Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
20112 June Paper 2 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. (a) Given that (2)

$$\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 2 & 5 \end{pmatrix},$$

find the inverse matrix $\mathbf{A}^{-1}$.

Solution
Well,

$$\det \mathbf{A} = 20 - (-6) = 26$$

and

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{26}\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix}}}.$$

(b) Use your answer to part (a) to solve the simultaneous equations (2)

$$4x - 3y = -10,$$
$$2x + 5y = 21.$$

Solution

Now,

$$\begin{aligned}
\begin{pmatrix} 4 & -3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -10 \\ 21 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 4 & -3 \\ 2 & 5 \end{pmatrix}^{-1}\begin{pmatrix} -10 \\ 21 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \tfrac{1}{26}\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -10 \\ 21 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \tfrac{1}{26}\begin{pmatrix} 13 \\ 104 \end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} \\ 4 \end{pmatrix};
\end{aligned}$$

hence,

$$\underline{\underline{x = \tfrac{1}{2},\, y = 4.}}$$

2. A cuboid has a square base of side (4)

$$(2 + \sqrt{3}) \text{ cm}$$

and a volume of

$$(16 + 9\sqrt{3}) \text{ cm}^3.$$

Without using a calculator, find the height of the cuboid in the form

$$(a + b\sqrt{3}) \text{ cm},$$

where a and b are integers.

Solution

Well,

$\times$	2	$+\sqrt{3}$
2	4	$+2\sqrt{3}$
$+\sqrt{3}$	$+2\sqrt{3}$	$+3$

so the area is $(7 + 4\sqrt{3})$ cm^2.

Finally, the height of the cuboid is

$$\begin{aligned}\frac{16+9\sqrt{3}}{(2+\sqrt{3})^2} &= \frac{16+9\sqrt{3}}{7+4\sqrt{3}}\\ &= \frac{16+9\sqrt{3}}{7+4\sqrt{3}} \times \frac{7-4\sqrt{3}}{7-4\sqrt{3}}\end{aligned}$$

$\times$	16	$+9\sqrt{3}$
7	112	$+63\sqrt{3}$
$-4\sqrt{3}$	$-64\sqrt{3}$	-108

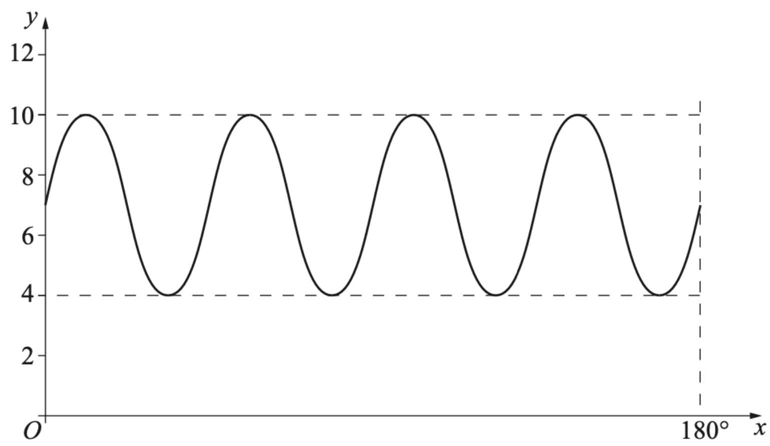
$\times$	7	$+4\sqrt{3}$
7	49	$+28\sqrt{3}$
$-4\sqrt{3}$	$-28\sqrt{3}$	-48

$$\begin{aligned}&= \frac{4-\sqrt{3}}{1}\\ &= \underline{\underline{4-\sqrt{3}}};\end{aligned}$$

hence, $\underline{\underline{a=4}}$ and $\underline{\underline{b=-1}}$.

3. (a) The diagram shows a sketch of the curve (3)

$$y = a\sin(bx) + c \text{ for } 0^\circ \leqslant x \leqslant 180^\circ.$$



Find the values of a, b, and c.

Solution

Well,

$$a = 10 - 7 = \underline{\underline{3}},$$
$$b = 4 \times 2 = \underline{\underline{8}},$$
$$c = \underline{\underline{7}}.$$

(b) Given that

$$\mathrm{f}(x) = 5\cos 3x + 1,$$

for all x, state

(i) the period of f, (1)

Solution

$$\frac{2\pi}{3} = \underline{\underline{\tfrac{2}{3}\pi}}.$$

(ii) the amplitude of f. (1)

Solution

$\underline{\underline{5}}$.

