Dr Oliver Mathematics
AQA Further Maths Level 2
June 2014 Paper 1
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.
You are **not** permitted to use a scientific or graphical calculator in this paper.

1. A straight line has gradient -2 and passes through the point $(-3, 10)$. (2)
   Work out the equation of the line.
   Give your answer in the form $y = mx + c$.

Solution

$y = -2x + c$, for some c. Now,

$$10 = -2(-3) + c \Rightarrow 10 = 6 + c$$
$$\Rightarrow c = 4$$

and so the line is

$$\underline{\underline{y = -2x + 4}}.$$

2. (2)

$$y = 4x^3 - 7x.$$

Work out $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = 4x^3 - 7x \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 - 7}}.$$

3. A transformation is given by the matrix $\mathbf{M}$, where (3)

$$\mathbf{M} = \begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}.$$

The image of the point $(b, 5)$ under $\mathbf{M}$ is $(5, b)$.
Work out the values of a and b.

Solution

$$\begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}\begin{pmatrix} b \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} b+5a \\ 10 \end{pmatrix} = \begin{pmatrix} 5 \\ b \end{pmatrix}.$$

Hence, $\underline{\underline{b = 10}}$ and

$$\begin{aligned} b + 5a = 5 &\Rightarrow 10 + 5a = 5 \\ &\Rightarrow 5a = -5 \\ &\Rightarrow \underline{\underline{a = -1}}. \end{aligned}$$

4. Solve (3)

$$20 + w < 3(w + 2).$$

Solution

$$\begin{aligned} 20 + w < 3(w+2) &\Rightarrow 20 + w < 3w + 6 \\ &\Rightarrow 2w > 14 \\ &\Rightarrow \underline{\underline{w > 7}}. \end{aligned}$$

5.

$$\mathrm{f}(x) = 10 - x^2, \text{ for all values of } x.$$
$$\mathrm{g}(x) = (x + 2a)(x + 3) \text{ for all values of } x.$$

(a) Circle the correct value of $\mathrm{f}(-4)$. (1)

26 $\quad -6 \quad$ 36 $\quad$ 16 $\quad$ 196

Solution

$$\begin{aligned} \mathrm{f}(-4) &= 10 - (-4)^2 \\ &= 10 - 16 \\ &= \underline{\underline{-6}}. \end{aligned}$$

(b) Write down the range of $\mathrm{f}(x)$. (1)

Solution

$\underline{\underline{\mathrm{f}(x) \leqslant 10}}$.

$\mathrm{g}(0) = 24$.

(c) Show that $a = 4$. (1)

Solution

$$\begin{aligned}(\mathrm{g}(0) = 24 &\Rightarrow (2a)(3) = 24\\ &\Rightarrow 6a = 24\\ &\Rightarrow \underline{\underline{a = 4}},\end{aligned}$$

as required.

(d) Hence solve (4)

$$\mathrm{f}(x) = \mathrm{g}(x).$$

Solution

$\times$	x	$+8$
x	x^2	$+8x$
$+3$	$+3x$	$+24$

$$\begin{aligned}\mathrm{f}(x) = \mathrm{g}(x) &\Rightarrow 10 - x^2 = (x+8)(x+3)\\ &\Rightarrow 10 - x^2 = x^2 + 11x + 24\\ &\Rightarrow 2x^2 + 11x + 14 = 0\end{aligned}$$

$$\begin{array}{ll} \text{add to:} & +11 \\ \text{multiply to:} & (+2)\times(+14)=+28 \end{array} \Big\} +4,\ +7$$

$$\begin{aligned} &\Rightarrow 2x^2+4x+7x+14=0 \\ &\Rightarrow 2x(x+2)+7(x+2)=0 \\ &\Rightarrow (2x+7)(x+2)=0 \\ &\Rightarrow 2x+7=0 \text{ or } x+2=0 \\ &\Rightarrow \underline{\underline{x=-3\tfrac{1}{2} \text{ or } x=-2}}. \end{aligned}$$

6. The nth term of a sequence is

$$\frac{2n^2+7}{3n^2-2}.$$

(a) Work out the 7th term. (2)
Give your answer as a fraction in its simplest form.

