Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2014 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. Solve the following: (3)

$$-6 < 2x - 1 < 7.$$

Solution

$$-6 < 2x - 1 < 7 \Rightarrow -5 < 2x < 8$$
$$\Rightarrow \underline{\underline{-2\tfrac{1}{2} < x < 4}}.$$

2. The gradient function of a curve that passes through the point $(1, 2)$ is given by (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 7.$$

Find the equation of the curve.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 7 \Rightarrow y = x^3 - 2x^2 + 7x + c,$$

for some constant c. Now,

$$x = 1, y = 2 \Rightarrow 2 = 1 - 2 + 7 + c$$
$$\Rightarrow c = -4.$$

Hence,

$$\underline{\underline{y = x^3 - 2x^2 + 7x - 4}}.$$

3. (a) Find the area enclosed between the curve $y = 8x^3$, the x-axis, and the line $x = 2$. (3)

Solution

$$\begin{aligned}\text{Area} &= \int_0^2 8x^3 \,\mathrm{d}x \\ &= \left[2x^4\right]_{x=0}^{2} \\ &= 32 - 0 \\ &= \underline{\underline{32}}.\end{aligned}$$

(b) Hence, or otherwise, deduce the area between the x-axis, y-axis, the line $x = 2$, and the curve $y = 8x^3$. (1)

Solution

$$\begin{aligned}\text{Area} &= \int_0^2 (8x^3 + 5) \,\mathrm{d}x \\ &= \int_0^2 8x^3 \,\mathrm{d}x + \int_0^2 5 \,\mathrm{d}x \\ &= 32 + (2 \times 5) \\ &= \underline{\underline{42}}.\end{aligned}$$

4. A train travels from station A to station B. It starts from rest at A and comes to rest again at B. The displacement of the train from A at time t seconds after starting from A is s metres where

$$s = 0.09t^2 - 0.000\,1t^3.$$

(a) Find the velocity at time t seconds after leaving A and hence find the time taken to reach B. (4)
Give the units of your answer.

Solution

$$s = 0.09t^2 - 0.000\,1t^3 \Rightarrow \underline{\underline{v = (0.18t - 0.000\,3t^2)\ \mathrm{ms}^{-1}}}.$$

$$\begin{aligned}0.18t - 0.000\,3t^2 = 0 &\Rightarrow 0.000\,3t(600 - t) = 0 \\ &\Rightarrow t = 0 \text{ or } t = 600.\end{aligned}$$

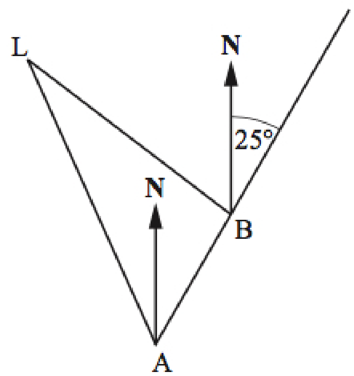
Hence, the time taken to reach B is $\underline{\underline{600\ \text{s}}}$ or $\underline{\underline{10\ \text{mins}}}$.

(b) Find the distance between A and B. (2)
Give the units of your answer.

Solution

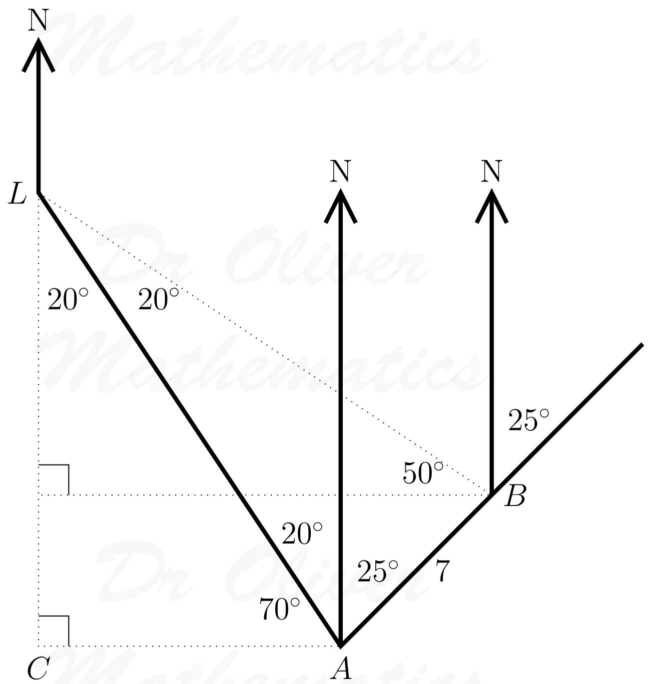
$$t = 600 \Rightarrow s = 0.09(600^2) - 0.000\,1(600^3)$$
$$\Rightarrow \underline{\underline{s = 10\,800 \text{ m}}}.$$