4. (a) Find (2)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \ln x).$$

Solution

Product rule:

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$v = \ln x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \ln x) &= (x^2)\left(\frac{1}{x}\right) + (2x)(\ln x)\\ &= \underline{\underline{x + 2x \ln x}}.\end{aligned}$$

(b) Hence, or otherwise, find (3)

$$\int x \ln x \, \mathrm{d}x.$$

Solution

Now,

$$\begin{aligned}2x \ln x = \frac{\mathrm{d}}{\mathrm{d}x}(x^2 \ln x) - x &\Rightarrow x \ln x = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \ln x) - \tfrac{1}{2}x\\ &\Rightarrow \int x \ln x \, \mathrm{d}x = \underline{\underline{\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + c}}.\end{aligned}$$

5. (a) Solve the equation (2)

$$3^{2x} = 1\,000,$$

giving your answer to 2 decimal places.

Solution

Well,

$$\begin{aligned}3^{2x} = 1\,000 &\Rightarrow 2x = \log_3 1\,000\\ &\Rightarrow x = \tfrac{1}{2}\log_3 1\,000\\ &\Rightarrow x = 3.143\,854\,911 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = 3.14 \text{ (2 dp)}}}.\end{aligned}$$

(b) Solve the equation (4)

$$\frac{36^{2y-5}}{6^{3y}} = \frac{6^{2y-1}}{216^{y+6}}.$$

Solution

Now,

$$\begin{aligned}\frac{36^{2y-5}}{6^{3y}} = \frac{6^{2y-1}}{216^{y+6}} &\Rightarrow \frac{(6^2)^{2y-5}}{6^{3y}} = \frac{6^{2y-1}}{(6^3)^{y+6}} \\ &\Rightarrow \frac{6^{4y-10}}{6^{3y}} = \frac{6^{2y-1}}{6^{3y+18}} \\ &\Rightarrow 6^{y-10} = 6^{-y-19}\end{aligned}$$

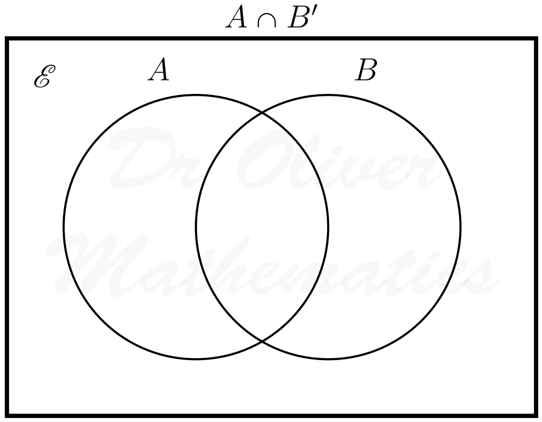
take $\log_6$:

$$\begin{aligned}&\Rightarrow y - 10 = -y - 19 \\ &\Rightarrow 2y = -9 \\ &\Rightarrow \underline{\underline{y = -\tfrac{9}{2}}}.\end{aligned}$$

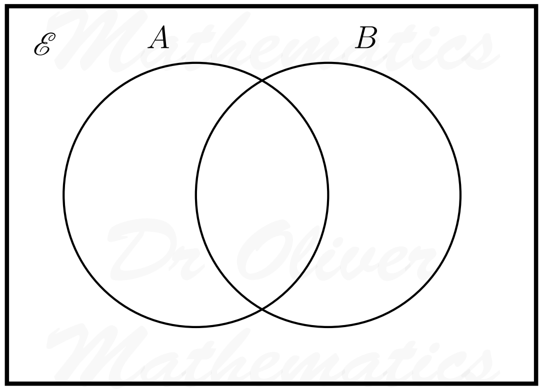
6. By shading the Venn diagrams below, investigate whether each of the following statements is true or false.
State your conclusions clearly.

(a) $A \cap B' = (A' \cap B)'$. (2)

