Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2011 November Paper 1 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Show that (3)

$$\frac{1}{\tan\theta + \cot\theta} \equiv \sin\theta\cos\theta.$$

Solution

Well,

$$\begin{aligned}\frac{1}{\tan\theta + \cot\theta} &\equiv \frac{1}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}} \\ &\equiv \frac{1}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}} \\ &\equiv \frac{1}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}} \\ &\equiv \frac{1}{\frac{1}{\sin\theta\cos\theta}} \\ &\equiv \underline{\underline{\sin\theta\cos\theta}},\end{aligned}$$

as required.

2. Find the coordinates of the points where the line $2y = x - 1$ meets the curve (5)

$$x^2 + y^2 = 29.$$

Solution

Well,

$$2y = x - 1 \Rightarrow x = 2y + 1$$

and insert this into the circle:

$$x^2 + y^2 = 29 \Rightarrow (2y+1)^2 + y^2 = 29$$

$\times$	$2y$	$+1$
$2y$	$4y^2$	$+2y$
$+1$	$+2y$	$+1$

$$\Rightarrow (4y^2 + 4y + 1) + y^2 = 29$$
$$\Rightarrow 5y^2 + 4y - 28 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +4 \\ \text{multiply to:} & (+5) \times (-28) = -140 \end{array}\right\} -10,\ +14$$

e.g.,

$$\Rightarrow 5y^2 + 14y - 10y - 28 = 0$$
$$\Rightarrow y(5y + 14) - 2(5y + 14) = 0$$
$$\Rightarrow (y - 2)(5y + 14) = 0$$
$$\Rightarrow y = 2 \text{ or } y = -2\tfrac{4}{5}$$
$$\Rightarrow x = 5 \text{ or } x = -4\tfrac{3}{5};$$

hence,

$$\underline{\underline{x = 5, y = 2 \text{ or } x = -4\tfrac{3}{5}, y = -2\tfrac{4}{5}}}.$$

3. (a) Express $\log_x 2$ in terms of a logarithm to base 2. (1)

Solution

Well,

$$\log_x 2 = \underline{\underline{\frac{1}{\log_2 x}}}.$$

(b) Using the result of part (a), and the substitution (4)

$$u = \log_2 x,$$

find the values of x which satisfy

$$\log_2 x = 3 - 2\log_x 2.$$

Solution

Now,

$$\begin{aligned}\log_2 x = 3 - 2\log_x 2 &\Rightarrow \log_2 x = 3 - \frac{2}{\log_2 x}\\ &\Rightarrow (\log_2 x)^2 = 3\log_2 x - 2\\ &\Rightarrow (\log_2 x)^2 - 3\log_2 x + 2 = 0\\ &\Rightarrow u^2 - 3u + 2 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -3\\ \text{multiply to:} & +2\end{array}\right\} -2,\ -1$$

$$\begin{aligned}&\Rightarrow (u-2)(u-1) = 0\\ &\Rightarrow u = 2 \text{ or } u = 1\\ &\Rightarrow \log_2 x = 2 \text{ or } \log_2 x = 1\\ &\Rightarrow x = 2^2 \text{ or } x = 2^1\\ &\Rightarrow \underline{\underline{x = 4 \text{ or } x = 2}}.\end{aligned}$$

4. A curve has equation (6)

$$y = (3x^2 + 15)^{\frac{2}{3}}.$$

Find the equation of the normal to the curve at the point where $x = 2$.

Solution

Well,

$$x = 2 \Rightarrow y = 9$$

so the point is $(2, 9)$. Now,

$$y = (3x^2 + 15)^{\frac{2}{3}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{2}{3}(3x^2 + 15)^{-\frac{1}{3}} \times 6x$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 4x(3x^2 + 15)^{-\frac{1}{3}}.$$

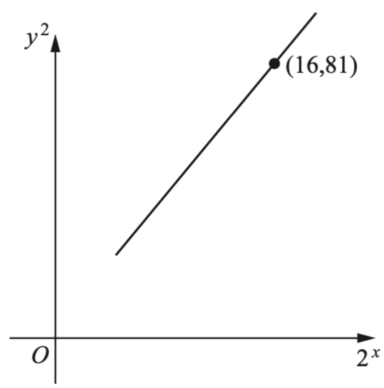
Next,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{8}{3}$$
$$\Rightarrow m_{\text{normal}} = -\tfrac{3}{8}.$$

Finally, the equation of the normal is

$$y - 9 = -\tfrac{3}{8}(x - 2) \Rightarrow y - 9 = \tfrac{3}{8}x + \tfrac{3}{4}$$
$$\Rightarrow \underline{\underline{y = \tfrac{3}{8}x + \tfrac{39}{4}}}.$$

5. Variables x and y are such that, when y^2 is plotted against 2^x, a straight line graph is obtained.



This line has a gradient of 5 and passes through the point $(16, 81)$.

