Dr Oliver Mathematics
Mathematics: Higher
2011 Paper 1: Non-Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

Section A

1. Given that (2)

$$\mathbf{p} = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \text{ and } \mathbf{r} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix},$$

express $2\mathbf{p} - \mathbf{q} - \frac{1}{2}\mathbf{r}$ in component form.

A. $\begin{pmatrix} 1 \\ 9 \\ -15 \end{pmatrix}$

B. $\begin{pmatrix} 1 \\ 11 \\ -13 \end{pmatrix}$

C. $\begin{pmatrix} 5 \\ 9 \\ -13 \end{pmatrix}$

D. $\begin{pmatrix} 5 \\ 11 \\ -15 \end{pmatrix}$

Solution

C

$$\begin{aligned}2\mathbf{p}-\mathbf{q}-\tfrac{1}{2}\mathbf{r} &= 2\begin{pmatrix}2\\5\\-7\end{pmatrix}-\begin{pmatrix}1\\0\\-1\end{pmatrix}-\tfrac{1}{2}\begin{pmatrix}-4\\2\\0\end{pmatrix}\\ &= \begin{pmatrix}4\\10\\-14\end{pmatrix}-\begin{pmatrix}1\\0\\-1\end{pmatrix}-\begin{pmatrix}-2\\1\\0\end{pmatrix}\\ &= \begin{pmatrix}5\\9\\-13\end{pmatrix}.\end{aligned}$$

2. A line l has equation (2)

$$3y + 2x = 6.$$

What is the gradient of any line parallel to l?

A. -2

B. $-\frac{2}{3}$

C. $\frac{3}{2}$

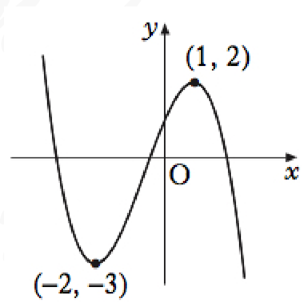
D. 2

Solution

B

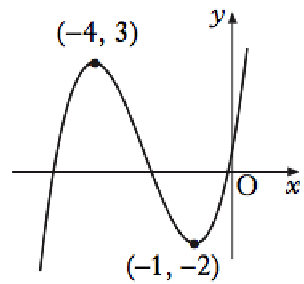
$$\begin{aligned}3y + 2x = 6 &\Rightarrow 3y = -2x + 6\\ &\Rightarrow y = -\tfrac{2}{3}x + 2.\end{aligned}$$

3. The diagram shows the graph of $y = \mathrm{f}(x)$. (2)

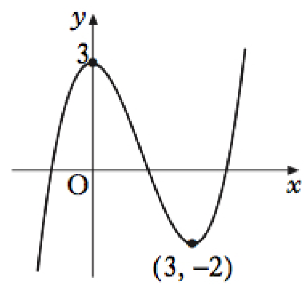


Which of the following shows the graph of $y = \mathrm{f}(x+2) - 1$?

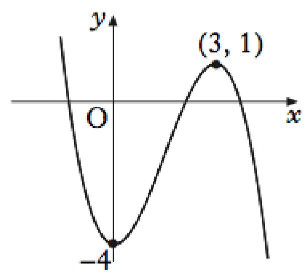
A



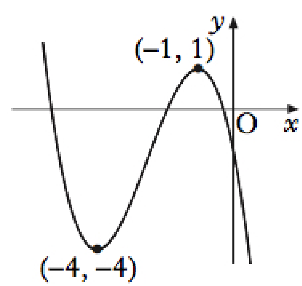
B



C



D



Solution

D

The graph of $y = \mathrm{f}(x+2) - 1$ goes through $(-4, -4)$ and $(-1, 1)$.

4. A tangent to the curve with equation (2)

$$y = x^3 - 2x$$

is drawn at the point $(2, 4)$.
What is the gradient of this tangent?

A. 2

B. 3

C. 4

D. 10

Solution

D

$$y = x^3 - 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3 \times 2^2 - 2 = 10.$$

5. If (2)

$$x^2 - 8x + 7$$

is written in the form

$$(x - p)^2 + q,$$

what is the value of q?