$A \cap B'$

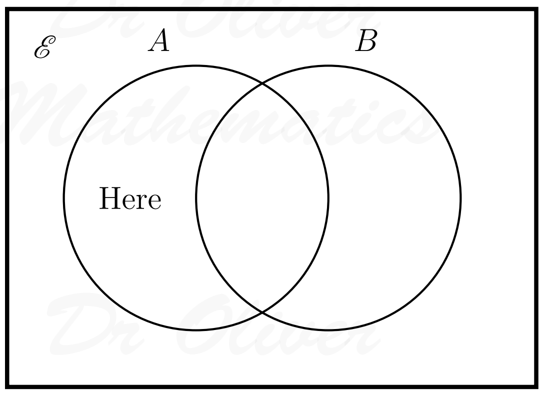


$(A' \cap B)'$

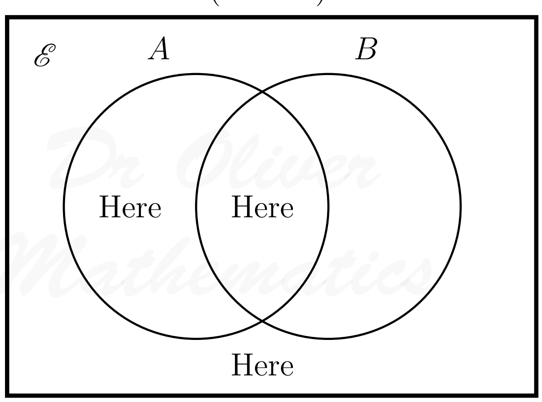


Solution

$A \cap B'$



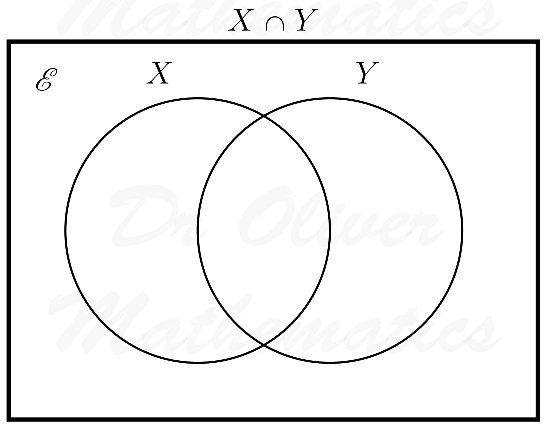
$(A' \cap B)'$

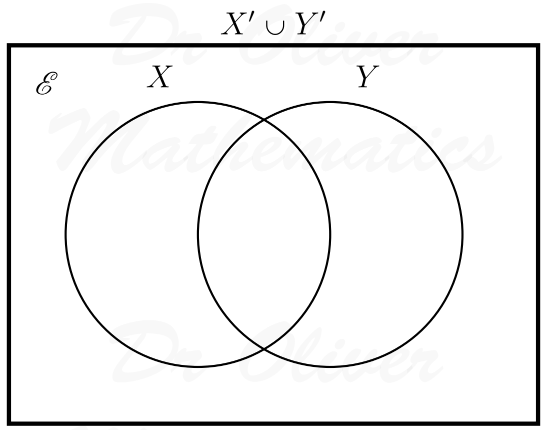


Hence,

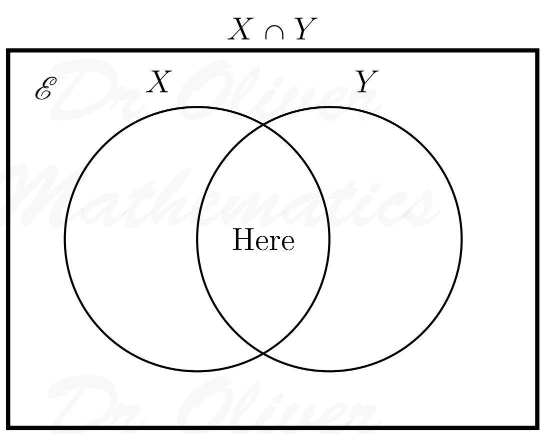
$$\underline{\underline{A \cap B' \neq (A' \cap B)'}}.$$

(b) $X \cap Y = X' \cup Y'$. (2)

