Dr Oliver Mathematics
Advance Level Mathematics
Core Mathematics 4: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. (a) Find the binomial series expansion of (5)

$$\sqrt{4 - 9x},\ |x| < \tfrac{4}{9},$$

in ascending powers of x, up to and including the term in x^2.
Give each coefficient in its simplest form.

(b) Use the expansion from part (a), with a suitable value of x, to find an approximate value for $\sqrt{310}$. (3)
Show all your working and give your answer to 3 decimal places.

2. (*Solutions based entirely on graphical or numerical methods are not acceptable.*)
The curve C has equation

$$x^2 + xy + y^2 - 4x - 5y + 1 = 0.$$

(a) Use implicit differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. (5)

(b) Find the x-coordinates of the two points on C where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$. (5)
Give exact answers in their simplest form.

3. (a) Given that

$$\frac{13 - 4x}{(2x+1)^2(x+3)} \equiv \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2} + \frac{C}{(x+3)},$$

(i) find the values of the constants A, B, and C. (4)

(ii) Hence find (3)

$$\int \frac{13 - 4x}{(2x+1)^2(x+3)}\,\mathrm{d}x\, x > -\tfrac{1}{2}.$$

(b) Find (3)

$$\int (\mathrm{e}^x + 1)^3\,\mathrm{d}x.$$

(c) Using the substitution $u^3 = x$, or otherwise, find (4)

$$\int \frac{1}{4x + 5x^{\frac{1}{3}}}\,\mathrm{d}x,\ x > 0.$$

4. A water container is made in the shape of a hollow inverted right circular cone with semi-vertical angle of 30°, as shown in Figure 1. The height of the container is 50 cm.

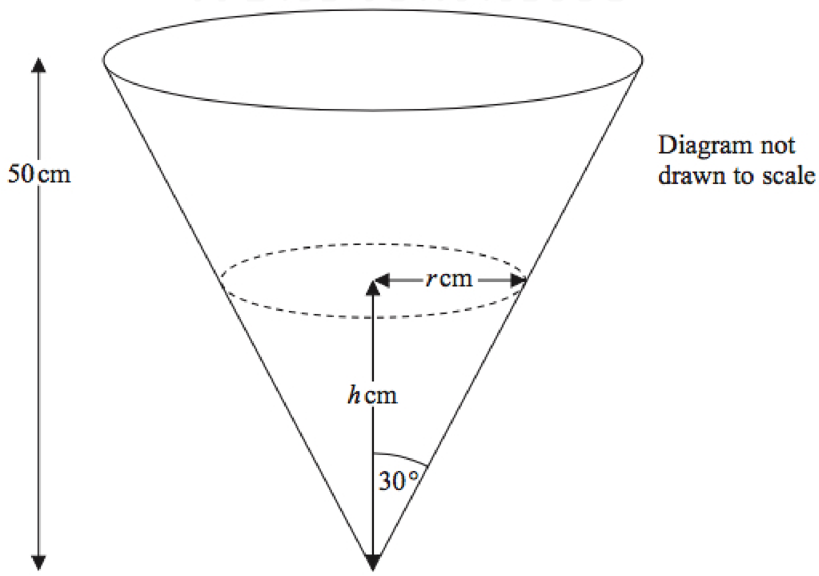

Figure 1: a water container

When the depth of the water in the container is h cm, the surface of the water has radius r cm, and the volume of water is V cm^3.

(a) Show that (2)

$$V = \frac{1}{9}\pi h^3.$$

[*You may assume the formula*

$$V = \frac{1}{3}\pi r^2 h$$

for the volume of a cone.]

Given that the volume of water in the container increases at a constant rate of 200 cm^3 s^{-1},

(b) find the rate of change of the depth of the water, in cm s^{-1}, when $h = 15$. (4)
Give your answer in its simplest form in terms of π.