A. -9

B. -1

C. 7

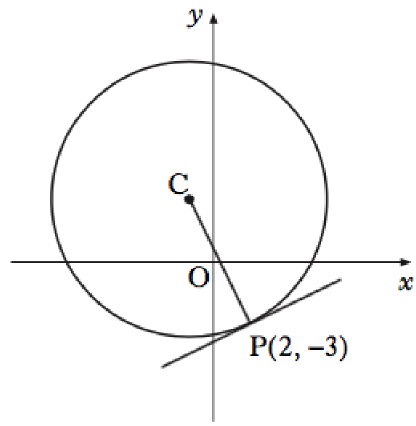
D. 23

Solution

A

$$\begin{aligned}(x^2 - 8x) + 7 &= (x^2 - 8x + 16) + 7 - 16 \\ &= (x - 4)^2 - 9.\end{aligned}$$

6. The point $P(2, -3)$ lies on the circle with centre C as shown. (2)



The gradient of CP is -2.
What is the equation of the tangent at P?

A. $y + 3 = -2(x - 2)$

B. $y - 3 = -2(x + 2)$

C. $y + 3 = \frac{1}{2}(x - 2)$

D. $y - 3 = \frac{1}{2}(x + 2)$

Solution

C

The gradient of the tangent is

$$-\frac{1}{-2} = \tfrac{1}{2}$$

and the equation of the tangent at P is

$$y - (-3) = \tfrac{1}{2}(x - 2) \Rightarrow y + 3 = \tfrac{1}{2}(x - 2).$$

7. A function f is defined on the set of real numbers by (2)

$$\mathrm{f}(x) = x^3 - x^2 + x + 3.$$

What is the remainder when $\mathrm{f}(x)$ is divided by $(x - 1)$?

A. 0

B. 2

C. 3

D. 4

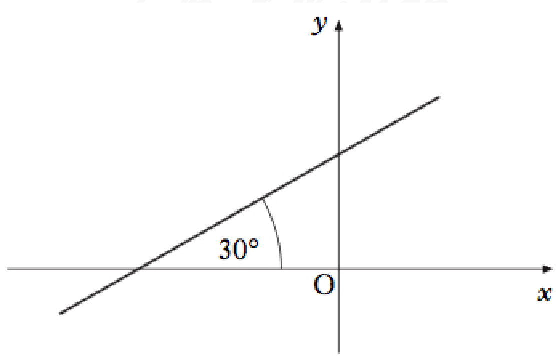
Solution

D

We use synthetic division:

1	1	−1	1	3
	↓	1	0	1
	1	0	1	4

8. A line makes an angle of 30° with the positive direction of the x-axis as shown. (2)



What is the gradient of the line?

A. $\frac{1}{\sqrt{3}}$

B. $\frac{1}{\sqrt{2}}$

C. $\frac{1}{2}$

D. $\frac{\sqrt{3}}{2}$

Solution

A

$\tan 30^\circ = \frac{1}{\sqrt{3}}$.

9. The discriminant of a quadratic equation is 23. (2)
Here are two statements about this quadratic equation:
(1) the roots are real;
(2) the roots are rational.

Which of the following is true?

A. Neither statement is correct.

B. Only statement (1) is correct.

C. Only statement (2) is correct.

D. Both statements are correct.

Solution

B

$\sqrt{23}$ is not a rational number but is a real one.

10. Solve (2)

$$2\cos x = \sqrt{3}$$

for x, where $0 \leqslant x < 2\pi$.

A. $\frac{1}{3}\pi$ and $\frac{5}{3}\pi$

B. $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$

C. $\frac{1}{6}\pi$ and $\frac{5}{6}\pi$

D. $\frac{1}{6}\pi$ and $\frac{11}{6}\pi$

Solution

D

$$\begin{aligned} 2\cos x = \sqrt{3} &\Rightarrow \cos x = \frac{\sqrt{3}}{2} \\ &\Rightarrow x = \tfrac{1}{6}\pi \text{ or } x = \tfrac{11}{6}\pi. \end{aligned}$$

11. Find (2)

$$\int \left(4x^{\frac{1}{2}} + x^{-3}\right)\,\mathrm{d}x,$$

where $x > 0$.