Solution

$$\begin{aligned} \frac{2(7^2)+7}{3(7^2)-2} &= \frac{2(49)+7}{3(49)-2} \\ &= \frac{98+7}{147-2} \\ &= \frac{105}{145} \\ &= \frac{5\times 21}{5\times 29} \\ &= \underline{\underline{\tfrac{21}{29}}}. \end{aligned}$$

(b) Show that the limiting value of (2)

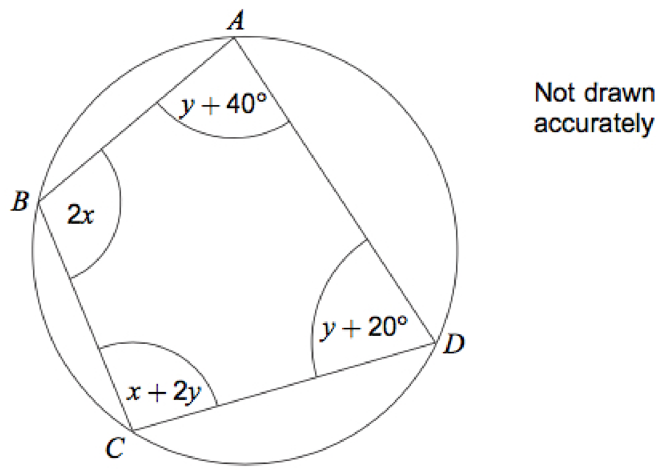
$$\frac{2n^2+7}{3n^2-2}$$

as $n \to \infty$ is $\frac{2}{3}$.

Solution

$$\frac{2n^2+7}{3n^2-2} = \frac{2+\frac{7}{n^2}}{3-\frac{2}{n^2}}$$
$$\to \frac{2+0}{3-0} \text{ (as } n \to \infty)$$
$$= \underline{\underline{\tfrac{2}{3}}}.$$

7. $ABCD$ is a cyclic quadrilateral. (5)



Work out the values of x and y.

Solution

Well, opposite angles in a cyclic quadrilateral add up to 180°:

$$2x + (y + 20) = 180 \Rightarrow y = 160 - 2x \quad (1)$$
$$(x + 2y) + (y + 40) = 180 \Rightarrow x + 3y = 140 \quad (2).$$

Substitute (1) into (2):

$$\begin{aligned} x + 3(160 - 2x) = 140 &\Rightarrow x + (480 - 6x) = 140 \\ &\Rightarrow 340 = 5x \\ &\Rightarrow \underline{\underline{x = 68^\circ}} \\ &\Rightarrow y = 160 - 2(68) \\ &\Rightarrow y = 160 - 136 \\ &\Rightarrow \underline{\underline{y = 24^\circ}}. \end{aligned}$$

8. (a) Factorise fully (2)

$$3x^2 - 12.$$

Solution

$$3x^2 - 12 = 3(x^2 - 4)$$

$$\left.\begin{array}{lr} \text{add to:} & 0 \\ \text{multiply to:} & -4 \end{array}\right\} -2, +2$$

$$= \underline{\underline{3(x-2)(x+2)}}.$$

(b) Factorise (3)

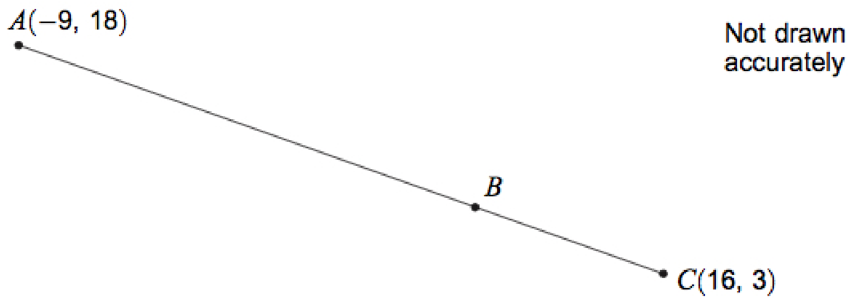
$$5x^2 + 4xy - 12y^2.$$

Solution

$$\left.\begin{array}{lr} \text{add to:} & +4 \\ \text{multiply to:} & (+5) \times (-12) = -60 \end{array}\right\} -6, +10$$

$$\begin{aligned} 5x^2 + 4xy - 12y^2 &= 5x^2 + 10xy - 6xy - 12y^2 \\ &= 5x(x + 2y) - 6y(x + 2y) \\ &= \underline{\underline{(5x - 6y)(x + 2y)}}. \end{aligned}$$

9. ABC is a straight line. (4)
BC is 20% of AC.



Work out the coordinates of B.

Solution

$$\begin{aligned}\overrightarrow{OB} &= \begin{pmatrix} -9 \\ 18 \end{pmatrix} + \tfrac{4}{5}\begin{pmatrix} 16-(-9) \\ 3-18 \end{pmatrix} \\ &= \begin{pmatrix} -9 \\ 18 \end{pmatrix} + \tfrac{4}{5}\begin{pmatrix} 25 \\ -15 \end{pmatrix} \\ &= \begin{pmatrix} -9 \\ 18 \end{pmatrix} + \begin{pmatrix} 20 \\ -12 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ 6 \end{pmatrix};\end{aligned}$$

hence, $\underline{\underline{B(11, 6)}}$.