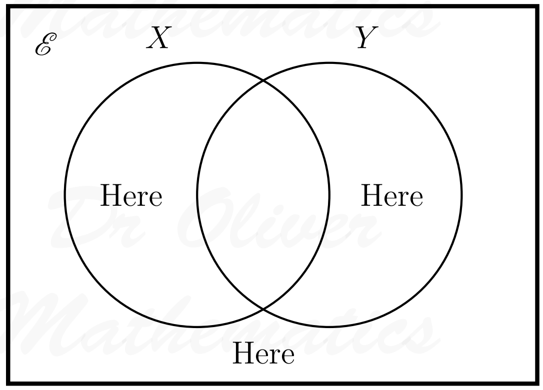




Solution



$X' \cup Y'$

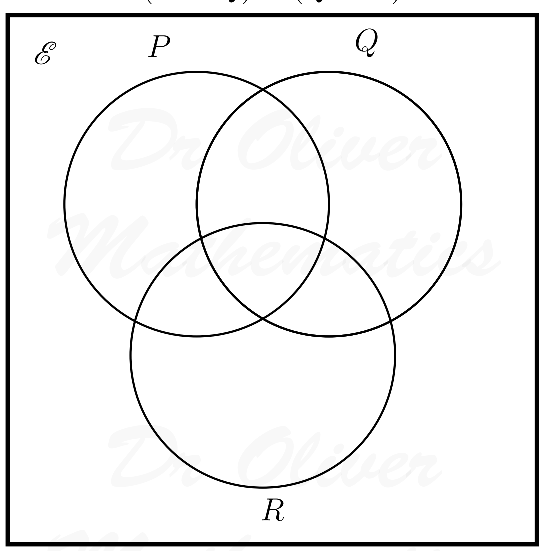


Hence,

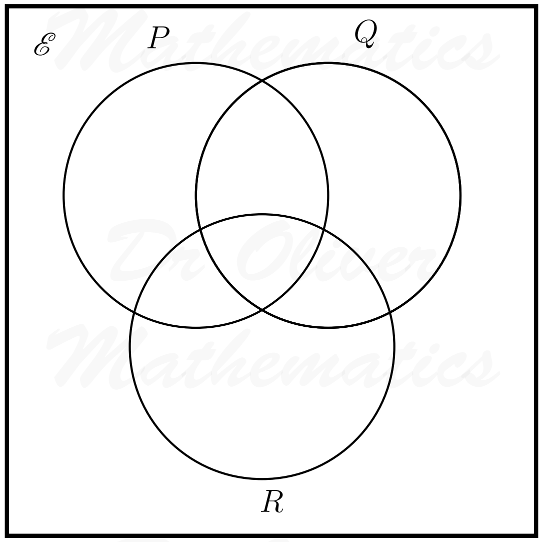
$$\underline{\underline{X \cap Y \neq X' \cup Y'}}.$$

(c) $(P \cap Q) \cup (Q \cap R) = Q \cap (P \cup R)$. (2)

$(P \cap Q) \cup (Q \cap R)$

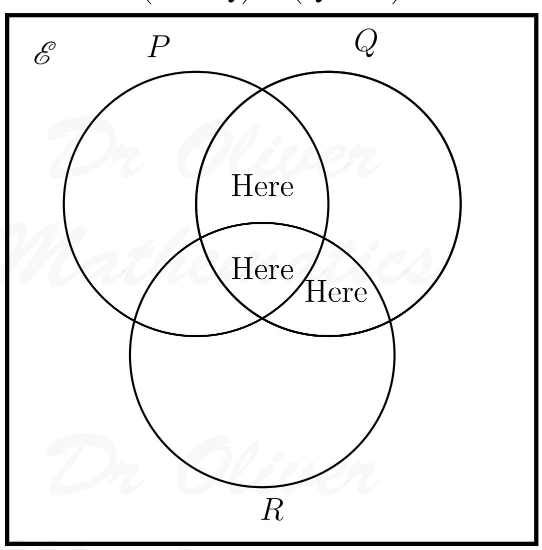


$Q \cap (P \cup R)$

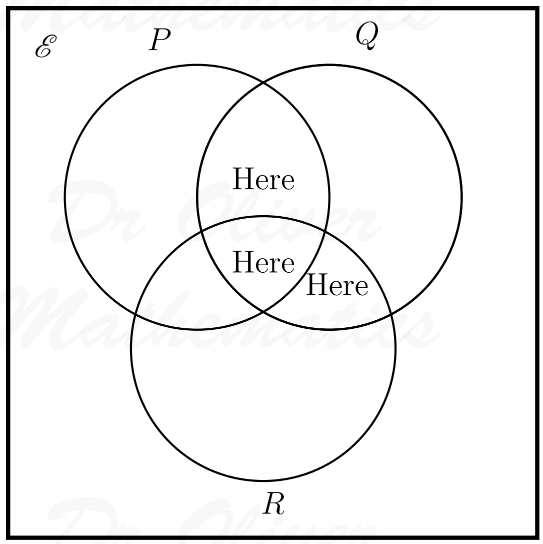


Solution

$(P \cap Q) \cup (Q \cap R)$



$$Q \cap (P \cup R)$$



Hence,

$$\underline{\underline{(P \cap Q) \cup (Q \cap R) = Q \cap (P \cup R).}}$$

7. Given that (6)

$$\mathrm{f}(x) = x^2 - \frac{648}{\sqrt{x}},$$

find the value of x for which

$$\mathrm{f}''(x) = 0.$$

Solution

$$\begin{aligned} \mathrm{f}(x) = x^2 - \frac{648}{\sqrt{x}} &\Rightarrow \mathrm{f}(x) = x^2 - \frac{648^{-\frac{1}{2}}}{x} \\ &\Rightarrow \mathrm{f}'(x) = 2x + \frac{324^{-\frac{3}{2}}}{x} \\ &\Rightarrow \mathrm{f}''(x) = 2 - \frac{486^{-\frac{5}{2}}}{x} \end{aligned}$$