A. $2x^{-\frac{1}{2}} - 3x^{-4} + c$

B. $2x^{-\frac{1}{2}} - \frac{1}{2}x^{-2} + c$

C. $\frac{8}{3}x^{\frac{3}{2}} - 3x^{-4} + c$

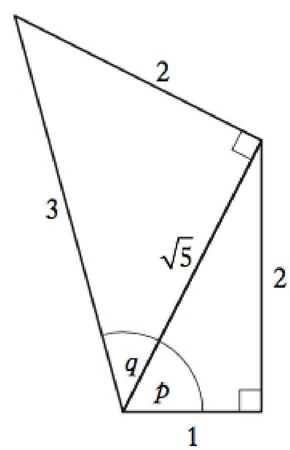
D. $\frac{8}{3}x^{\frac{3}{2}} - \frac{1}{2}x^{-2} + c$

Solution

D

$$\int \left(4x^{\frac{1}{2}} + x^{-3}\right) \, \mathrm{d}x = \frac{8}{3}x^{\frac{3}{2}} - \frac{1}{2}x^{-2} + c.$$

12. The diagram shows two right-angled triangles with sides and angles as given. (2)



What is the value of $\sin(p+q)$?

A. $\frac{2}{\sqrt{5}} + \frac{2}{3}$

B. $\frac{2}{\sqrt{5}} + \frac{\sqrt{5}}{3}$

C. $\frac{2}{3} + \frac{2}{3\sqrt{5}}$

D. $\frac{4}{3\sqrt{5}} + \frac{1}{3}$

Solution

C

$$\begin{aligned}\sin(p+q) &= \sin p \cos q + \cos p \sin q \\ &= \left(\tfrac{2}{\sqrt{5}} \times \tfrac{\sqrt{5}}{3}\right) + \left(\tfrac{1}{\sqrt{5}} \times \tfrac{2}{3}\right) \\ &= \tfrac{2}{3} + \tfrac{2}{3\sqrt{5}}.\end{aligned}$$

13. Given that (2)

$$f(x) = 4\sin 3x,$$

find $f'(0)$.

A. 0

B. 1

C. 12

D. 36

Solution

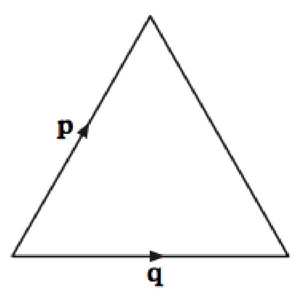
C

$$f(x) = 4\sin 3x \Rightarrow f(x) = 12\cos 3x$$

and

$$f'(0) = 12 \times 1 = 12.$$

14. An equilateral triangle of side 3 units is shown. (2)



The vectors **p** and **q** are as represented in the diagram.
What is the value of $\mathbf{p} \cdot \mathbf{q}$?

A. 9

B. $\frac{9}{2}$

C. $\frac{9}{\sqrt{2}}$

D. 0

Solution

B

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= \begin{pmatrix} 3\cos 60^\circ \\ 3\sin 60^\circ \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 0 \end{pmatrix}\\ &= \tfrac{3}{2}\times 3\\ &= \tfrac{9}{2}.\end{aligned}$$

15. Given that the points (2)

$$S(-4,5,1), T(-16,-4,16), \text{ and } U(-24,-10,26)$$

are collinear, calculate the ratio in which T divides SU.

A. 2 : 3

B. 3 : 2

C. 2 : 5

D. 3 : 5

Solution

B

$$\begin{aligned}ST &= \sqrt{12^2+9^2+15^2}\\ &= \sqrt{144+81+225}\\ &= \sqrt{144+306}\\ &= \sqrt{450}\\ &= \sqrt{225\times 2}\\ &= \sqrt{225}\times\sqrt{2}\\ &= 15\sqrt{2}\end{aligned}$$

and

$$\begin{aligned} TU &= \sqrt{8^2 + 6^2 + 10^2} \\ &= \sqrt{64 + 36 + 100} \\ &= \sqrt{200} \\ &= \sqrt{100 \times 2} \\ &= \sqrt{100} \times \sqrt{2} \\ &= 10\sqrt{2}. \end{aligned}$$

Finally,

$$ST : TU = 15\sqrt{2} : 10\sqrt{2} = 3 : 2.$$

16. Find (2)

$$\int \frac{1}{3x^4} \, \mathrm{d}x,$$

where $x \neq 0$.

