Dr Oliver Mathematics
Further Mathematics
Conic Sections: Ellipses
Past Examination Questions

This booklet consists of 17 questions across a variety of examination topics.
The total number of marks available is 155.

1. The ellipse C has equation
$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$
   (a) Find an equation of the normal to C at the point $P(5\cos\theta, 3\sin\theta)$. (5)

   The normal to C at P meets the coordinate axes at Q and R.
   Given that $ORSQ$ is a rectangle, where O is the origin,

   (b) show that, as θ varies, the locus of S is an ellipse and find its equation in Cartesian form. (8)

2. The ellipse E has equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
   where a and b are positive constants and $a > b$.

   (a) Find an equation of the normal to C at the point $P(a\cos\theta, b\sin\theta)$. (5)

   The normal to C at P meets the x-axis at Q. R is the foot of the perpendicular from P to the x-axis.

   (b) Show that $\dfrac{OQ}{OR} = e^2$, where e is the eccentricity of C. (7)

3. The line with equation $y = mx + c$ is a tangent to the ellipse with equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$
   (a) Show that $c^2 = a^2m^2 + b^2$. (8)

   (b) Hence, or otherwise, find the equations of the tangents from the point $(3, 4)$ to the ellipse with equation (7)
$$\frac{x^2}{16} + \frac{y^2}{25} = 1.$$

4. The ellipse C has parametric equations (5)
$$x = 4\cos\theta,\ y = \sqrt{7}\sin\theta,\ 0 \leqslant \theta < 2\pi.$$
   Show that the foci of C are at the points $(3, 0)$ and $(-3, 0)$.

5. An ellipse has equation

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(a) Sketch the ellipse. (1)

(b) Find the eccentricity e. (2)

(c) State the coordinates of the foci of the ellipse. (2)

6. An ellipse, with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

has foci S and S'.

(a) Find the coordinates of the foci of the ellipse. (4)

(b) Using the focus-directrix property of the ellipse, show that, for any point P on the ellipse, (3)

$$SP + S'P = 6.$$

7. The point S, which lies on the positive x-axis, is a focus of the ellipse with equation

$$\frac{x^2}{4} + y^2 = 1.$$

Given that S is also the focus of a parabola P, with vertex at the origin, find

(a) a cartesian equation for P, (4)

(b) an equation for the directrix of P. (1)

8. The ellipse D has equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

and the ellipse E has equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

(a) Sketch D and E on the same diagram, showing the coordinates of the point where each curve crosses the axes. (3)

The point S is a focus for D and the point T is a focus for E.

(b) Find the length of ST. (5)

9. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{8} = 1,$$

where $a > 2\sqrt{2}$. The eccentricity of E is $\frac{1}{\sqrt{2}}$.

(a) Calculate the value of a. (2)

The ellipse E cuts the y-axis at the points D and D'. The foci of E are S and S'.

(b) Calculate the area of the quadrilateral $SDS'D'$. (3)

10. The line $x = 8$ is a directrix of the ellipse E with equation (5)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > 0, b > 0,$$

and the point $(2, 0)$ is the corresponding focus.
Find the value of a and the value of b.

11. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The line l_1 is a tangent to E at the point $P(a\cos\theta, b\sin\theta)$.

(a) Using calculus, show that an equation for l_1 is (4)

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

The circle C has equation

$$x^2 + y^2 = a^2.$$

The line l_2 is a tangent to C at the point $Q(a\cos\theta, a\sin\theta)$.

(b) Find an equation for the line l_2. (2)

Given that l_1 and l_2 meet at the point R,

(c) find, in terms of a, b, and θ, the coordinates of R. (3)

(d) Find the locus of R, as θ varies. (2)

12. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > b > 0$. The line l is a normal to E at the point $P(a\cos\theta, b\sin\theta)$, $0 < \theta < \frac{\pi}{2}$.

(a) Using calculus, show that an equation for l is (5)

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta.$$

The line l meets the x-axis at A and the y-axis at B.

(b) Show that the are of the triangle OAB, where O is the origin, may be written as $k\sin 2\theta$, giving the value of the constant k in terms of a and b. (4)

(c) Find, in terms of a and b, the exact coordinates of the point P, for which the area of the triangle OAB is a maximum. (3)

13. The point P lies on the ellipse E with equation

$$\frac{x^2}{36} + \frac{y^2}{9} = 1.$$

N is the foot of the perpendicular from the point P to the line $x = 8$. M is the midpoint of PN.

(a) Sketch the graph of the ellipse E, showing also the line $x = 8$ and a possible position for the line PN. (1)

(b) Find an equation of the locus of M as the point P moves around the ellipse. (4)

(c) Show that this locus is a circle and state its centre and radius. (3)

14. The points $P(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$, where $\alpha \neq \beta$, lie on the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

(a) Show the equation of the chord PQ is (4)

$$\frac{x}{3}\cos\frac{(\alpha+\beta)}{2} + \frac{y}{2}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}.$$

(b) Write down the coordinates of the mid-point of PQ. (1)

Given the that gradient, m, of the chord PQ is a constant,

(c) show that the centre of the chord lies on a line $y = -km$, expressing k in terms of m. (5)

15. The ellipse E has equation

$$x^2 + 9y^2 = 9.$$

The point $P(3\cos\theta, 2\sin\theta)$ is a general point on the ellipse E.

(a) Write down the value of a and the value of b. (1)

The line L is a tangent to E at the point P.

(b) Show that an equation of the line L is given by (3)

$$3y\sin\theta + x\cos\theta = 3.$$

The line L meets the x-axis at the point Q and meets the y-axis at the point R.

(c) Show that the area of the triangle OQR, where O is the origin, is given by $k \operatorname{cosec} 2\theta$, where k is a constant to be found. (3)

The point M is the midpoint of QR.

(d) Find a cartesian equation of the locus of M, giving your answer in the form $y^2 = \mathrm{f}(x)$. (4)

16. The ellipse E has equation $x^2 + 4y^2 = 4$.

(a) (i) Find the coordinates of the foci, F_1 and F_2, of E. (4)

(ii) Write down the equations of the directrices of E.

(b) Given that the point P lies of the ellipse, show that (4)

$$|PF_1| + |PF_2| = 4.$$

A chord of an ellipse is a line segment joining two points on the ellipse. The set of midpoints of the parallel chords of E with gradient m, where m is a constant, lie on the straight line l.

(c) Find an equation of l. (6)

17. The ellipse E has equation

$$\frac{x^2}{36} + \frac{y^2}{25} = 1.$$

The line l is the normal to E at the point $P(6\cos\theta, 5\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$.

(a) Use calculus to show that an equation of l is (5)

$$6x\sin\theta - 5y\cos\theta = 11\sin\theta\cos\theta.$$

The line l meets the x-axis at the point Q. The point R is the foot of the perpendicular from P to the x-axis.

(b) Show that $\dfrac{OQ}{OR} = e^2$, where e is the eccentricity of the ellipse E. (4)