(a) Express y^2 in terms of 2^x. (3)

Solution

Well,

$$y^2 - 81 = 5(2^x - 16) \Rightarrow y^2 - 81 = 5(2^x) - 80$$
$$\Rightarrow \underline{\underline{y^2 = 5(2^x) + 1}}.$$

(b) Find the value of x when $y = 6$. (3)

Solution

Now,

$$\begin{aligned} y = 6 &\Rightarrow 6^2 = 5(2^x) + 1 \\ &\Rightarrow 36 = 5(2^x) + 1 \\ &\Rightarrow 35 = 5(2^x) \\ &\Rightarrow 2^x = 7 \\ &\Rightarrow \underline{\underline{x = \log_2 7}}. \end{aligned}$$

6. (a) Given that (4)

$$(3 + x)^5 + (3 - x)^5 \equiv A + Bx^2 + Cx^4,$$

find the value of A, of B, and of C.

Solution

Well,

$$\begin{aligned} & (3 + x)^5 + (3 - x)^5 \\ \equiv\; & \left[3^5 + \binom{5}{1}(3)^4(x)^1 + \binom{5}{2}(3)^3(x)^2 + \binom{5}{3}(3)^2(x)^3 + \binom{5}{4}(3)^1(x)^4 + (x)^5\right] \\ & + \left[3^5 + \binom{5}{1}(3)^4(-x)^1 + \binom{5}{2}(3)^3(-x)^2 \right. \\ & \qquad \left. + \binom{5}{3}(3)^2(-x)^3 + \binom{5}{4}(3)^1(-x)^4 + (-x)^5\right] \\ \equiv\; & \left(243 + 405x + 270x^2 + 90x^3 + 15x^4 + x^5\right) \\ & + \left(243 - 405x + 270x^2 - 90x^3 + 15x^4 - x^5\right) \\ \equiv\; & \underline{\underline{486 + 540x^2 + 30x^4}}; \end{aligned}$$

hence, $\underline{\underline{A = 486}}$, $\underline{\underline{B = 540}}$, and $\underline{\underline{C = 30}}$.

(b) Hence, using the substitution (4)

$$y = x^2,$$

solve, for x, the equation

$$(3 + x)^5 + (3 - x)^5 = 1\,086.$$

Solution

Now,

$$\begin{aligned}(3 + x)^5 + (3 - x)^5 = 1\,086 &\Rightarrow 486 + 540x^2 + 30x^4 = 1\,086\\ &\Rightarrow 486 + 540y + 30y^2 = 1\,086\\ &\Rightarrow 30y^2 + 540y - 600 = 0\\ &\Rightarrow 30(y^2 + 18y - 20) = 0.\end{aligned}$$

Next, $a = 1$, $b = 18$, and $c = -20$:

$$\begin{aligned} & y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ \Rightarrow\quad & y = \frac{-18 \pm \sqrt{18^2 - 4 \times 1 \times (-20)}}{2 \times 1}\\ \Rightarrow\quad & y = \frac{-18 \pm \sqrt{404}}{2}\\ \Rightarrow\quad & y = -19.049\ldots \text{(cannot happen)}, 1.049\,875\,621 \text{ (FCD)}\\ \Rightarrow\quad & x = 1.024\,634\,384 \text{ (FCD)};\end{aligned}$$

hence, the solution is $x = 1.02$ (3 sf).

7. (a) Show that (3)

$$\frac{(4 - \sqrt{x})^2}{\sqrt{x}}$$

can be written in the form

$$px^{-\frac{1}{2}} + q + rx^{\frac{1}{2}},$$

where p, q, and r are integers to be found.

Solution

Well,

$\times$	4	$-\sqrt{x}$
4	16	$-4\sqrt{x}$
$-\sqrt{x}$	$-4\sqrt{x}$	$+x$

and so

$$\frac{(4-\sqrt{x})^2}{\sqrt{x}} \Rightarrow \frac{16-8\sqrt{x}+x}{\sqrt{x}}$$
$$\Rightarrow \underline{\underline{16x^{-\frac{1}{2}}-8+x^{\frac{1}{2}}}};$$

hence, $\underline{\underline{p=16}}$, $\underline{\underline{q=-8}}$, and $\underline{\underline{r=1}}$

(b) A curve is such that (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4-\sqrt{x})^2}{\sqrt{x}} \text{ for } x > 0.$$

Given that the curve passes through the point $(9, 30)$, find the equation of the curve.