A. $-\dfrac{1}{9x^3} + c$

B. $-\dfrac{1}{x^3} + c$

C. $\dfrac{1}{x^3} + c$

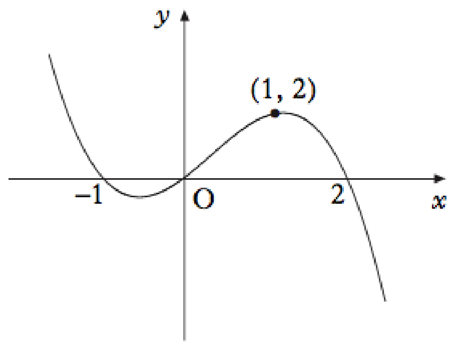
D. $\dfrac{1}{12x^3} + c$

Solution

A

$$\begin{aligned} \int \frac{1}{3x^4} \, \mathrm{d}x &= \int \tfrac{1}{3} x^{-4} \, \mathrm{d}x \\ &= -\tfrac{1}{9} x^{-3} + c. \end{aligned}$$

17. The diagram shows the graph of a cubic. (2)



What is the equation of this cubic?

A. $y = -x(x+1)(x-2)$

B. $y = -x(x-1)(x+2)$

C. $y = x(x+1)(x-2)$

D. $y = x(x-1)(x+2)$

Solution

A

It goes through $-(1,0)$, $(0,0)$, and $(2,0)$ and is clearly negative (why?).

18. If (2)

$$f(x) = (x-3)(x+5),$$

for what values of x is the graph of $y = f(x)$ above the x-axis?

A. $-5 < x < 3$

B. $-3 < x < 5$

C. $x < -5, x > 3$

D. $x < -3, x > 5$

Solution

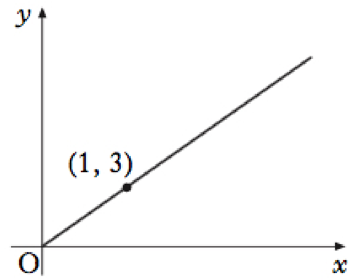
C

$$(x-3)(x+5) = 0 \Rightarrow x = 3 \text{ or } x = -5.$$

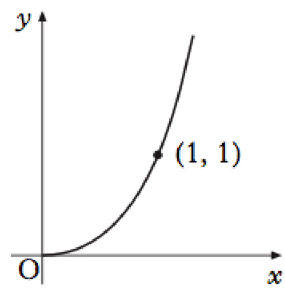
Finally, it has the standard parabola shape (not inverted) and the values are $x < -5$ or $x > 3$.

19. Which of the following diagrams represents the graph with equation $\log_3 y = x$? (2)

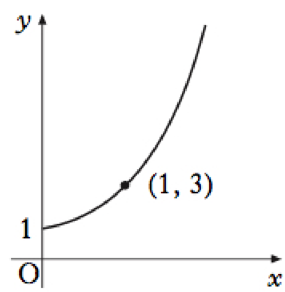
A



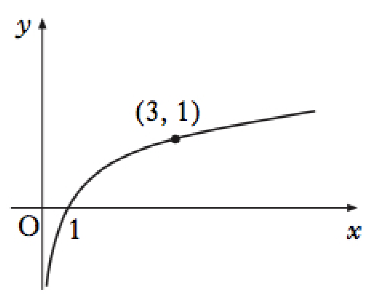
B



C



D



Solution

C

It does not go through the origin and it is D reflected in the line $y = x$.

20. On a suitable domain, D, a function g is defined by (2)

$$g(x) = \sin^2 \sqrt{x-2}.$$

Which of the following gives the real values of x in D and the corresponding values of $g(x)$?