5. A ship is moving on a bearing of 025° at 14 knots. (1 knot = 1 nautical mile per hour). As it passes point A, a lighthouse L is seen on a bearing of 340°. After 30 minutes, the ship passes point B from where the lighthouse is seen on a bearing of 320°.



(a) Find the angle BAL and the angle ALB. (3)

Solution



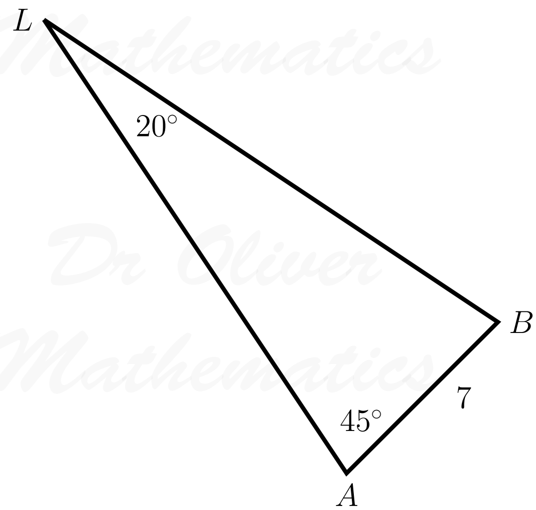
$$\begin{aligned}\angle BAL &= \angle BAN + \angle NAL \\ &= 25 + (360 - 340) \\ &= \underline{\underline{45^\circ}}\end{aligned}$$

and

$$\begin{aligned}\angle ALB &= 90^\circ - 50^\circ - \angle CLA \\ &= 90^\circ - 50^\circ - 20^\circ \\ &= \underline{\underline{20^\circ}}.\end{aligned}$$

(b) Hence, or otherwise, calculate the distance BL in nautical miles. (3)

Solution



We use the sine rule:

$$\frac{BL}{\sin 45^\circ} = \frac{7}{\sin 20^\circ} \Rightarrow BL = \frac{7\sin 45^\circ}{\sin 20^\circ}$$
$$\Rightarrow BL = 14.472\,093\,43 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{BL = 14.5 \text{ nautical miles (3 sf)}}}.$$

6. The function

$$f(x) = x^3 - 4x^2 + ax + b$$

is such that

- $x = 3$ is a root of the equation $f(x) = 0$,
- when $f(x)$ is divided by $(x - 1)$, there is a remainder of 4.

(a) Find the value of a and the value of b. (4)

Solution

$$f(3) = 0 \Rightarrow 27 - 36 + 3a + b = 0$$
$$\Rightarrow \underline{\underline{3a + b = 9 \quad (1)}}$$

and

$$f(1) = 4 \Rightarrow 1 - 4 + a + b = 4$$
$$\Rightarrow \underline{\underline{a + b = 7 \quad (2)}}.$$

Do (1) − (2):

$$2a = 2 \Rightarrow \underline{\underline{a = 1}}$$

$$\Rightarrow \underline{\underline{b = 6}}.$$

(b) Solve the equation $\mathrm{f}(x) = 0$. (3)

Solution

We use synthetic division:

3	1	−4	1	6
	↓	3	−3	−6
	1	−1	−2	0

$$\mathrm{f}(x) = 0 \Rightarrow (x-3)(x^2 - x - 2) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & -2\end{array}\right\} -2,\ +1$$

$$\Rightarrow (x-3)(x-2)(x+1) = 0$$

$$\Rightarrow \underline{\underline{x = -1,\ x = 2,\ \text{or}\ x = 3}}.$$

7. The points A and B have coordinates $(3, 7)$ and $(5, 11)$ respectively.

(a) Find the exact length of AB. (2)

Solution

$$\begin{aligned} AB &= \sqrt{(5-3)^2 + (11-7)^2} \\ &= \sqrt{4 + 16} \\ &= \sqrt{20} \\ &= \underline{\underline{2\sqrt{5}}}. \end{aligned}$$

(b) Find the equation of the circle with diameter AB. (3)

Solution
The midpoint of AB is

$$\left(\frac{3+5}{2}, \frac{7+11}{2}\right) = (4, 9)$$

and the equation is

$$\underline{\underline{(x-4)^2 + (y-9)^2 = 5}}.$$

8. Four points have coordinates $A(-5,-1)$, $B(0,4)$, $C(7,3)$ and $D(2,-2)$.

(a) Using gradients of lines, prove that $ABCD$ is a parallelogram. (2)

Solution

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 0-(-5) \\ 4-(-1) \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 5 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{DC} &= \begin{pmatrix} 2-7 \\ -2-3 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ -5 \end{pmatrix};\end{aligned}$$

hence, AB is parallel to DC which makes $ABCD$ is a $\underline{\underline{\text{parallelogram}}}$.

(b) Using lengths of lines, prove further that $ABCD$ is a rhombus. (2)

Solution

$$\begin{aligned}\overrightarrow{BC} &= \begin{pmatrix} 0-7 \\ 4-3 \end{pmatrix} \\ &= \begin{pmatrix} -7 \\ 1 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{AD} &= \begin{pmatrix} 2-(-5) \\ -2-(-1) \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ -1 \end{pmatrix}.\end{aligned}$$

Now,

$$AB = CD = \sqrt{5^2 + 5^2} = 5\sqrt{2}$$

and

$$BC = AB = \sqrt{7^2 + (-1)^2} = 5\sqrt{2}.$$

Hence, $ABCD$ is a $\underline{\underline{\text{rhombus}}}$.

(c) Prove that $ABCD$ is not a square. (2)

Solution

$$\begin{aligned}AC &= \sqrt{[7-(-5)]^2 + [3-(-1)]^2} \\ &= \sqrt{12^2 + 4^2} \\ &= 4\sqrt{10};\end{aligned}$$

now,

$$AB^2 + BC^2 = 2(5^2 + 5^2) = 100$$

whereas

$$AC^2 = 160.$$

Thus, $ABCD$ is not a $\underline{\underline{\text{square}}}$.

9. (a) Show that (1)

$$\frac{1-\cos^2 x}{1-\sin^2 x} \equiv \tan^2 x.$$

Solution

$$\begin{aligned}\frac{1-\cos^2 x}{1-\sin^2 x} &\equiv \frac{\sin^2 x}{\cos^2 x} \\ &\equiv \underline{\underline{\tan^2 x}},\end{aligned}$$

as required.

(b) Hence solve the equation (4)

$$\frac{1-\cos^2 x}{1-\sin^2 x} = 3 - 2\tan x$$

for values of x in the range $0° \leqslant x \leqslant 180°$.

Solution

$$\begin{aligned}\frac{1-\cos^2 x}{1-\sin^2 x} = 3 - 2\tan x &\Rightarrow \tan^2 x = 3 - 2\tan x \\ &\Rightarrow \tan^2 x + 2\tan x - 3 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +2 \\ \text{multiply to:} & -3\end{array}\right\} +3,\ -1$$

$$\begin{aligned}&\Rightarrow (\tan x + 3)(\tan x - 1) = 0 \\ &\Rightarrow \tan x = -3 \text{ or } \tan x = 1.\end{aligned}$$

Now,

$$\begin{aligned}\tan x = -3 \Rightarrow x &= 108.434\,948\,8 \text{ (FCD)} \\ \Rightarrow \underline{\underline{x = 108° \text{ (3 sf)}}}\end{aligned}$$

and

$$\tan x = 1 \Rightarrow \underline{\underline{x = 45° \text{ (exact!)}}}.$$

10. (a) Find the coordinates of the point P on the curve (3)

$$y = 2x^2 + x - 5$$

where the gradient of the curve is 5.