Solution

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4-\sqrt{x})^2}{\sqrt{x}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 16x^{-\frac{1}{2}}-8+x^{\frac{1}{2}}$$
$$\Rightarrow y = 32x^{\frac{1}{2}} - 8x + \tfrac{2}{3}x^{\frac{3}{2}} + c.$$

Next,

$$x=9, y=30 \Rightarrow 30 = 96-72+18+c$$
$$\Rightarrow c=-12;$$

hence,

$$\underline{\underline{y = 32x^{\frac{1}{2}} - 8x + \tfrac{2}{3}x^{\frac{3}{2}} - 12}}.$$

8. The line CD is the perpendicular bisector of the line joining the point $A(-1,-5)$ and the point $B(5,3)$.

(a) Find the equation of the line CD. (4)

Solution

The midpoint, M, is

$$\left(\frac{-1+5}{2}, \frac{-5+3}{2}\right) = (2, -1).$$

Now,

$$\begin{aligned} m_{AB} &= \frac{3-(-5)}{5-(-1)} \\ &= \frac{8}{6} \\ &= \tfrac{4}{3} \end{aligned}$$

and

$$m_{CD} = -\tfrac{3}{4}.$$

Finally, the equation of the line CD is

$$\begin{aligned} y + 1 = -\tfrac{3}{4}(x-2) &\Rightarrow y + 1 = -\tfrac{3}{4}x + \tfrac{3}{2} \\ &\Rightarrow \underline{\underline{y = -\tfrac{3}{4}x + \tfrac{1}{2}}}. \end{aligned}$$

(b) Given that (3)

- M is the midpoint of AB,
- that $2CM = MD$, and that
- the x-coordinate of C is -2,

find the coordinates of D.

Solution

Well,

$$x = -2 \Rightarrow y = 2$$

so $\underline{\underline{C(-2, 2)}}$.

Now,

$$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{OM} + \overrightarrow{MD}\\ &= \overrightarrow{OM} + 2\overrightarrow{CM}\\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 2-(-2) \\ -1-2 \end{pmatrix}\\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 4 \\ -3 \end{pmatrix}\\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 8 \\ -6 \end{pmatrix}\\ &= \begin{pmatrix} 10 \\ -7 \end{pmatrix};\end{aligned}$$

so, $\underline{\underline{D(10, -7)}}$.

(c) Find the area of the triangle CAD. (2)

Solution
Well,

$$\begin{aligned}CD &= \sqrt{[10-(-2)]^2 + (-7-2)^2}\\ &= \sqrt{12^2 + (-9)^2}\\ &= 15\end{aligned}$$

and

$$\begin{aligned}AM &= \sqrt{(-1-2)^2 + [-5-(-1)]^2}\\ &= \sqrt{(-3)^2 + (-4)^2}\\ &= 5.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area of the triangle } CAD &= \tfrac{1}{2}bh\\ &= \tfrac{1}{2} \times 15 \times 5\\ &= \underline{\underline{37\tfrac{1}{2}}}.\end{aligned}$$

9. (a) Given that (3)

$$y = x \sin 4x,$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution
Product rule:

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = \sin 4x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 4\cos 4x$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x)(4\cos 4x) + (1)(\sin 4x)\\ &= \underline{\underline{4x\cos 4x + \sin 4x}}.\end{aligned}$$

(b) Hence find (6)

$$\int x \cos 4x \,\mathrm{d}x$$

and evaluate

$$\int_0^{\frac{1}{8}\pi} x \cos 4x \,\mathrm{d}x.$$

Solution
Well,

$$\begin{aligned}4x\cos 4x = \frac{\mathrm{d}y}{\mathrm{d}x} - \sin 4x &\Rightarrow \int x\cos 4x \,\mathrm{d}x = \tfrac{1}{4}\int \left(\frac{\mathrm{d}y}{\mathrm{d}x} - \sin 4x\right) \mathrm{d}x\\ &\Rightarrow \int x\cos 4x \,\mathrm{d}x = \tfrac{1}{4}x\sin 4x - \tfrac{1}{4}\int \sin 4x \,\mathrm{d}x\\ &\Rightarrow \int x\cos 4x \,\mathrm{d}x = \underline{\underline{\tfrac{1}{4}x\sin 4x + \tfrac{1}{16}\cos 4x + c}}.\end{aligned}$$

