Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2018 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. Solve the inequality (3)

$$2 - x < 1 + 3(x - 2).$$

Solution

$$\begin{aligned} 2 - x < 1 + 3(x - 2) &\Rightarrow 2 - x < 1 + (3x - 6) \\ &\Rightarrow 7 < 4x \\ &\Rightarrow \underline{\underline{x > 1\tfrac{3}{4}}}. \end{aligned}$$

2. The gradient function of a curve is given by (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + 2x - 3x^2.$$

Find the equation of the curve given that it passes through the point $(2, 3)$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + 2x - 3x^2 \Rightarrow y = c + 2x + x^2 - x^3,$$

for some constant c. Now,

$$\begin{aligned} x = 2,\ y = 3 &\Rightarrow 3 = c + 4 + 4 - 8 \\ &\Rightarrow c = 3 \end{aligned}$$

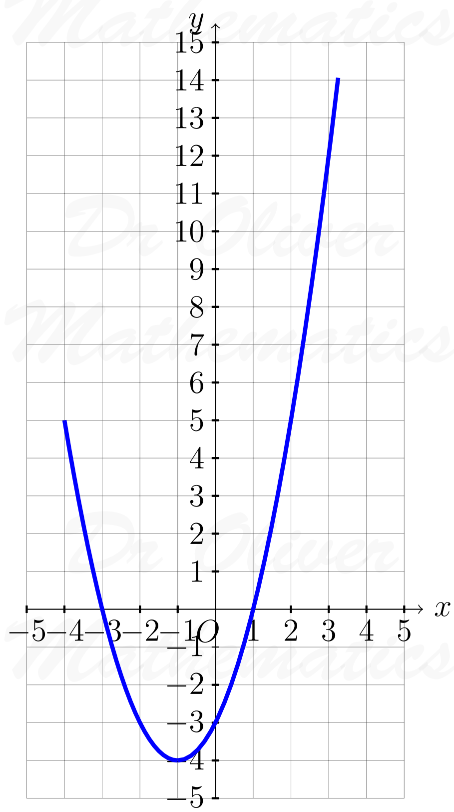
and hence the equation is

$$\underline{\underline{y = 3 + 2x + x^2 - x^3}}.$$

3. The graph of

$$y = x^2 + 2x - 3$$

is given below.



(a) Write down the solution to the equation (1)

$$x^2 + 2x - 3 = 0.$$

Solution

$\underline{\underline{x = -3 \text{ or } x = 1}}$.

(b) By plotting the appropriate straight line on the grid, find the solution to the equation (3)

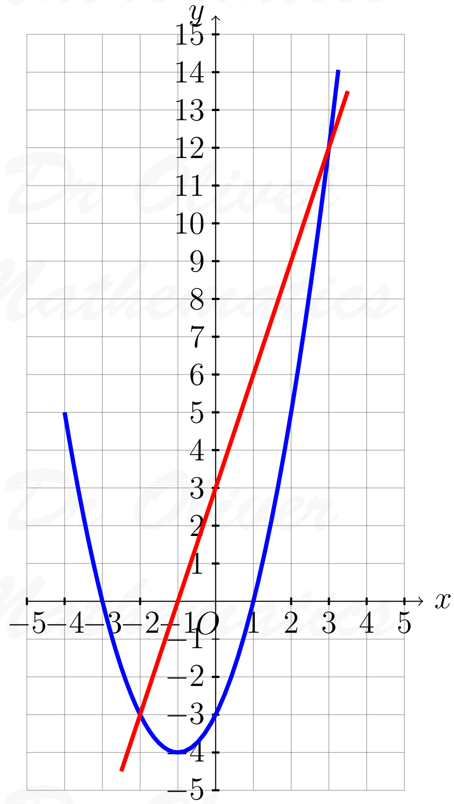
$$x^2 - x - 6 = 0.$$

Solution

Well,

$$x^2 - x - 6 = 0 \Rightarrow x^2 + 2x - 3 = 3x + 3$$

so let us draw the line $y = 3x + 3$:



Read-off the intercepts: $x = -2$, $(y = -3)$ or $x = 3$, $(y = 4)$.