Solution

$$y = 2x^2 + x - 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 4x + 1$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 \Rightarrow 4x + 1 = 5$$
$$\Rightarrow 4x = 4$$
$$\Rightarrow x = 1$$
$$\Rightarrow y = 2 + 1 - 5 = -2;$$

hence, $\underline{\underline{P(1, -2)}}$.

(b) Find the equation of the normal to the curve at the point P. (3)

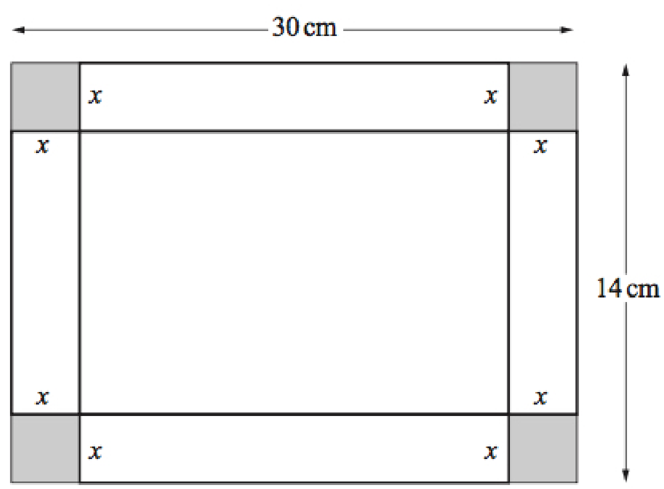
Solution

The gradient of the normal is $-\frac{1}{5}$ and the equation is

$$y + 2 = -\tfrac{1}{5}(x - 1) \Rightarrow y + 2 = -\tfrac{1}{5}x + \tfrac{1}{5}$$
$$\Rightarrow \underline{\underline{y = -\tfrac{1}{5}x - \tfrac{9}{5}}}.$$

Section B

11. Kala is making an open box out of a rectangular piece of card measuring 30 cm by 14 cm. She cuts squares of side x cm out of each corner and turns up the sides to form the box.



(a) Find an expression in terms of x for the volume, V cm^3, of the box and show that this reduces to (4)

$$V = 4x^3 - 88x^2 + 420x.$$

Solution

$$V = x(30 - 2x)(14 - 2x)$$

$\times$	30	$-2x$
14	420	$-28x$
$-2x$	$-60x$	$+4x^2$

$$= x(420 - 88x + 4x^2)$$
$$= \underline{\underline{4x^3 - 88x^2 + 420x}},$$

as required.

(b) Find the two values of x that give $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$. (5)

Solution

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow 12x^2 - 176x + 420 = 0$$
$$\Rightarrow 4(3x^2 - 44x + 105) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -44 \\ \text{multiply to:} & (+3) \times (+105) = +315 \end{array}\right\} -35,\ -9$$

$$\Rightarrow 4[3x^2 - 35x - 9x + 105] = 0$$
$$\Rightarrow 4[x(3x - 35) - 3(3x - 35)] = 0$$
$$\Rightarrow 4(x - 3)(3x - 35) = 0$$
$$\Rightarrow \underline{\underline{x = 3 \text{ or } x = 11\tfrac{2}{3}}}.$$

(c) Explain why one of these values should be rejected and find the maximum volume of the box using the other value. (3)

Solution

Clearly, $x = 11\frac{2}{3}$ should be rejected as we cannot make a box:

$$14 - 2 \times 11\tfrac{2}{3} = -9\tfrac{1}{3}.$$

Hence, the maximum volume is

$$\begin{aligned} V &= 3(30 - 2 \times 3)(14 - 2 \times 3) \\ &= 3(24)(8) \\ &= \underline{\underline{576 \text{ cm}^3}}. \end{aligned}$$

12. Paul walked from Anytown to Nexttown, a distance of 15 km. When he got there he then walked back. His average speed on the return journey was 2 km per hour less than on the outward journey.