5. Figure 2 shows a sketch of the curve C with parametric equations

$$x = 1 + t - 5\sin t,\ y = 2 - 4\cos t,\ -\pi \leqslant t \leqslant \pi.$$

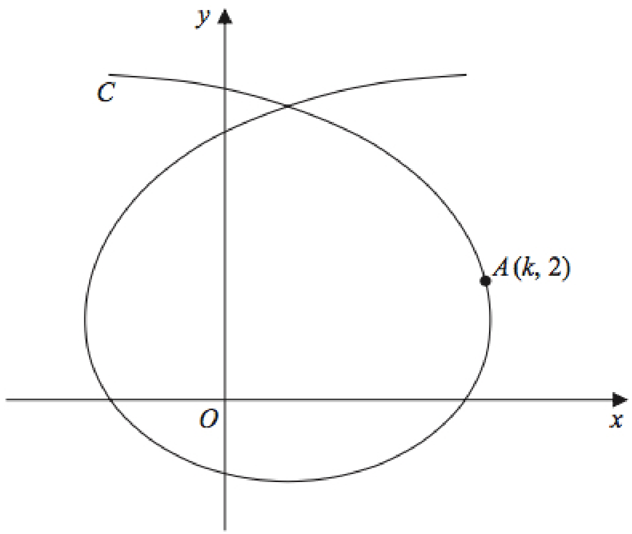

Diagram not drawn to scale

Figure 2: $x = 1 + t - 5\sin t$, $y = 2 - 4\cos t$

The point A lies on the curve C.
Given that the coordinates of A are $(k, 2)$, where $k > 0$,

(a) find the exact value of k, giving your answer in a fully simplified form. (2)

(b) Find the equation of the tangent to C at the point A. (5)
Give your answer in the form $y = px + q$, where p and q are exact real values.

6. Given that $y = 2$ when $x = -\frac{1}{8}\pi$, solve the differential equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{3\cos^2 2x}, \ -\tfrac{1}{2} < x < \tfrac{1}{2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

7. The point A with coordinates $(-3, 7, 2)$ lies on a line l_1.
The point B also lies on the line l_1.
Given that $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$,

(a) find the coordinates of point B. (2)

The point P has coordinates $(9, 1, 8)$.

(b) Find the cosine of the angle PAB, giving your answer as a simplified surd. (3)

(c) Find the exact area of triangle PAB, giving your answer in its simplest form. (3)

The line l_2 passes through the point P and is parallel to the line l_1.

(d) Find a vector equation for the line l_2. (2)

The point Q lies on the line l_2.
Given that the line segment AP is perpendicular to the line segment BQ,

(e) find the coordinates of the point Q. (5)

8. (*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(a) Find (3)

$$\int x \cos 4x \, \mathrm{d}x.$$

Figure 3 shows part of the curve with equation

$$y = \sqrt{x} \sin 2x, \ x \geqslant 0.$$

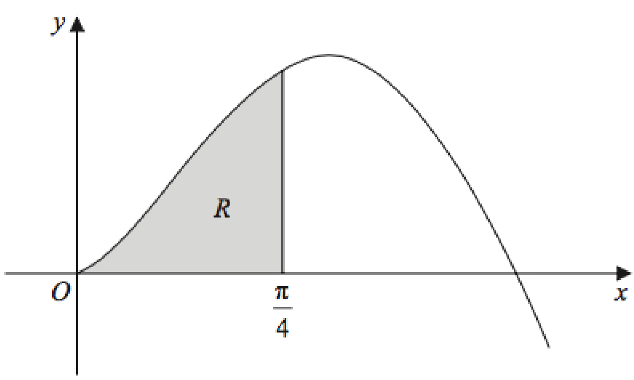

Diagram not drawn to scale

Figure 3: $y = \sqrt{x} \sin 2x$

The finite region R, shown shaded in Figure 3, is bounded by the curve, the x-axis, and the line with equation $x = \frac{1}{4}\pi$.

The region R is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact value of the volume of this solid of revolution, giving your answer in its simplest form. (6)