4. You are given that the acute angle θ is such that

$$\sin\theta = \tfrac{1}{5}.$$

Find the exact value of each of the following.

(a) $\cos\theta$, (2)

Solution

If θ is acute, that means $0 \leqslant \cos\theta \leqslant 1$ and so

$$\begin{aligned}\cos\theta &= \sqrt{1 - \sin^2\theta} \\ &= \sqrt{1 - (\tfrac{1}{5})^2} \\ &= \sqrt{1 - \tfrac{1}{25}} \\ &= \sqrt{\tfrac{24}{25}} \\ &= \underline{\underline{\tfrac{2\sqrt{6}}{5}}}.\end{aligned}$$

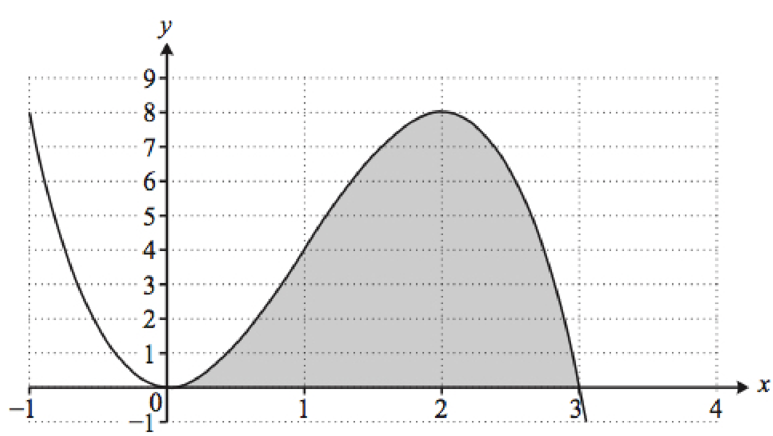
(b) $\tan\theta$. (2)

Solution

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{\frac{1}{5}}{\frac{2\sqrt{6}}{5}} \\ &= \underline{\underline{\tfrac{\sqrt{6}}{12}}}.\end{aligned}$$

5. The figure below shows part of the graph of the curve with equation (4)

$$y = 6x^2 - 2x^3.$$



Find the area of the shaded region enclosed by the curve and the x-axis.

Solution

$$\begin{aligned}\text{Area} &= \int_0^3 (6x^2 - 2x^3)\,\mathrm{d}x \\ &= \left[2x^3 - \tfrac{1}{2}x^4\right]_{x=0}^3 \\ &= \left(54 - 40\tfrac{1}{2}\right) - (0 - 0) \\ &= \underline{\underline{13\tfrac{1}{2}}}.\end{aligned}$$

6. (a) Solve these simultaneous equations. (4)

$$\begin{aligned}3x + 4y &= 18 \\ 7x - 3y &= 5.\end{aligned}$$

Solution

$$\begin{aligned}3x + 4y &= 18 \quad (1) \\ 7x - 3y &= 5 \quad (2)\end{aligned}$$

Do $3 \times (1)$ and $4 \times (2)$:

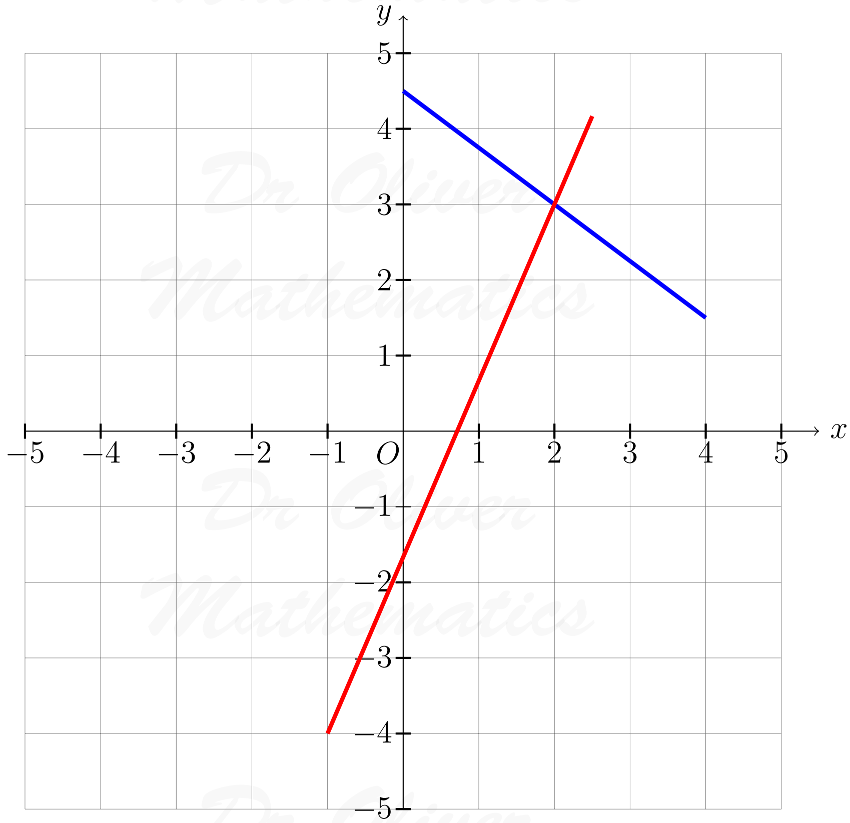
$$\begin{aligned}9x + 12y &= 54 \quad (3) \\ 28x - 12y &= 20 \quad (4)\end{aligned}$$

and $(3) + (4)$:

$$\begin{aligned}37x = 74 &\Rightarrow \underline{\underline{x = 2}} \\ &\Rightarrow 3(2) + 4y = 18 \\ &\Rightarrow 4y = 12 \\ &\Rightarrow \underline{\underline{y = 3}}.\end{aligned}$$

(b) Draw a rough sketch of the lines to demonstrate graphically the solution to part (a). (2)

Solution



7. (a) Find the coordinates of the points where the line (4)

$$y = 7x - 9$$

cuts the curve

$$y = x^2 + 2x - 5.$$

Solution

$$x^2 + 2x - 5 = 7x - 9 \Rightarrow x^2 - 5x + 4 = 0$$

$$\begin{array}{ll} \text{add to:} & -5 \\ \text{multiply to:} & +4 \end{array} \Bigg\} -4, -1$$

$$\begin{aligned} &\Rightarrow (x-1)(x-4) = 0 \\ &\Rightarrow x = 1 \text{ or } x = 4 \\ &\Rightarrow y = -2 \text{ or } y = 19; \end{aligned}$$

hence, the coordinates are $\underline{\underline{(1, -2)}}$ and $\underline{\underline{(4, 19)}}$.

(b) Determine whether the line is a normal to the curve at either of the points of intersection. (3)

Solution

$$y = x^2 + 2x - 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2$$

and

$$\begin{aligned} x = 1 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 4 \\ x = 4 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 10. \end{aligned}$$

Now, the gradients of the normal are

$$-\tfrac{1}{4} \text{ or } -\tfrac{1}{10}$$

respectively. So, as neither of those equals 7, the gradient of the straight line, the line is a $\underline{\underline{\text{not}}}$ normal to the curve.

8. (a) Simplify the equation (3)

$$\frac{x+a}{x} + \frac{x-2}{4} = 0,$$

leaving your answer in the form $(x+p)^2 = q$, where p is an integer and q is given in terms of the constant a.

Solution

Multiply by $4x$:

$$\begin{aligned}\frac{x+a}{x} + \frac{x-2}{4} = 0 &\Rightarrow 4(x+a) + x(x-2) = 0 \\ &\Rightarrow (4x + 4a) + (x^2 - 2x) = 0 \\ &\Rightarrow x^2 + 2x = -4a \\ &\Rightarrow x^2 + 2x + 1 = 1 - 4a \\ &\Rightarrow \underline{\underline{(x-1)^2 = 1 - 4a}};\end{aligned}$$

hence, $\underline{\underline{p = -1}}$ and $\underline{\underline{q = 1 - 4a}}$.