Let Paul's average speed on the outward journey be x km hr^{-1}.

(a) Write down an expression for the time, in hours, taken for the whole journey. (2)

Solution

The whole journey takes

$$\underline{\underline{\left(\frac{15}{x} + \frac{15}{x-2}\right) \text{ hours.}}}$$

The time taken by Paul for the whole journey was 6 hours.

(b) Use your expression in (a) to form an equation in x and show that it simplifies to (4)

$$x^2 - 7x + 5 = 0.$$

Solution

$$
\begin{aligned}
\frac{15}{x} + \frac{15}{x-2} = 6 &\Rightarrow 15(x-2) + 15x = 6x(x-2) \\
&\Rightarrow 15x - 30 + 15x = 6x^2 - 12x \\
&\Rightarrow 6x^2 - 42x + 30 = 0 \\
&\Rightarrow 6(x^2 - 7x + 5) = 0 \\
&\Rightarrow \underline{\underline{x^2 - 7x + 5 = 0}},
\end{aligned}
$$

as required.

(c) Solve this equation to find Paul's average speed on the outward journey. (3)

Solution

$a = 1$, $b = -7$, and $c = 5$:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{7 \pm \sqrt{29}}{2} \\
&= 0.807\,417\,596\,4,\ 6.192\,582\,404 \text{ (FCD)}.
\end{aligned}
$$

Now,

$$
\frac{15}{0.807\ldots} = 18.577\ldots
$$

which is obviously not correct. So, Paul's average speed on the outward journey is $\underline{\underline{6.19 \text{ km hr}^{-1} \text{ (3 sf)}}}$.

(d) Find the difference in time between the outward and return journeys. Give your answer to the nearest minute. (3)

Solution

$$
\begin{aligned}
\text{Difference} &= \frac{15}{6.192\ldots - 2} - \frac{15}{6.192\ldots} \\
&= 1.155\,494\,421 \text{ hours (FCD)} \\
&= 69.329\,665\,28 \text{ mins (FCD)} \\
&= \underline{\underline{69 \text{ mins (nearest minute)}}}.
\end{aligned}
$$

13. A company needs to buy some storage units. There are two types of unit available,

type X and type Y. The cost of each type of unit, the floor space required and the volume for storage are given in the following table.

	Cost per unit (£C)	Floor space required (m^2)	Volume for storage (m^3)
X	100	2	3.5
Y	120	1.5	3

The maximum cost allowed for the purchase of the units is £1 200 and the maximum floor space available is 18 m^2. The company wants to maximise the volume for storage.

Let x and y be the number of each type of unit, X and Y, respectively.

(a) Write down an inequality for the total cost and an inequality for the total floor space required. (3)