A. $x \geqslant 0$ and $-1 \leqslant g(x) \leqslant 1$

B. $x \geqslant 0$ and $0 \leqslant g(x) \leqslant 1$

C. $x \geqslant 2$ and $-1 \leqslant g(x) \leqslant 1$

D. $x \geqslant 2$ and $0 \leqslant g(x) \leqslant 1$

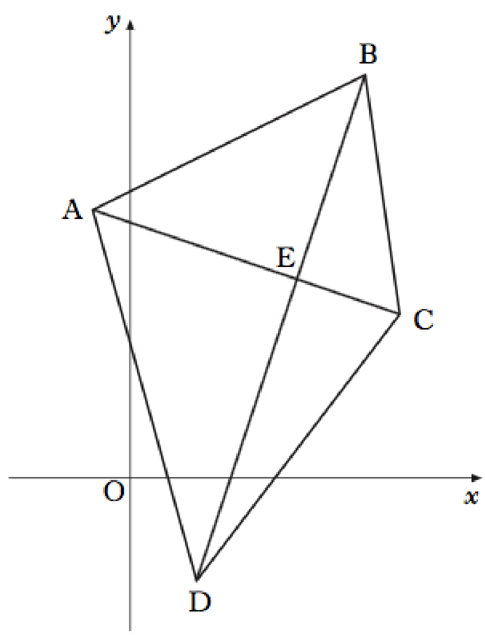
Solution

D

Clearly, the answer is C or D (why?). But can $g(x)$ be negative?

Section B

21. A quadrilateral has vertices $A(-1,8)$, $B(7,12)$, $C(8,5)$, and $D(2,-3)$ as shown in the diagram.



(a) Find the equation of diagonal BD. (2)

Solution

$$\begin{aligned}\text{Gradient} &= \frac{12-(-3)}{7-2} \\ &= \frac{15}{5} \\ &= 3\end{aligned}$$

and the equation of diagonal BD is

$$\begin{aligned}y-12 = 3(x-7) &\Rightarrow y-12 = 3x-21 \\ &\Rightarrow \underline{\underline{y = 3x-9}}.\end{aligned}$$

The equation of diagonal AC is $x+3y=23$.

(b) Find the coordinates of E, the point of intersection of the diagonals. (3)

Solution

$$\begin{aligned}x+3(3x-9) = 23 &\Rightarrow x+(9x-27) = 23 \\ &\Rightarrow 10x = 50 \\ &\Rightarrow x = 5 \\ &\Rightarrow y = 6;\end{aligned}$$

hence, $\underline{\underline{E(5,6)}}$.

(c) (i) Find the equation of the perpendicular bisector of AB. (5)

Solution
Well,

$$\begin{aligned}\text{gradient} &= \frac{12-8}{7-(-1)} \\ &= \frac{4}{8} \\ &= \tfrac{1}{2}\end{aligned}$$

and the gradient of the perpendicular bisector is -2. How, the midpoint of

AB is $(3, 10)$ and the equation of the perpendicular bisector of AB is

$$y - 10 = -2(x - 3) \Rightarrow y - 10 = -2x + 6$$
$$\Rightarrow \underline{\underline{y = -2x + 16}}.$$

(ii) Show that this line passes through E.

Solution

$$x = 5 \Rightarrow y = (-2 \times 5) + 16 = 6$$

and so this line passes through E

22. A function f is defined on the set of real numbers by

$$\mathrm{f}(x) = (x - 2)(x^2 + 1).$$

(a) Find where the graph of $y = \mathrm{f}(x)$ cuts: (2)

(i) the x-axis,

Solution

$\underline{\underline{(2, 0)}}$.

(ii) the y-axis

Solution

$\underline{\underline{(0, -2)}}$.

(b) Find the coordinates of the stationary points on the curve with equation $y = \mathrm{f}(x)$ and determine their nature. (8)

Solution

$\times$	x^2	$+1$
x	x^3	$+x$
-2	$-2x^2$	-2

Now,

$$y = (x-2)(x^2+1) \Rightarrow y = x^3 - 2x^2 + x - 2$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 1$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 4.$$

Next,

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & (+3) \times (+1) = +3 \end{array}\right\} -3,\ -1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3x^2 - 4x + 1 = 0$$
$$\Rightarrow 3x^2 - 3x - x + 1 = 0$$
$$\Rightarrow 3x(x-1) - (x-1) = 0$$
$$\Rightarrow (3x-1)(x+1) = 0$$
$$\Rightarrow x = \tfrac{1}{3} \text{ or } x = 1$$
$$\Rightarrow y = -\tfrac{50}{27} \text{ or } y = -2.$$

So, the coordinates of the stationary points are $(-\frac{1}{3}, -\frac{50}{27})$ and $(-1, -2)$.

$(\frac{1}{3}, -\frac{50}{27})$:

$$x = \tfrac{1}{3} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4 < 0$$

and this is a minimum turning point.