10. Rationalise the denominator of (3)

$$\frac{8}{3-\sqrt{5}}.$$

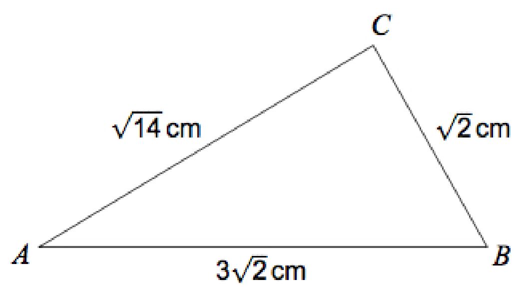
Give your answer in the form $a + b\sqrt{5}$, where a and b are integers.

Solution

$$\begin{aligned}\frac{8}{3-\sqrt{5}} &= \frac{8}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} \\ &= \frac{8(3+\sqrt{5})}{3^2-(\sqrt{5})^2} \text{ (using the difference of two squares)} \\ &= \frac{8(3+\sqrt{5})}{9-5} \\ &= \frac{8(3+\sqrt{5})}{4} \\ &= 2(3+\sqrt{5}) \\ &= \underline{\underline{6+2\sqrt{5}}};\end{aligned}$$

hence, $\underline{\underline{a=6}}$ and $\underline{\underline{b=2}}$.

11. Here is triangle ABC.



(a) Show that angle $B = 60^\circ$. (3)

Solution

We use the cosine rule:

$$\begin{aligned} \cos B &= \frac{(3\sqrt{2})^2 + (\sqrt{2})^2 - (\sqrt{14})^2}{2(3\sqrt{2})(\sqrt{2})} \\ \Rightarrow \quad \cos B &= \frac{18+2-14}{12} \\ \Rightarrow \quad \cos B &= \tfrac{1}{2} \\ \Rightarrow \quad \underline{\underline{B = 60^\circ}}&, \end{aligned}$$

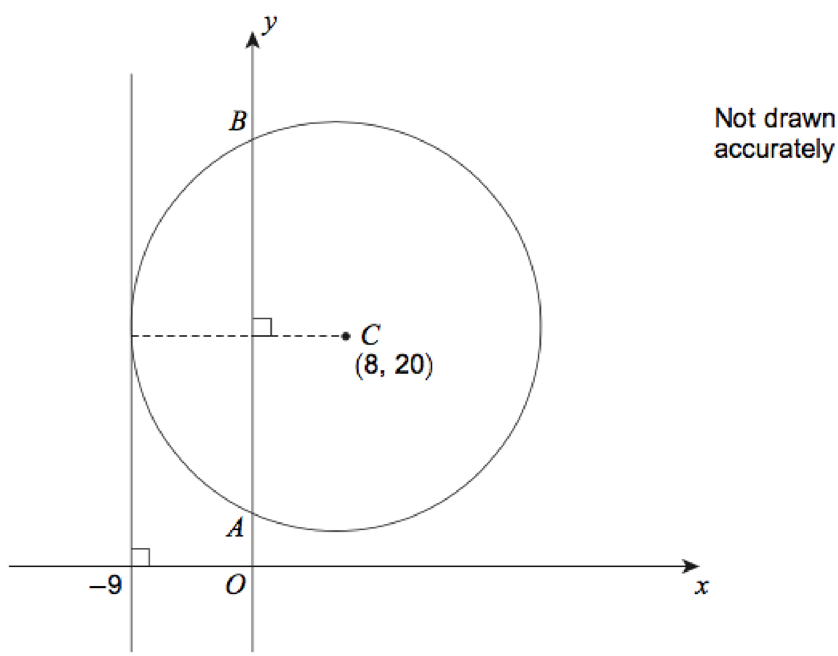
as required.

(b) Hence work out the area of triangle ABC. (3)

Solution

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}(3\sqrt{2})(\sqrt{2})(\sin 60^\circ)\\ &= \tfrac{1}{2}(6)(\tfrac{\sqrt{3}}{2})\\ &= (3)(\tfrac{\sqrt{3}}{2})\\ &= \underline{\underline{\tfrac{3\sqrt{3}}{2}}}.\end{aligned}$$

12. The line $x = -9$ is a tangent to the circle, centre $C(8, 20)$.



(a) Show that the radius of the circle is 17. (1)

Solution

$$8 - (-9) = \underline{\underline{17}}.$$

The circle intersects the y-axis at A and B.

(b) Show that the length AB is 30. (3)

Solution

$$\begin{aligned} x = 0 &\Rightarrow (0-8)^2 + (y-20)^2 = 17^2 \\ &\Rightarrow 64 + (y-20)^2 = 289 \\ &\Rightarrow (y-20)^2 = 225 \\ &\Rightarrow y - 20 = \pm 15 \\ &\Rightarrow y = 5 \text{ or } 35; \end{aligned}$$

hence, the length AB is $35 - 5 = \underline{\underline{30}}$.