(b) Hence write down the range of values of a for which the equation has real roots. (2)

Solution

$$\begin{aligned}(x-1)^2 \geqslant 0 &\Rightarrow 1 - 4a \geqslant 0 \\ &\Rightarrow 4a \leqslant 1 \\ &\Rightarrow \underline{\underline{a \leqslant \tfrac{1}{4}}}.\end{aligned}$$

(c) Using your answer to part (a), solve the equation when $a = -1$, giving your answers **exactly**. (2)

Solution

$$\begin{aligned}a = -1 &\Rightarrow (x-1)^2 = 5 \\ &\Rightarrow x - 1 = \pm\sqrt{5} \\ &\Rightarrow \underline{\underline{x = 1 \pm \sqrt{5}}}.\end{aligned}$$

9. The proportion of people who are left-handed is 20%.

(a) For a group of 10 students chosen at random, use the binomial distribution to find the probability that

(i) no student is left-handed, (2)

Solution

$$\begin{aligned}\text{P(no student is left-handed)} &= \left(\tfrac{4}{5}\right)^{10} \\ &= 0.107\,374\,182\,4 \text{ (FCD)} \\ &= \underline{\underline{0.107 \text{ (3 sf)}}}.\end{aligned}$$

(ii) exactly 4 students are left-handed. (3)

Solution

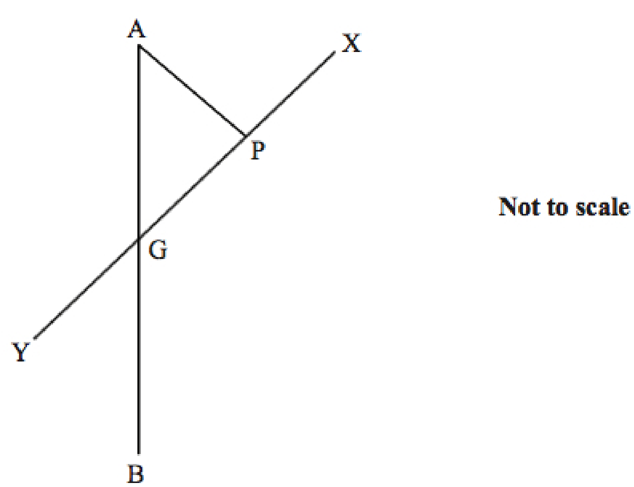
$$\begin{aligned}\text{P(exactly 4 students are left-handed)} &= \binom{10}{4}\left(\tfrac{1}{5}\right)^4\left(\tfrac{4}{5}\right)^6 \\ &= 0.088\,080\,384 \text{ (FCD)} \\ &= \underline{\underline{0.088\,1 \text{ (3 sf)}}}.\end{aligned}$$

(b) State the conditions necessary for the binomial distribution to be valid. (2)

Solution

E.g., fixed number of trials, independent trials, constant of those left-handed students is constant, each trial has two outcomes.

10. The diagram shows an "up and over" garage door, XY, that is 200 cm long. There is a small wheel at the point G on the door. The wheel runs freely up a groove in a fixed vertical door frame, AB. A metal rod AP is fixed to the top of the door frame, A, and is also fixed to the point P on the door. The rod is hinged at both ends.



$GP = PX = AP = 60$ cm and $YG = 80$ cm.
When the door is closed, Y is at B and X is at A. When the door is fully open, G is at A and the door is horizontal, 200 cm above the horizontal ground.

(a) Explain why P is the centre of the circle through A, G, and X. (1)

Solution
$AP = XP = GP = \text{radius}$.

(b) Hence show that AX is horizontal whatever the position of the garage door. (1)

Solution
Angle GAX is an angle in a semi-circle and BA is vertical, XA must be horizontal.

(c) Find the height of Y above the ground when angle $AGP = 40°$. (4)

Solution

$$\cos = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 40° = \frac{AG}{200-80} \Rightarrow AG = 120\cos 40°.$$

Now, finding depth of Y below G is

$$\cos = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 40° = \frac{\text{depth}}{80} \Rightarrow \text{depth} = 80\cos 40°.$$

Finally,

$$\begin{aligned}\text{height} &= 200 - AG - \text{depth} \\ &= 200 - 120\cos 40° - 80\cos 40° \\ &= 200 - 200\cos 40° \\ &= 46.791\,111\,138 \text{ (FCD)} \\ &= 46.8 \text{ cm (3 sf)}.\end{aligned}$$

Section B

11. A circle has centre $(0, 3)$ and radius 3.

(a) Show that the equation of the circle is (2)

$$x^2 + y^2 - ky = 0,$$

where k is to be determined.

