Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2012 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. (a) Write down and simplify the general term in the expansion of (3)

$$(x^2 + 3x)^8.$$

Solution
The general term is

$$\begin{aligned}\binom{8}{r}(x^2)^r(3x)^{8-r} &= \binom{8}{r}(x^{2r})(3^{8-r}x^{8-r}) \\ &= \underline{\underline{\binom{8}{r}3^{8-r}x^{8+r}}}.\end{aligned}$$

(b) Hence, or otherwise, obtain the coefficient of x^{13}. (2)

Solution
Now,

$$8 + r = 13 \Rightarrow r = 5$$

and the coefficient of x^{13} is

$$\binom{8}{5}3^3 = \underline{\underline{1\,512}}.$$

2. (a) Given the curve (3)

$$y = \frac{x}{x^2 + 4},$$

calculate the gradient when $x = 2$.

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = x^2 + 4 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

$$y = \frac{x}{x^2+4} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+4)(1) - (x)(2x)}{(x^2+4)^2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-x^2}{(x^2+4)^2}.$$

Finally,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-2^2}{(4^2+4)^2} = \underline{\underline{0}}.$$

(b) Determine (2)

$$\int \mathrm{e}^{-2t}\,\mathrm{d}t.$$

Solution

$$\int \mathrm{e}^{-2t}\,\mathrm{d}t = \underline{\underline{-\tfrac{1}{2}\mathrm{e}^{-2t} + c}}.$$

3. Given

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}:$$

(a) Calculate $\mathbf{M}^2$. (2)

Solution

$$\mathbf{M}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}}}.$$

(b) Calculate $\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3$. (2)

Solution

$$\begin{aligned}\mathbf{M}^3 &= \mathbf{M}^2\mathbf{M} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 3 & 0 & 0 \\ 18 & 3 & 0 \\ 0 & 0 & \lambda + \lambda^2 + \lambda^3 \end{pmatrix}}}.\end{aligned}$$

(c) For what values of λ does $\mathbf{M}$ have an inverse? (2)

Solution

$$\begin{aligned}\det \mathbf{M} &= 1(\lambda - 0) - 0 + 0 \\ &= \lambda;\end{aligned}$$

hence, $\mathbf{M}$ has an inverse if $\underline{\underline{\lambda \neq 0}}$.

4. (a) Express (3)

$$\frac{1}{x^2 + x}$$

in partial fractions, where x is neither 0 nor -1.

Solution

$$\begin{aligned}\frac{1}{x^2+x} &\equiv \frac{1}{x(x+1)} \\ &\equiv \frac{A}{x} + \frac{B}{x+1} \\ &\equiv \frac{A(x+1)+Bx}{x(x+1)}\end{aligned}$$

and so

$$1 \equiv A(x+1) + Bx.$$

$\underline{x=0}$: $1 = A$.
$\underline{x=-1}$: $1 = -B \Rightarrow B = -1$.
Hence,

$$\frac{1}{x^2+x} \equiv \underline{\underline{\frac{1}{x} - \frac{1}{x+1}}}.$$

A region is enclosed by the curve with equation

$$y = \frac{1}{\sqrt{x^2+x}},$$

the x-axis, and the lines $x = 1$ and $x = 3$.

(b) Calculate the volume of the solid of revolution formed by rotating this region through 360° about the x-axis. (4)

Solution

$$\begin{aligned}\text{Volume} &= \int_1^3 \pi \left(\frac{1}{\sqrt{x^2+x}}\right)^2 \mathrm{d}x \\ &= \pi \int_1^3 \frac{1}{x^2+x}\,\mathrm{d}x \\ &= \pi \int_1^3 \left(\frac{1}{x} - \frac{1}{x+1}\right) \mathrm{d}x \\ &= \pi \left[\ln|x| - \ln|x+1|\right]_{x=1}^3 \\ &= \pi\left[(\ln 3 - \ln 4) - (0 - \ln 2)\right] \\ &= \pi \ln\left(\frac{3\times 2}{4}\right) \\ &= \underline{\underline{\pi \ln \tfrac{3}{2}}}.\end{aligned}$$

5. A turkey is taken from a refrigerator to be cooked. Its temperature is 4°C when it is placed in an oven preheated to 180°C.

Its temperature, T°C, after a time of x hours in the oven satisfies the equation

$$\frac{\mathrm{d}T}{\mathrm{d}x} = k(180 - T).$$

(a) Show that (4)

$$T = 180 - 176\mathrm{e}^{-kx}.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}x} = k(180 - T) &\Rightarrow \frac{1}{(180 - T)}\,\mathrm{d}T = k\,\mathrm{d}x \\ &\Rightarrow \int \frac{1}{(180 - T)}\,\mathrm{d}T = \int k\,\mathrm{d}x \\ &\Rightarrow -\ln(180 - T) = kx + c \\ &\Rightarrow \ln(180 - T) = -kx - c \\ &\Rightarrow 180 - T = \mathrm{e}^{-kx-c} \\ &\Rightarrow T = 180 - \mathrm{e}^{-kx}\mathrm{e}^{-c} \\ &\Rightarrow T = 180 - A\mathrm{e}^{-kx}\end{aligned}$$

for some constant A. Now,

$$x = 0, T = 4 \Rightarrow 4 = 180 - A \Rightarrow T = 176$$

and

$$\underline{\underline{T = 180 - 176\mathrm{e}^{-kx}}}.$$

After an hour in the oven the temperature of the turkey is 30°C.

(b) Calculate the value of k correct to 2 decimal places. (2)

Solution

$$
\begin{aligned}
x = 1, T = 30 &\Rightarrow 30 = 180 - 176\mathrm{e}^{-k} \\
&\Rightarrow 176\mathrm{e}^{-k} = 150 \\
&\Rightarrow \mathrm{e}^{-k} = \tfrac{75}{88} \\
&\Rightarrow -k = \ln \tfrac{75}{88} \\
&\Rightarrow k = -\ln \tfrac{75}{88} \\
&\Rightarrow k = 0.159\,848\,700\,9 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{k = 0.16 \text{ (2 dp)}}}.
\end{aligned}
$$

The turkey will be cooked when it reaches a temperature of 80°C.

(c) After how long (to the nearest minute) will the turkey be cooked? (3)

Solution

$$
\begin{aligned}
80 = 180 - 176\mathrm{e}^{-kx} &\Rightarrow 176\mathrm{e}^{-kx} = 100 \\
&\Rightarrow \mathrm{e}^{-kx} = \tfrac{100}{176} \\
&\Rightarrow -kx = \ln \tfrac{100}{176} \\
&\Rightarrow x = -\tfrac{1}{k} \ln \tfrac{100}{176} \\
&\Rightarrow x = 3.536\,555\,41 \text{ hours (FCD)} \\
&\Rightarrow x = \underline{\underline{3 \text{ hours } 32 \text{ mins (nearest minute)}}}.
\end{aligned}
$$