$(1, -2)$:

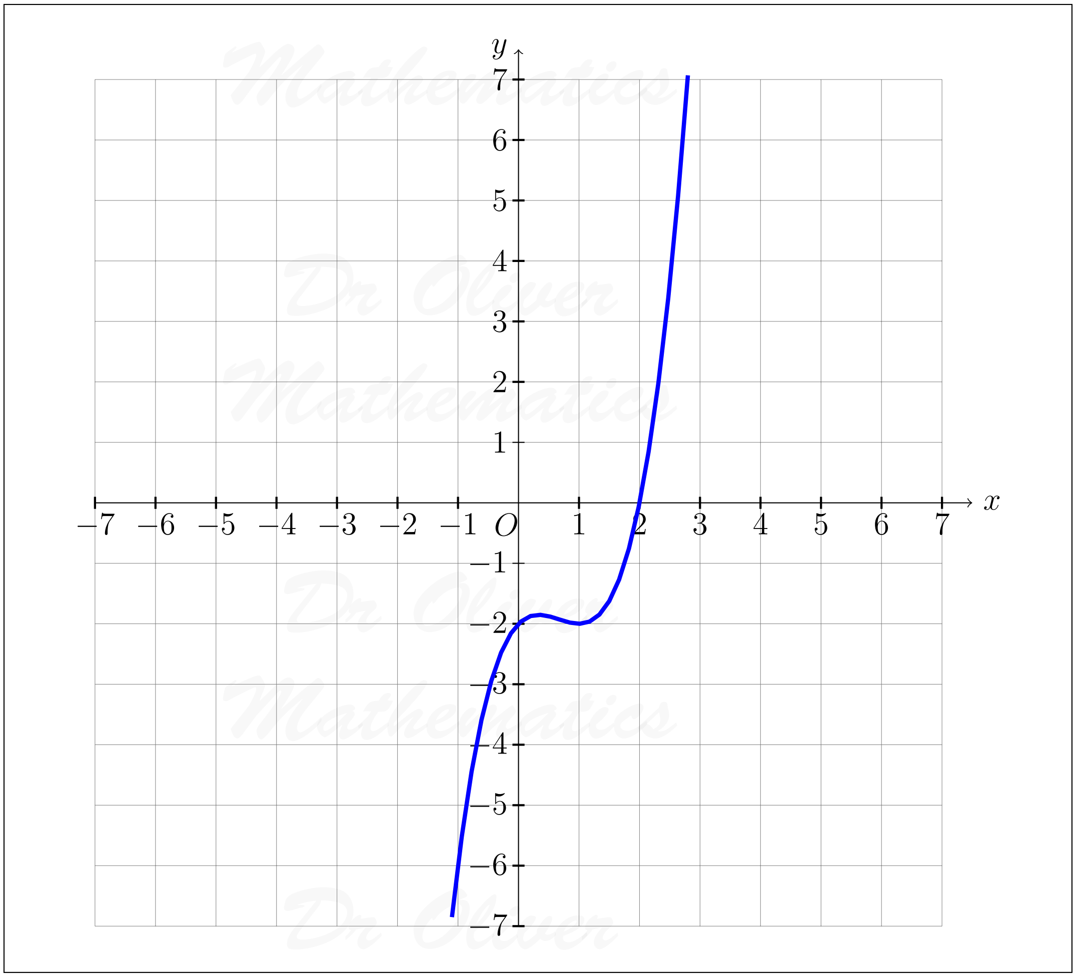
$$x = 1 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4 > 0$$

and this is a maximum turning point.

(c) On separate diagrams sketch the graphs of: (3)