and

$$
\begin{aligned}
\mathrm{f}''(x) = 0 &\Rightarrow 2 - \frac{486^{-\frac{5}{2}}}{x} = 0 \\
&\Rightarrow 2 = \frac{486^{-\frac{5}{2}}}{x} \\
&\Rightarrow x^{\frac{5}{2}} = 243 \\
&\Rightarrow (x^{\frac{1}{2}})^5 = 243 \\
&\Rightarrow x^{\frac{1}{2}} = 3 \\
&\Rightarrow \underline{\underline{x = 9}}.
\end{aligned}
$$

8. Relative to an origin O, the position vectors of the points A and B are $2\mathbf{i} - 3\mathbf{j}$ and $11\mathbf{i} + 42\mathbf{j}$ respectively.

   (a) Write down an expression for $\overrightarrow{AB}$. (2)

Solution

Well,

$$
\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\
&= -\overrightarrow{OA} + \overrightarrow{OB} \\
&= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 11 \\ 42 \end{pmatrix} \\
&= \underline{\underline{\begin{pmatrix} 9 \\ 45 \end{pmatrix}}}.
\end{aligned}
$$

The point C lies on AB such that $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$.

   (b) Find the length of $\overrightarrow{OC}$. (4)

Solution

Now,

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + \tfrac{1}{3}\overrightarrow{AB} \\ &= \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \tfrac{1}{3}\begin{pmatrix} 9 \\ 45 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 15 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 12 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}OC &= \sqrt{5^2 + 12^2} \\ &= \underline{\underline{13}}.\end{aligned}$$

The point D lies on $\overrightarrow{OA}$ such that $\overrightarrow{DC}$ is parallel to $\overrightarrow{OB}$.

(c) Find the position vector of D. (2)

Solution

Well,

$$\overrightarrow{OD} = \begin{pmatrix} 2\lambda \\ -3\lambda \end{pmatrix},$$

for some λ. Now,

$$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{OC} + \overrightarrow{CD} \\ &= \overrightarrow{OC} - \overrightarrow{DC} \\ &= \begin{pmatrix} 5 \\ 12 \end{pmatrix} - \begin{pmatrix} 2\lambda \\ -3\lambda \end{pmatrix} \\ &= \begin{pmatrix} 5 - 2\lambda \\ 12 + 3\lambda \end{pmatrix}.\end{aligned}$$

Next, $\overrightarrow{DC}$ is parallel to $\overrightarrow{OB}$ so

$$\begin{aligned}\frac{5-2\lambda}{12+3\lambda} = \frac{11}{42} &\Rightarrow 42(5-2\lambda) = 11(12+3\lambda)\\ &\Rightarrow 210 - 84\lambda = 132 + 33\lambda\\ &\Rightarrow 78 = 117\lambda\\ &\Rightarrow \lambda = \tfrac{2}{3}.\end{aligned}$$

Hence, the position vector of D is

$$\underline{\underline{\tfrac{4}{3}\mathbf{i} - 2\mathbf{j}}}.$$

9. A particle moves in a straight line so that, t s after passing through a fixed point O, its velocity, v ms^{-1}, is given by (7)

$$v = 2t - 11 + \frac{6}{t+1}.$$

Find the acceleration of the particle when it is at instantaneous rest.

Solution

Well,

$$v = 0 \Rightarrow 2t - 11 + \frac{6}{t+1} = 0$$

multiply by $(t+1)$:

$$\begin{aligned}&\Rightarrow 2t(t+1) - 11(t+1) + 6 = 0\\ &\Rightarrow 2t^2 + 2t - 11t - 11 + 6 = 0\\ &\Rightarrow 2t^2 - 9t - 5 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -9\\ \text{multiply to:} & (+2)\times(-5) = -10\end{array}\right\} -10,\ +1$$

e.g.,

$$\begin{aligned}&\Rightarrow 2t^2 - 10t + t - 5 = 0\\ &\Rightarrow 2t(t-5) + 1(t-5) = 0\\ &\Rightarrow (2t+1)(t-5) = 0\\ &\Rightarrow 2t+1 = 0 \text{ or } t - 5 = 0\\ &\Rightarrow t = -\tfrac{1}{2} \text{ or } t = 5;\end{aligned}$$

so, the time is $t = 5$.