Finally,

$$\begin{aligned}\int_0^{\frac{1}{8}\pi} x\cos 4x\,\mathrm{d}x &= \left[\tfrac{1}{4}x\sin 4x + \tfrac{1}{16}\cos 4x\right]_{x=0}^{\frac{1}{8}\pi}\\ &= \left(\tfrac{1}{32}\pi + 0\right) - \left(0 + \tfrac{1}{16}\right)\\ &= \underline{\underline{\tfrac{1}{32}\pi - \tfrac{1}{16}}}.\end{aligned}$$

10. (a) Solve (5)

$$2\sec^2 x = 5\tan x + 5, \text{ for } 0^\circ < x < 360^\circ.$$

Solution

$$\begin{aligned}2\sec^2 x = 5\tan x + 5 &\Rightarrow 2(\tan^2 x + 1) = 5\tan x + 5\\ &\Rightarrow 2\tan^2 x + 2 = 5\tan x + 5\\ &\Rightarrow 2\tan^2 x - 5\tan x - 3 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & \quad -5\\ \text{multiply to:} & (+2)\times(-3) = -6\end{array}\right\}\ -6,\ +1$$

e.g.,

$$\begin{aligned}&\Rightarrow 2\tan^2 x - 6\tan x + \tan x - 3 = 0\\ &\Rightarrow 2\tan x(\tan x - 3) + 1(\tan x - 3) = 0\\ &\Rightarrow (\tan x - 3)(2\tan x + 1) = 0\\ &\Rightarrow \tan x = 3 \text{ or } \tan x = -\tfrac{1}{2}.\end{aligned}$$

$\underline{\tan x = 3}$:

$$\begin{aligned}\tan x = 3 &\Rightarrow x = 71.565\,051\,18,\ 251.565\,051\,2\ \text{(FCD)}\\ &\Rightarrow \underline{\underline{x = 71.6,\ 252\ (3\text{ sf})}}.\end{aligned}$$

$\underline{\tan x = -\tfrac{1}{2}}$:

$$\begin{aligned}\tan x = -\tfrac{1}{2} &\Rightarrow x = 153.434\,948\,8,\ 333.434\,948\,8\ \text{(FCD)}\\ &\Rightarrow \underline{\underline{x = 153,\ 333\ (3\text{ sf})}}.\end{aligned}$$

(b) Solve (5)

$$\sqrt{2}\sin(\tfrac{1}{2}y + \tfrac{1}{3}\pi) = 1 \text{ for } 0 < y < 4\pi \text{ radians.}$$

Solution

Well,

$$\begin{aligned} 0 < y < 4\pi &\Rightarrow 0 < \tfrac{1}{2}y < 2\pi \\ &\Rightarrow \tfrac{1}{3}\pi < (\tfrac{1}{2}y + \tfrac{1}{3}\pi) < \tfrac{7}{3}\pi \end{aligned}$$

and so

$$\begin{aligned} \sqrt{2}\sin(\tfrac{1}{2}y + \tfrac{1}{3}\pi) = 1 &\Rightarrow \sin(\tfrac{1}{2}y + \tfrac{1}{3}\pi) = \frac{\sqrt{2}}{2} \\ &\Rightarrow \tfrac{1}{2}y + \tfrac{1}{3}\pi = \tfrac{3}{4}\pi,\ \tfrac{9}{4}\pi \\ &\Rightarrow \tfrac{1}{2}y = \tfrac{5}{12}\pi,\ \tfrac{23}{12}\pi \\ &\Rightarrow \underline{\underline{y = \tfrac{5}{6}\pi,\ \tfrac{23}{6}\pi}}. \end{aligned}$$

EITHER

11. A curve has equation

$$y = \mathrm{e}^{-x}(A\cos 2x + B\sin 2x).$$

At the point $(0, 4)$ on the curve, the gradient of the tangent is 6.