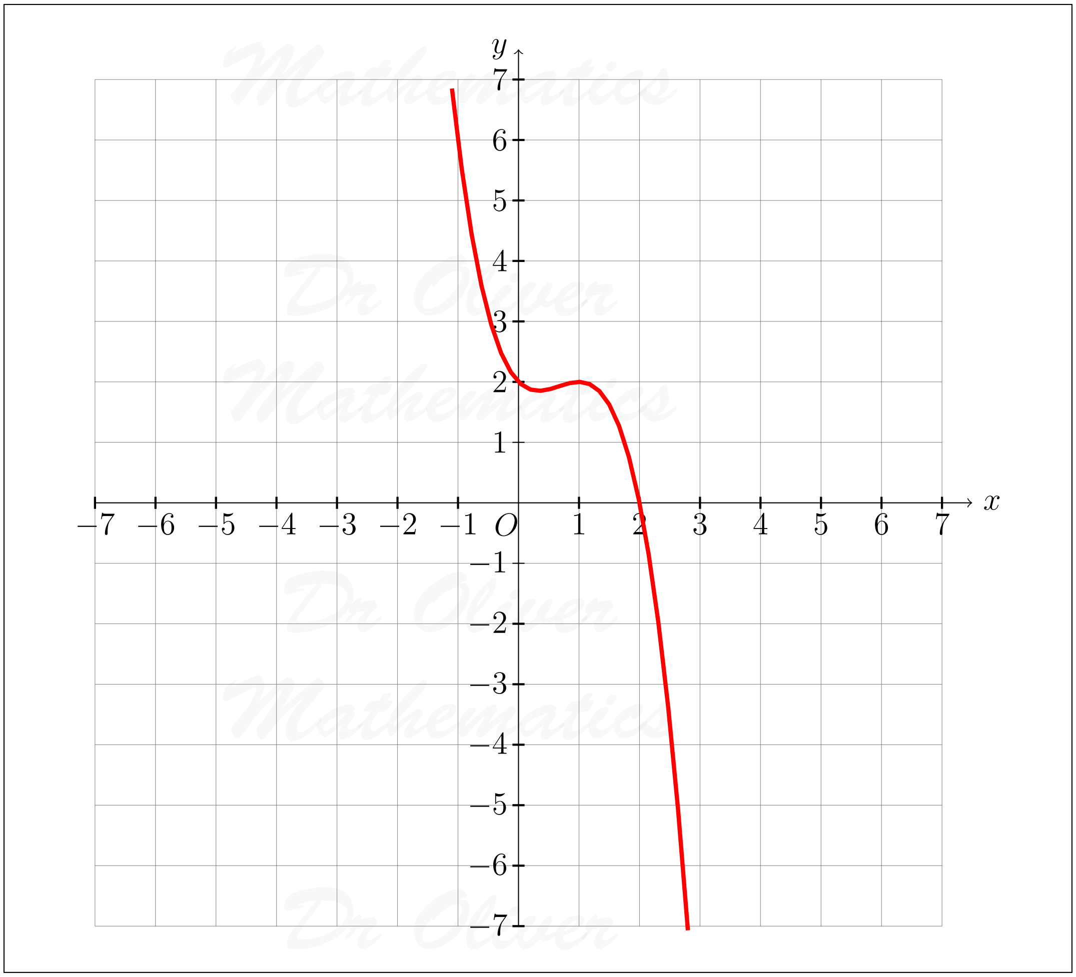
(i) $y = \mathrm{f}(x)$,

Solution



(ii) $y = -\mathrm{f}(x)$.

Solution



23. (a) Solve (5)

$$\cos 2x^\circ - 3\cos x^\circ + 2 = 0$$

for $0 \leqslant x < 360$.

Solution

$$\begin{aligned}\cos 2x^\circ - 3\cos x^\circ + 2 = 0 &\Rightarrow (2\cos^2 x^\circ - 1) - 3\cos x^\circ + 2 = 0\\ &\Rightarrow 2\cos^2 x^\circ - 3\cos x^\circ + 1 = 0\\ &\Rightarrow (2\cos x^\circ - 1)(\cos x^\circ - 1) = 0\\ &\Rightarrow 2\cos x^\circ - 1 = 0 \text{ or } \cos x^\circ - 1 = 0\\ &\Rightarrow \cos x^\circ = \tfrac{1}{2} \text{ or } \cos x^\circ = 1.\end{aligned}$$

$\underline{\cos x^\circ = \frac{1}{2}}$:

$$\cos x^\circ = \tfrac{1}{2} \Rightarrow \underline{\underline{x = 60, 300}}.$$

$\underline{\cos x^\circ = 1}$:

$$\cos x^\circ = 1 \Rightarrow \underline{\underline{x = 0}}.$$

(b) Hence solve (2)

$$\cos 4x^\circ - 3\cos 2x^\circ + 2 = 0$$

for $0 \leqslant x < 360$.

Solution

$$2x = 0, 60, 300, 360, 420, 660 \Rightarrow \underline{\underline{x = 0, 30, 150, 180, 210, 330}}.$$