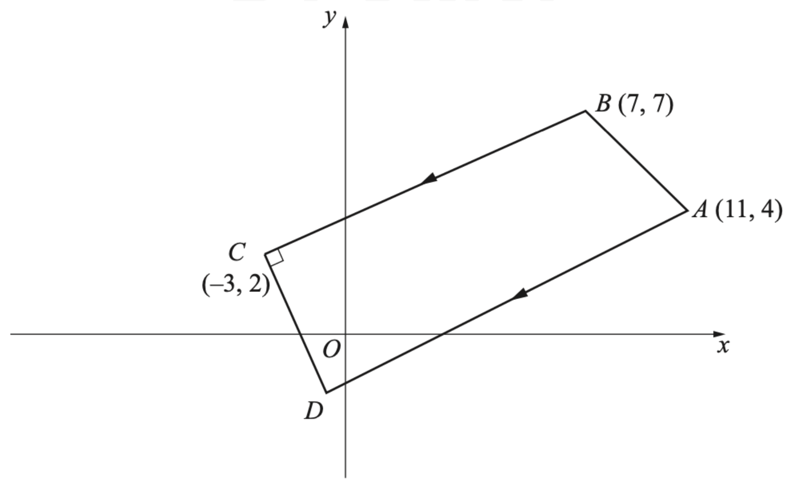
Now,

$$v = 2t - 11 + \frac{6}{t+1} \Rightarrow v = 2t - 11 + 6(t+1)^{-1}$$
$$\Rightarrow a = 2 - 6(t+1)^{-2}$$

and, finally,

$$t = 5 \Rightarrow \underline{\underline{a = 1\tfrac{5}{6} \text{ ms}^{-2}}}.$$

10. **Solutions to this question by accurate drawing will not be accepted.** (9)
The diagram shows a trapezium $ABCD$ with vertices $A(11, 4)$, $B(7, 7)$, $C(-3, 2)$, and D.



The side AD is parallel to BC and the side CD is perpendicular to BC.

Find the area of the trapezium $ABCD$.

Solution
Well,

$$m_{BC} = \frac{7-2}{7-(-3)}$$
$$= \tfrac{5}{10}$$
$$= \tfrac{1}{2}$$

and

$$m_{CD} = -2.$$

Now, the equation of CD is

$$\begin{aligned} y - 2 = -2(x+3) &\Rightarrow y - 2 = -2x - 6 \\ &\Rightarrow y = -2x - 4 \quad (1) \end{aligned}$$

and the equation of AD is

$$\begin{aligned} y - 4 = \tfrac{1}{2}(x - 11) &\Rightarrow y - 4 = \tfrac{1}{2}x - \tfrac{11}{2} \\ &\Rightarrow y = \tfrac{1}{2}x - \tfrac{3}{2} \quad (2). \end{aligned}$$

Remove y from each of equations (1) and (2):

$$\begin{aligned} -2x - 4 = \tfrac{1}{2}x - \tfrac{3}{2} &\Rightarrow -\tfrac{5}{2}x = \tfrac{5}{2} \\ &\Rightarrow x = -1 \\ &\Rightarrow y = -2; \end{aligned}$$

hence, $D(-1, -2)$.

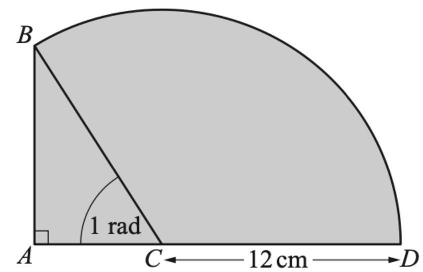
Next,

$$\begin{aligned} BC &= \sqrt{[7 - (-3)]^2 + (7-2)^2} \\ &= \sqrt{10^2 + 5^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5}, \\ AD &= \sqrt{[11 - (-1)]^2 + [4 - (-2)]^2} \\ &= \sqrt{12^2 + 6^2} \\ &= \sqrt{180} \\ &= 6\sqrt{5}, \text{ and} \\ CD &= \sqrt{[-3 - (-1)]^2 + [2 - (-2)]^2} \\ &= \sqrt{2^2 + 4^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area of the trapezium } ABCD &= \tfrac{1}{2} \times 2\sqrt{5} \times (5\sqrt{5} + 6\sqrt{5}) \\ &= \tfrac{1}{2} \times 2\sqrt{5} \times 11\sqrt{5} \\ &= \underline{\underline{55}}. \end{aligned}$$

11. The diagram shows a right-angled triangle ABC and a sector $CBDC$ of a circle with centre C and radius 12 cm.



Angle $ACB = 1$ radian and ACD is a straight line.