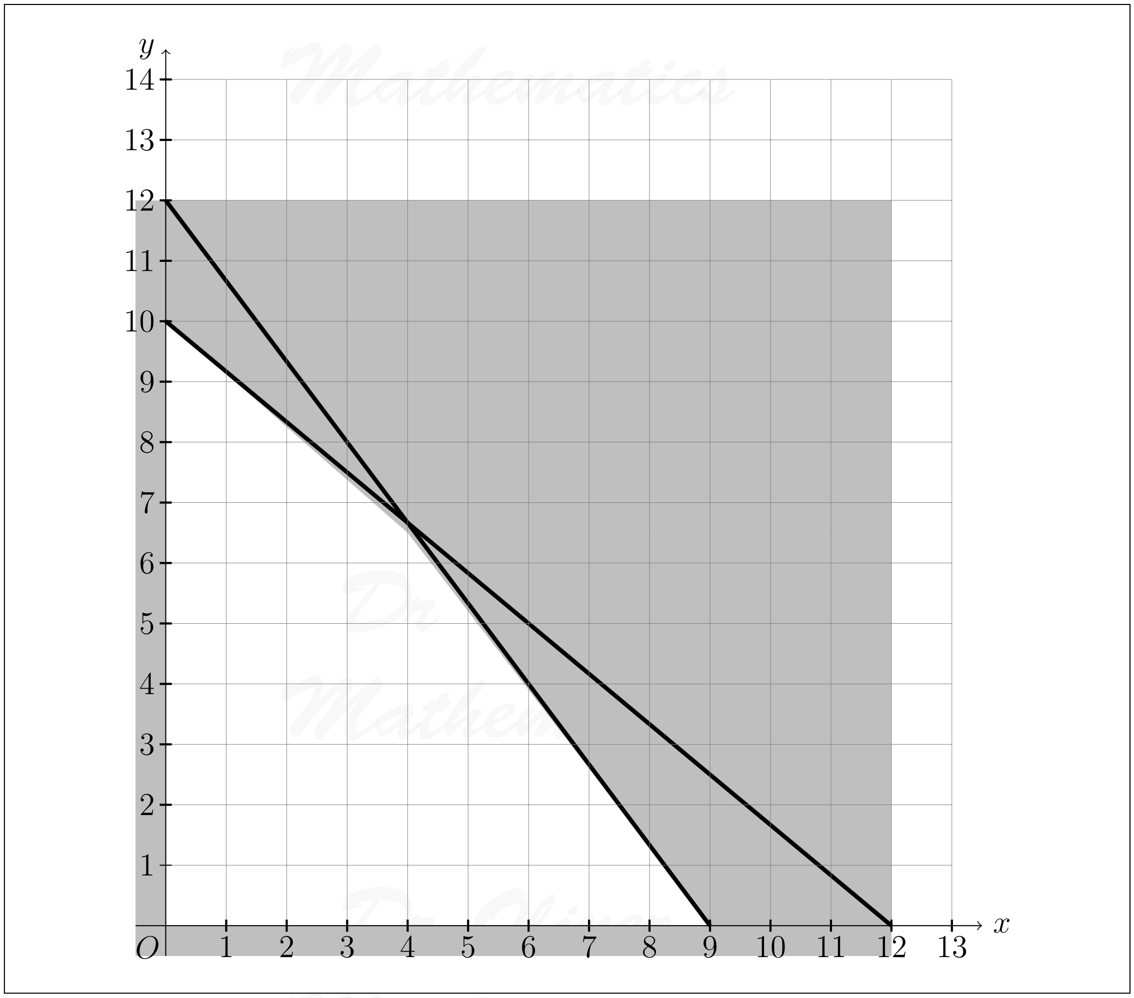
Solution

Total cost: $100x + 120y = 1\,200 \Rightarrow \underline{\underline{10x + 12y = 120}}$.

Floor space: $2x + 1.5y = 18 \Rightarrow \underline{\underline{4x + 3y = 36}}$.

(b) Draw the inequalities you gave in (a). Given that $x \geqslant 0$ and $y \geqslant 0$, indicate the region for which the inequalities hold by shading the area that is **not** required. (4)