(a) Find the value of A. (1)

Solution

Well,

$$\begin{aligned} x = 0, y = 4 &\Rightarrow 4 = A + 0 \\ &\Rightarrow \underline{\underline{A = 4}}. \end{aligned}$$

(b) Show that $B = 5$. (5)

Solution

Product rule:

$$u = \mathrm{e}^{-x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\mathrm{e}^{-x}$$
$$v = A\cos 2x + B\sin 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -2A\sin 2x + 2B\cos 2x$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\mathrm{e}^{-x})(-2A\sin 2x + 2B\cos 2x) + (-\mathrm{e}^{-x})(A\cos 2x + B\sin 2x).$$

Now,

$$\begin{aligned} x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 6 \Rightarrow 6 &= (1)(0 + 2B) + (-1)(4 + 0) \\ &\Rightarrow -4 + 2B = 6 \\ &\Rightarrow 2B = 10 \\ &\Rightarrow \underline{\underline{B = 5}}, \end{aligned}$$

as required.

(c) Find the value of x, where $0 < x < \frac{1}{2}\pi$ radians, for which y has a stationary value. (5)

Solution

Well,

$$\begin{aligned} & \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ \Rightarrow\quad & (\mathrm{e}^{-x})(-8\sin 2x + 10\cos 2x) + (-\mathrm{e}^{-x})(4\cos 2x + 5\sin 2x) = 0 \\ \Rightarrow\quad & \mathrm{e}^{-x}\left[(-8\sin 2x + 10\cos 2x) - (4\cos 2x + 5\sin 2x)\right] = 0 \\ \Rightarrow\quad & \mathrm{e}^{-x}(-13\sin 2x + 6\cos 2x) = 0 \end{aligned}$$

but $\mathrm{e}^{-x} > 0$:

$$\begin{aligned} \Rightarrow\quad & -13\sin 2x + 6\cos 2x = 0 \\ \Rightarrow\quad & 13\sin 2x = 6\cos 2x \\ \Rightarrow\quad & \tan 2x = \tfrac{6}{13} \\ \Rightarrow\quad & 2x = 0.432\,407\,775\,6 \text{ (FCD)} \\ \Rightarrow\quad & x = 0.216\,203\,887\,8 \text{ (FCD)} \\ \Rightarrow\quad & \underline{\underline{x = 0.216 \text{ (3 sf)}}}. \end{aligned}$$

OR

12. A curve has equation

$$y = \frac{\ln(x^2-1)}{x^2-1} \text{ for } x > 1.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx\left[1-\ln(x^2-1)\right]}{(x^2-1)^2},$$

where k is a constant to be found.

Solution
Quotient rule:

$$u = \ln(x^2-1) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2x}{x^2-1}$$
$$v = x^2-1 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x^2-1)\left(\frac{2x}{x^2-1}\right)-(2x)[\ln(x^2-1)]}{(x^2-1)^2}\\ &= \frac{2x-2x[\ln(x^2-1)]}{(x^2-1)^2}\\ &= \underline{\underline{\frac{2x[1-\ln(x^2-1)]}{(x^2-1)^2}}},\end{aligned}$$

hence, $\underline{\underline{k=2}}$.

(b) Hence find the approximate change in y when x increases from $\sqrt{5}$ to $(\sqrt{5}+p)$, where p is small. (2)

Solution
Well,

$$x = \sqrt{5} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -0.107\,972\,556\,63 \text{ (FCD)}$$

and, finally,

$$\begin{aligned}\delta y &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x\\ &= -0.107\,972\,556\,63 \times p\\ &= \underline{\underline{-0.108p \text{ (3 sf)}}}.\end{aligned}$$

(c) Find, in terms of e, the coordinates of the stationary point on the curve. (5)

Solution

Now,

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{2x[1 - \ln(x^2 - 1)]}{(x^2 - 1)^2} = 0 \\
&\Rightarrow 1 - \ln(x^2 - 1) = 0 \\
&\Rightarrow \ln(x^2 - 1) = 1 \\
&\Rightarrow x^2 - 1 = \mathrm{e} \\
&\Rightarrow x^2 = \mathrm{e} + 1 \\
&\Rightarrow x = \sqrt{\mathrm{e} + 1} \\
&\Rightarrow y = \frac{\ln(\sqrt{\mathrm{e} + 1}^2 - 1)}{\sqrt{\mathrm{e} + 1}^2 - 1} \\
&\Rightarrow y = \frac{\ln(\mathrm{e} + 1 - 1)}{\mathrm{e} + 1 - 1} \\
&\Rightarrow y = \frac{1}{\mathrm{e}};
\end{aligned}
$$

hence, the coordinates of the stationary point on the curve are

$$\underline{\underline{\left(\sqrt{\mathrm{e} + 1}, \frac{1}{\mathrm{e}}\right)}}.$$