(a) Show that the length of AB is approximately 10.1 cm. (1)

Solution

Well,

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin 1 = \frac{AB}{12} \\ &\Rightarrow AB = 12\sin 1 \\ &\Rightarrow AB = 10.097\,651\,82 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{AB = 10.1 \text{ cm (3 sf)}}}. \end{aligned}$$

(b) Find the perimeter of the shaded region. (5)

Solution

Now,

$$\begin{aligned} \cos = \frac{\text{adj}}{\text{hyp}} &\Rightarrow \cos 1 = \frac{AC}{12} \\ &\Rightarrow AC = 12\cos 1 \end{aligned}$$

and

$$\begin{aligned} \text{perimeter} &= AB + BD + CD + AC \\ &= 12\sin 1 + \left(\frac{\pi - 1}{2\pi} \times 2 \times \pi \times 12\right) + 12 + 12\cos 1 \\ &= 54.280\,391\,33 \text{ (FCD)} \\ &= \underline{\underline{54.3 \text{ cm (3 sf)}}}. \end{aligned}$$

(c) Find the area of the shaded region. (4)

Solution
Well,

$$\begin{aligned}\text{area} &= \text{area of } ABC + \text{area of } CBDC \\ &= \left(\tfrac{1}{2} \times 12\sin 1 \times 12\cos 1\right) + \left(\frac{\pi - 1}{2\pi} \times \pi \times 12^2\right) \\ &= 186.929\,378\,4 \text{ (FCD)} \\ &= \underline{\underline{187 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

EITHER

12. The equation of a curve is

$$y = 2x^2 - 20x + 37.$$

(a) Express y in the form (3)

$$a(x + b)^2 + c,$$

where a, b, and c are integers.

Solution
Well,

$$\begin{aligned}y &= 2x^2 - 20x + 37 \\ &= 2[x^2 - 10x] + 37 \\ &= 2[(x^2 - 10x + 25) - 25] + 37 \\ &= 2[(x - 5)^2 - 25] + 37 \\ &= 2(x - 5)^2 - 50 + 37 \\ &= \underline{\underline{2(x - 5)^2 - 13}};\end{aligned}$$

hence, $\underline{\underline{a = 2}}$, $\underline{\underline{b = -5}}$, and $\underline{\underline{c = -13}}$.

(b) Write down the coordinates of the stationary point on the curve. (1)

Solution
$\underline{\underline{(5, -13)}}$.

A function f is defined by

$$\mathrm{f} : x \mapsto 2x^2 - 20x + 37 \text{ for } x > k.$$

Given that the function $\mathrm{f}^{-1}(x)$ exists,

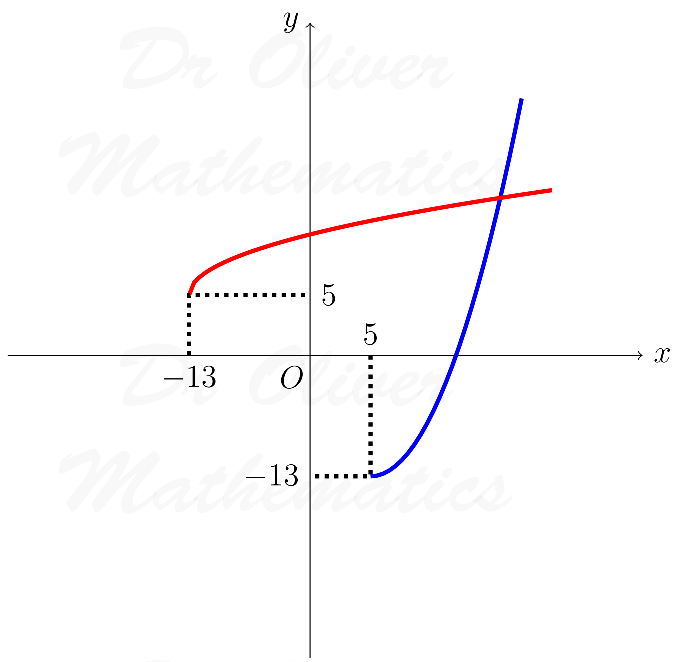
(c) write down the least possible value of k, (1)

Solution

$\underline{\underline{5}}$.