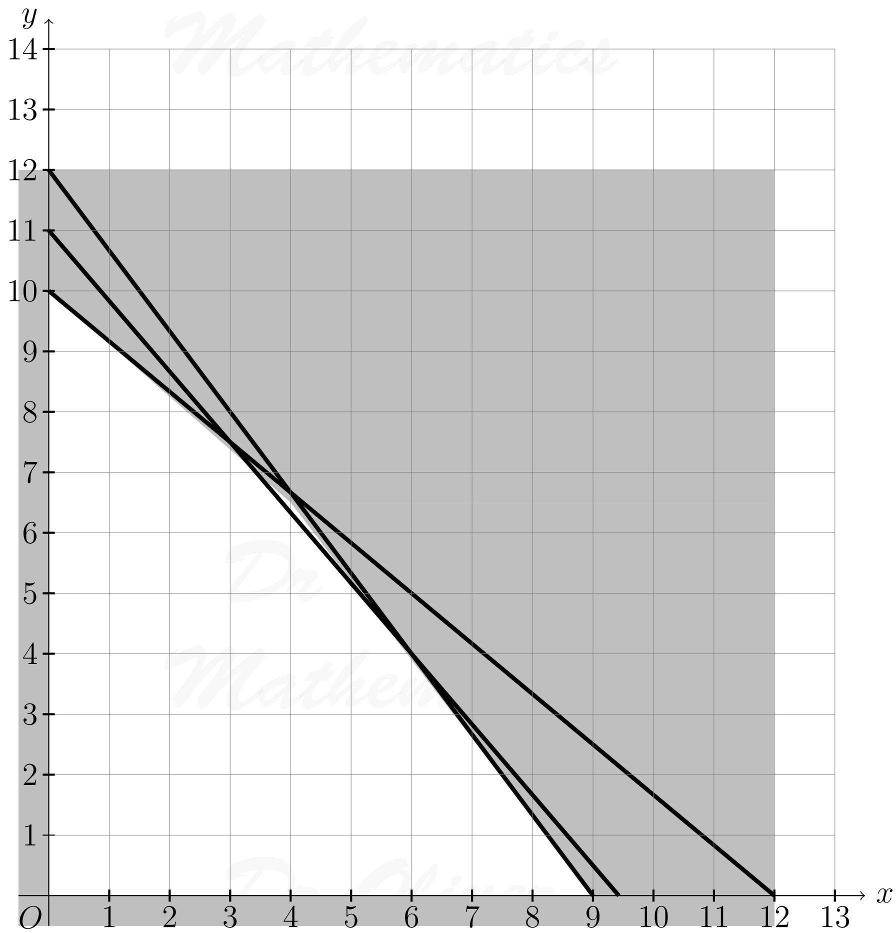
Solution



(c) Write down the objective function for the volume for storage and find the combination of units that should be bought to maximise the volume for storage. (5)
Write down this maximum volume.

Solution

$\underline{\underline{P = 3.5x + 3y}}$.



The first is to go around the polygon (integers only) and determine $3.5x + 3y$ at its vertices (thus, $(1, 9)$, $(2, 8)$, $\ldots$). The second is to go $3.5x + 3y = ?$ and translate this line out and upwards: $3.5x+3y = ?$, $3.5x+3y = 0.5$, $3.5x+3y = 1$, $\ldots$, until we reach the very last point on its vertices. Hence, it is $x = 6$, $y = 4$ and the minimum is

$$3.5(6) + 3(4) = \underline{\underline{\underline{33}}}.$$

14. Mugs are packed in boxes of 10. On average, 5% of the mugs are imperfect. A box of mugs is classified as "unsatisfactory" if it contains two or more imperfect mugs.

 (a) State two conditions that must be satisfied for the number of imperfect mugs in a box to have a binomial distribution. (2)

Solution

E.g., The number of observations is <u>fixed</u>; each observation is <u>independent</u>;

each observation represents one of two outcomes ("success" or "failure"); the probability of "success" p is the same for each outcome.

(b) Assuming that these two conditions are satisfied, calculate the probability that a box chosen at random is "unsatisfactory." (6)

Solution

$$\begin{aligned} \mathrm{P}(\text{unsatisfactory}) &= \mathrm{P}(2 \text{ or more imperfect}) \\ &= 1 - \mathrm{P}(\text{at most 1 imperfect}) \\ &= 1 - \left[(0.95)^{10} + \binom{10}{1}(0.95)^9(0.05)\right] \\ &= 0.086\,138\,355\,9 \text{ (FCD)} \\ &= \underline{\underline{0.086\,1 \text{ (3 sf)}}}. \end{aligned}$$

A shop receives a delivery of a large number of boxes of mugs. The delivery is checked as follows. A box is chosen at random.

- If there are no imperfect mugs in the box then the whole delivery is accepted.
- If the box is "unsatisfactory" then the whole delivery is rejected.
- If there is exactly one imperfect mug in the box then a second box is chosen at random. The delivery is accepted only if this box contains no imperfect mugs.

(c) Calculate the probability that the delivery is accepted. (4)

Solution

$$\begin{aligned} \mathrm{P}(\text{accepted}) &= (0.95)^{10} + \left[\binom{10}{1}(0.95)^9(0.05) \cdot (0.95)^{10}\right] \\ &= 0.787\,413\,740\,5 \text{ (FCD)} \\ &= \underline{\underline{0.787 \text{ (3 sf)}}}. \end{aligned}$$