13. A curve has equation

$$y = x^3 - 3x^2 + 5.$$

(a) Show that the curve has a minimum point when $x = 2$. (4)

Solution

$$y = x^3 - 3x^2 + 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3x^2 - 6x = 0 \\ &\Rightarrow 3x(x-2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2. \end{aligned}$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 6$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 > 0$$

and the curve has a minimum point when $x = 2$.

(b) Show that the tangent at the minimum point meets the curve again when $x = -1$. (3)

Solution

Now,

$$\begin{aligned} x = 2 \Rightarrow y &= 2^3 - 3(2^2) + 5 \\ \Rightarrow y &= 8 - 12 + 5 \\ \Rightarrow y &= 1. \end{aligned}$$

Next,

$$x^3 - 3x^2 + 5 = 1 \Rightarrow x^3 - 3x^2 + 4 = 0$$

and we use synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & \downarrow & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

Hence,

$$x^3 - 3x^2 + 4 = (x + 1)(x^2 - 4x + 4)$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & +4 \end{array}\right\} -2, -2$$

$$= (x + 1)(x - 2)^2.$$

Hence, the tangent at the minimum point meets the curve again when $x = -1$.

14. $(x - a)$ is a factor of

$$x^3 + 2ax^2 - a^2x - 16.$$

(a) Show that $a = 2$. (2)

Solution

Let

$$f(x) = x^3 + 2ax^2 - a^2x - 16.$$

Then

$$\begin{aligned} f(a) = 0 &\Rightarrow (a^3) + 2a(a^2) - a^2(a) - 16 = 0 \\ &\Rightarrow a^3 + 2a^3 - a^3 - 16 = 0 \\ &\Rightarrow 2a^3 - 16 = 0 \\ &\Rightarrow 2(a^3 - 8) = 0 \\ &\Rightarrow a^3 = 8 \\ &\Rightarrow \underline{\underline{a = 2}}, \end{aligned}$$

as required.

(b) Solve (4)

$$x^3 + 4x^2 - 4x - 16.$$

Solution

We use synthetic division:

2	1	4	−4	−16
	↓	2	12	16
	1	6	8	0

How,

$$x^3 + 4x^2 - 4x - 16 = 0 \Rightarrow (x - 2)(x^2 + 6x + 8) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +6 \\ \text{multiply to:} & +8 \end{array}\right\} + 2, +4$$

$$\begin{aligned} &\Rightarrow (x - 2)(x + 2)(x + 4) = 0 \\ &\Rightarrow \underline{\underline{x = -4,\ x = -2,\ \text{or } x = 2}}. \end{aligned}$$

15. Prove that (3)

$$\frac{\sin\theta - \sin^3\theta}{\cos^3\theta} \equiv \tan\theta.$$

Solution

$$\frac{\sin\theta - \sin^3\theta}{\cos^3\theta} \equiv \frac{\sin\theta(1-\sin^2\theta)}{\cos^3\theta} \equiv \frac{\sin\theta\cos^2\theta}{\cos^3\theta} \equiv \frac{\sin\theta}{\cos\theta} \equiv \underline{\underline{\tan\theta}},$$

as required.

16. (6)

$$2x^2 - 2bx + 7a \equiv 2(x-a)^2 + 3.$$

Work out the **two** possible pairs of values of a and b.

Solution

$$\begin{aligned} 2x^2 - 2bx + 7a &\equiv 2(x^2 - bx) + 7a \\ &\equiv 2\left[(x^2 - bx + (\tfrac{1}{2}b)^2) - (\tfrac{1}{2}b)^2\right] + 7a \\ &\equiv 2(x - \tfrac{1}{2}b)^2 - \tfrac{1}{2}b^2 + 7a \end{aligned}$$

and

$$a = \tfrac{1}{2}b \quad (1)$$
$$7a - \tfrac{1}{2}b^2 = 3 \quad (2).$$

Substitute (1) into (2):

$$\begin{aligned} 7(\tfrac{1}{2}b) - \tfrac{1}{2}b^2 = 3 &\Rightarrow \tfrac{7}{2}b - \tfrac{1}{2}b^2 = 3 \\ &\Rightarrow 7b - b^2 = 6 \\ &\Rightarrow b^2 - 7b + 6 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -7 \\ \text{multiply to:} & +6 \end{array}\right\} -6,\ -1$$

$$\begin{aligned} &\Rightarrow (b-6)(b-1) = 0 \\ &\Rightarrow b = 6 \text{ or } b = 1 \\ &\Rightarrow a = 3 \text{ or } a = \tfrac{1}{2}; \end{aligned}$$

hence,

$$\underline{\underline{a = 3,\ b = 6}} \text{ or } \underline{\underline{a = \tfrac{1}{2},\ b = 1}}.$$