Solution

$\times$	y	-3
y	y^2	$-3y$
-3	$-3y$	$+9$

$$x^2 + (y-3)^2 = 3^2 \Rightarrow x^2 + (y^2 - 6y + 9) = 9$$
$$\Rightarrow \underline{\underline{x^2 + y^2 - 6y = 0}};$$

hence, $\underline{\underline{k = 6}}$.

The line $y = mx - 2$ passes through the point $P(0, -2)$ and is a tangent to the circle.

(b) Find the two possible values of m. (6)

Solution

$\times$	mx	-2
mx	m^2x^2	$-2mx$
-2	$-2mx$	$+4$

$$x^2 + y^2 - 6y = 0 \Rightarrow x^2 + (m^2x^2 - 4mx + 4) - 6(mx - 2) = 0$$
$$\Rightarrow (1 + m^2)x^2 - 10mx + 16 = 0.$$

Now, it is a tangent if $b^2 - 4ac = 0$ where $a = 1 + m^2$, $b = -10m$, and $c = 16$:

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (-10m)^2 - 4(1+m^2)(16) = 0 \\ &\Rightarrow 100m^2 = 64(1+m^2) \\ &\Rightarrow 100m^2 = 64 + 64m^2 \\ &\Rightarrow 36m^2 = 64 \\ &\Rightarrow m^2 = \tfrac{16}{9} \\ &\Rightarrow \underline{\underline{m = \pm\tfrac{4}{3}}}. \end{aligned}$$

The two tangents from P meet the circle at the points A and B respectively.

(c) Find the lengths PA and PB. (4)

Solution

We will call the centre $C(0,3)$. Then

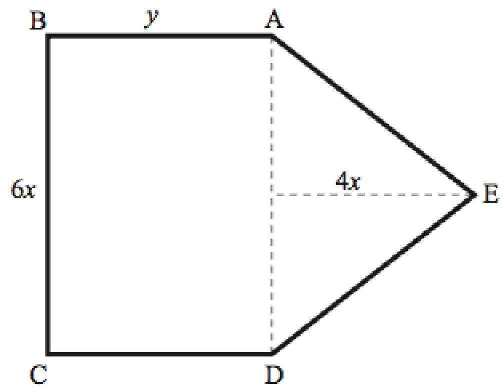
$$\begin{aligned} CP &= \sqrt{(0-0)^2 + [3-(-2)]^2} \\ &= \sqrt{5^2} \\ &= 5. \end{aligned}$$

Clearly, $CA = 3$ (it is the radius) and

$$\begin{aligned} CP^2 = CA^2 + AP^2 &\Rightarrow 5^2 = 3^2 + AP^2 \\ &\Rightarrow 25 = 9 + AP^2 \\ &\Rightarrow AP^2 = 16 \\ &\Rightarrow AP = 4; \end{aligned}$$

hence, $\underline{\underline{PA = PB = 4}}$.

12. The shape shown in the figure is made of metal rods.



Not to scale

$ABCD$ is a rectangle. $AB = CD = y$ cm and $BC = DA = 6x$ cm.
AED is an isosceles triangle with height $4x$ cm and $AE = ED$.

(a) Show that the perimeter, p cm, can be written as (3)

$$p = 16x + 2y.$$

Solution

$$\begin{aligned} p &= 6x + 2y + 2\sqrt{(3x)^2 + (4x)^2} \\ &= 6x + 2y + 10x \\ &= \underline{\underline{16x + 2y}}, \end{aligned}$$

as required.

You are given that $p = 96$.

