{6}\pi)) &= \mathrm{f}(\tfrac{1}{3}\pi) \\ &= 2\sin(\tfrac{1}{3}) \\ &= \underline{\underline{\sqrt{3}}}. \end{aligned}$$

(ii) Determine an expression for $\mathrm{f}(\mathrm{g}(x))$. (2)

Solution

$$\begin{aligned} \mathrm{f}(\mathrm{g}(x)) &= \mathrm{f}(2x) \\ &= \underline{\underline{2\sin 2x}}. \end{aligned}$$

(b) (i) Given that (1)

$$\mathrm{f}(p) = \tfrac{1}{3},$$

determine the exact value of $\sin p$.

Solution

$$\begin{aligned} \mathrm{f}(p) = \tfrac{1}{3} &\Rightarrow 2\sin p = \tfrac{1}{3} \\ &\Rightarrow \underline{\underline{\sin p = \tfrac{1}{6}}}. \end{aligned}$$

(ii) Hence, determine the exact value of $\mathrm{f}(\mathrm{g}(p))$. (3)

Solution

Well,

$$\mathrm{f}(\mathrm{g}(p)) = 2\sin(2\sin p)$$

recall $\sin 2A = 2\sin A\cos A$

$$\begin{aligned} &= 2(2\sin p\cos p) \\ &= 4 \times \tfrac{1}{6} \times \tfrac{\sqrt{35}}{6} \\ &= 4 \times \tfrac{\sqrt{35}}{36} \\ &= \underline{\underline{\tfrac{\sqrt{35}}{9}}}. \end{aligned}$$