(d) sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, (2)

Solution



(e) obtain an expression for $\mathrm{f}^{-1}(x)$. (3)

Solution

Well,

$$\begin{aligned} y = 2x^2 - 20x + 37 &\Rightarrow y = 2(x-5)^2 - 13 \\ &\Rightarrow y + 13 = 2(x-5)^2 \\ &\Rightarrow \frac{y+13}{2} = (x-5)^2 \\ &\Rightarrow \sqrt{\frac{y+13}{2}} = x - 5 \\ &\Rightarrow 5 + \sqrt{\frac{y+13}{2}} = x; \end{aligned}$$

hence,

$$\underline{\underline{\mathrm{f}^{-1}(x) = 5 + \sqrt{\frac{x+13}{2}}}}.$$

OR

13. A function g is defined by

$$\mathrm{g} : x \mapsto 5x^2 + px + 72,$$

where p is a constant.

The function can also be written as

$$\mathrm{g} : x \mapsto 5(x-4)^2 + q.$$

(a) Find the value of p and of q. (3)

Solution

Well,

$\times$	x	-4
x	x^2	$-4x$
-4	$-4x$	$+16$

so

$$\begin{aligned} 5(x-4)^2 + q &= 5(x^2 - 8x + 16) + q \\ &= 5x^2 - 40x + (80 + q). \end{aligned}$$

As the two expressions are equal, $\underline{\underline{p = -40}}$ and

$$80 + q = 72 \Rightarrow \underline{\underline{q = -8}}.$$

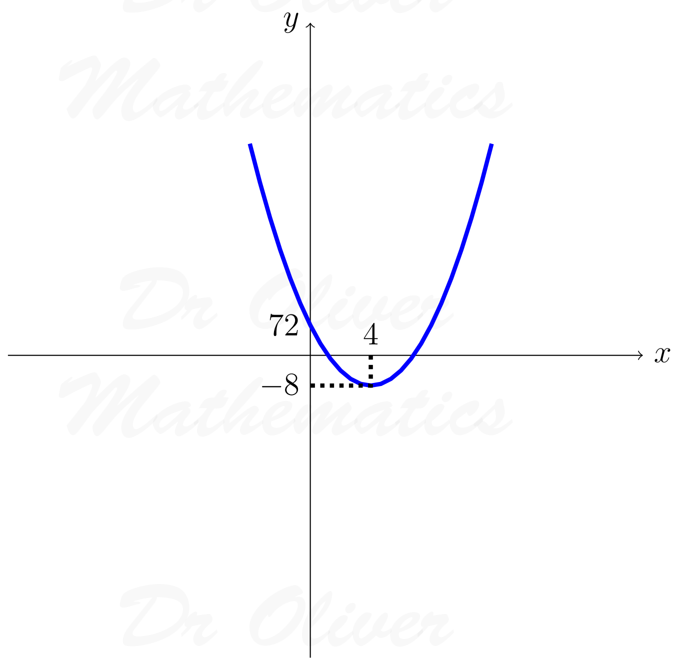
(b) Find the range of the function g. (1)

Solution

$\underline{\underline{\mathrm{g}(x) \geqslant -8}}$.

(c) Sketch the graph of the function. (2)

Solution



(d) Given that the function h is defined by (4)

$$\mathrm{h} : x \mapsto \ln x,$$

where $x > 0$, solve the equation

$$\mathrm{g}\,\mathrm{h}(x) = 12.$$

Solution

Now,

$$
\begin{aligned}
\mathrm{g}\,\mathrm{h}(x) = 12 &\Rightarrow \mathrm{g}(\mathrm{h}(x)) = 12 \\
&\Rightarrow \mathrm{g}(\ln x) = 12 \\
&\Rightarrow 5(\ln x - 4)^2 - 8 = 12 \\
&\Rightarrow 5(\ln x - 4)^2 = 20 \\
&\Rightarrow (\ln x - 4)^2 = 4 \\
&\Rightarrow \ln x - 4 = -\sqrt{4} \text{ or } \ln x - 4 = \sqrt{4} \\
&\Rightarrow \ln x = 4 - \sqrt{4} \text{ or } \ln x = 4 + \sqrt{4} \\
&\Rightarrow \underline{\underline{x = \mathrm{e}^{4-\sqrt{4}} \text{ or } x = \mathrm{e}^{4+\sqrt{4}}}} \\
&\Rightarrow \underline{\underline{x = 7.39 \text{ or } x = 403 \text{ (both to 3 sf)}}}
\end{aligned}
$$