(b) Show that the area of the shape, A cm^2, can be written as (3)

$$A = 288x - 36x^2.$$

Solution

$$\begin{aligned} 96 = 16x + 2y &\Rightarrow 2y = 96 - 16x \\ &\Rightarrow y = 48 - 8x. \end{aligned}$$

Now,

$$\begin{aligned}A &= \text{rectangle} + \text{triangle}\\ &= (6x \times y) + (\tfrac{1}{2} \times 6x \times 4x)\\ &= 6xy + 12x^2\\ &= 6x(48 - 8x) + 12x^2\\ &= (288x - 48x^2) + 12x^2\\ &= \underline{\underline{288x - 36x^2}},\end{aligned}$$

as required.

(c) Find the maximum area of the shape as x and y vary and find the values of x and y for this area. (6)

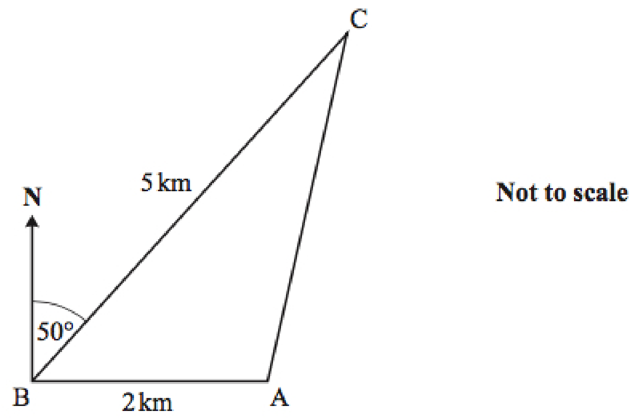
Solution

$$A = 288x - 36x^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = 288 - 72x$$

and

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 &\Rightarrow 288 - 72x = 0\\ &\Rightarrow 72x = 288\\ &\Rightarrow \underline{\underline{x = 4}}\\ &\Rightarrow \underline{\underline{y = 16}}\\ &\Rightarrow \underline{\underline{A = 576 \text{ cm}^2}}.\end{aligned}$$

13. Jessie walks at 3 km per hour in a straight line from a point B to a point C, a distance of 5 km. C is on a bearing 050° from B, as shown in the figure below.



Brandon sets out at the same time as Jessie. He starts from a point A which is 2 km due East of B. He walks at 2 km per hour directly to C.

(a) Calculate the distance AC, correct to 3 significant figures. (4)

Solution

$$\angle ABC = 90 - 50 = 40^\circ$$

and now we apply the cosine rule:

$$\begin{aligned} AC &= \sqrt{AB^2 + BC^2 - 2 \times AB \times BC \times \cos ABC} \\ &= \sqrt{2^2 + 5^2 - 2 \times 2 \times 5 \times \cos 40^\circ} \\ &= 3.698\,528\,239 \text{ (FCD)} \\ &= \underline{\underline{3.70 \text{ km (3 sf)}}}. \end{aligned}$$

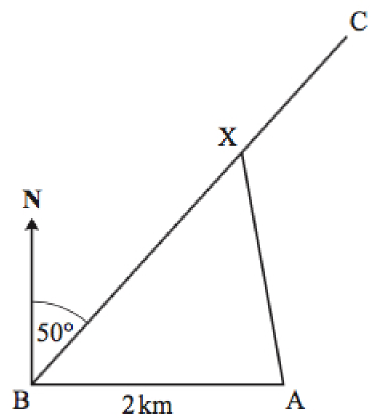
(b) Show that Brandon arrives at C approximately 11 minutes after Jessie arrives. (3)

Solution

$$\begin{aligned} \text{Time} &= \frac{3.698\ldots}{2} - \frac{5}{3} \\ &= 0.182\,597\,453 \text{ hours (FCD)} \\ &= 10.955\,847\,18 \text{ mins (FCD)}; \end{aligned}$$

hence, Brandon arrives at C approximately $\underline{\underline{\text{11 minutes}}}$ after Jessie arrives.

Charlie also sets out at the same time as Jessie. He walks in a straight line from A at 2 km per hour to meet Jessie at a point X on BC, as shown in the figure below.



He arrives at the point X at the same time as Jessie.

(c) Show that there are two possible positions for X and find the bearing on which Charlie must walk in each case. (5)

Solution

Let $\theta°$ be the Charlie walks and we use the sine rule:

$$\frac{\sin\theta°}{3} = \frac{\sin 40°}{2} \Rightarrow \sin\theta° = \frac{3\sin 40°}{2}$$
$$\Rightarrow \theta = 74.618\,568\,31,\ 105.381\,431\,7 \text{ (FCD)}.$$

Finally, he walks on a bearing of

$$270 + 74.618\ldots = 345.618\,568\,3 \text{ (FCD)}$$
$$= \underline{\underline{346° \text{ (3 sf)}}}$$

or

$$105.381\ldots - 90 = 15.381\,431\,69 \text{ (FCD)}$$
$$= \underline{\underline{015° \text{ (3 sf)}}}.$$

14. Two cars, P and Q, accelerate from rest from a point O at the same time.

(a) P accelerates uniformly at 2 ms^{-2}.

(i) Write down the formula for the displacement, s metres, of P at time t seconds after leaving O. (1)

Solution

$$\begin{aligned} a_P = 2 \Rightarrow v_P &= 2t + c_P \\ \Rightarrow s_P &= t^2 + c_P t + d_P \end{aligned}$$

for some constants c_P and d_P. Now,

$$t = 0 \Rightarrow s_P = v_P = 0 \Rightarrow \underline{\underline{s_P = t^2 \text{ m}}}.$$

(ii) Using appropriate units, find the time taken for P to reach a speed of 90 km h^{-1}. (3)

Solution
Well,

$$\begin{aligned} 90 \text{ kmh}^{-1} &= \frac{90 \text{ km}}{1 \text{ hr}} \\ &= \frac{90\,000 \text{ m}}{60 \text{ mins}} \\ &= \frac{90\,000 \text{ m}}{3\,600 \text{ s}} \\ &= 25 \text{ ms}^{-1} \end{aligned}$$

and

$$\begin{aligned} \text{time} &= \frac{25}{2} \\ &= \underline{\underline{12.5 \text{ s}}}. \end{aligned}$$

(b) Q accelerates from rest with variable acceleration a ms^{-2} where, at time t seconds, $a = 1 + kt$, where k is a positive constant. Q passes P when $t = 10$.

(i) Find the value of k. (5)

Solution

$$\begin{aligned} a_Q = (1 + kt) \Rightarrow v_Q &= t + \tfrac{1}{2}kt^2 + c_Q \\ \Rightarrow s_Q &= \tfrac{1}{2}t^2 + \tfrac{1}{6}kt^3 + c_Q t + d_Q \end{aligned}$$

for some constants c_Q and d_Q. Now,

$$t = 0 \Rightarrow s_Q = v_Q = 0 \Rightarrow s_Q = \left(\tfrac{1}{2}t^2 + \tfrac{1}{6}kt^3\right) \text{ m}.$$

Next,

$$\begin{aligned} t = 10 \Rightarrow 10^2 &= \tfrac{1}{2}(10^2) + \tfrac{1}{6}k(10^3) \\ \Rightarrow 100 &= 50 + \tfrac{1\,000}{6}k \\ \Rightarrow \tfrac{1\,000}{6}k &= 50 \\ \Rightarrow k &= \tfrac{3}{10}. \end{aligned}$$

(ii) Show that at the time when P reaches 90 $\mathrm{km\,h^{-1}}$, Q is travelling at a speed just less than 130 $\mathrm{km\,h^{-1}}$. (3)

Solution

$$\begin{aligned} t = 12.5 \Rightarrow v_Q &= 12.5 + \tfrac{1}{2}(\tfrac{3}{10})(12.5^2) \\ \Rightarrow v_Q &= 35.937\,5 \text{ ms}^{-1} \\ \Rightarrow v_Q &= 2\,156.25 \text{ m min}^{-1} \\ \Rightarrow v_Q &= 129\,375 \text{ m h}^{-1} \\ \Rightarrow v_Q &= 129.375 \text{ km h}^{-1}; \end{aligned}$$

hence, Q is travelling at a speed just less than 130 $\mathrm{km\,h^{-1}}$.